Homework 5: Elementary linear logic
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to be returned by nov 25, 2015

Notations: Here application in λ-calculus will be denoted as $(t\ u)$. We will also write $\lambda x_1 x_2.t$ pour $\lambda x_1.\lambda x_2.t$ (please indicate in case you use other notations).

We write $\underline{n}$ the Church (unary) integer $\lambda f x.(f\ (f\ \ldots (f\ x) \ldots)$ (with n occurrences of f).

We want to type the following λ-term in IELL:

$$t = \lambda n f x.(n\ f\ (n\ f\ (f\ x)))$$

1. Show that t is typable in simple types (or if you prefer in system F, or in LJ2) with the type:

$$[(\alpha \to \alpha) \to (\alpha \to \alpha)] \to (\alpha \to \alpha) \to (\alpha \to \alpha)$$

   where α is a base type.

   What does the term t compute when it is applied to the Church integer $\underline{n}$?

2. We write $\mathbf{N}_\alpha$ the following IELL type for Church integers:

$$\mathbf{N}_\alpha = !(\alpha \multimap \alpha) \multimap !(\alpha \multimap \alpha).$$

   In the following whenever we speak of an IELL derivation, we mean an IELL derivation decorated with terms. Give an IELL derivation for the following judgements:

$$\begin{array}{rcl} y_1 : !(\alpha \multimap \alpha), y_2 : !(\alpha \multimap \alpha), f_3 : !(\alpha \multimap \alpha) & \vdash & \lambda x.(y_1\ (y_2\ (f_3\ x))) : !(\alpha \multimap \alpha) \\ n_1 : N_\alpha, n_2 : N_\alpha & \vdash & \lambda f x.(n_1\ f\ (n_2\ f\ (f\ x))) : \mathbf{N}_\alpha \end{array}$$

   Give an IELL derivation $\mathcal{D}$ of $\vdash t : !N_\alpha \multimap !N_\alpha$, by using if necessary the previous derivations.

3. Translate the derivation $\mathcal{D}$ into an ELL proof-net.

   Denote as R the proof-net corresponding to the application $(t\ \underline{1})$. Perform the normalisation of R and explain if the normal proof-net obtained does correspond to the expected result.