# Subject 1: Intuitionistic Linear Logic

We focus here on the multiplicative exponential parts of the systems.

## The System ILL

Intuitionistic linear logic (ILL) is obtained from (two-sided) LL through a restriction of sequents to the "intuitionistic shape": $\Gamma \vdash A$ where $\Gamma$ is a (possibly empty) sequence of formulas and $A$ is a formula.
Formulas are given by:

$$A ::= X \mid A \otimes A \mid 1 \mid A \multimap A \mid !A$$

Rules are:

$$\frac{}{A \vdash A}\,ax \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\,cut \qquad \frac{\Gamma \vdash C}{\sigma(\Gamma) \vdash C}\,ex(\sigma)$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes\mathrm{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes\mathrm{L} \qquad \frac{}{\vdash 1}\,1\mathrm{R} \qquad \frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}\,1\mathrm{L}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap\mathrm{R} \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,\multimap\mathrm{L}$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,!\mathrm{R} \qquad \frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C}\,!\mathrm{L} \qquad \frac{\Gamma, !A, !A \vdash C}{\Gamma, !A \vdash C}\,ctr\mathrm{L} \qquad \frac{\Gamma \vdash C}{\Gamma, !A \vdash C}\,wk\mathrm{L}$$

**Question 1.** Using the following translation of ILL formulas into LL formulas:

$$\begin{aligned}\overline{X} &= X\\ \overline{A \otimes B} &= \overline{A} \otimes \overline{B}\\ \overline{1} &= 1\\ \overline{A \multimap B} &= \overline{A}^{\perp} \parr \overline{B}\\ \overline{!A} &= !\overline{A}\end{aligned}$$

Show that $\Gamma \vdash A$ is provable in ILL implies that $\vdash \overline{\Gamma}^{\perp}, \overline{A}$ is provable in LL.

**Question 2.** Using the following translation of ILL formulas into intuitionistic formulas:

$$\begin{aligned}\underline{X} &= X\\ \underline{A \otimes B} &= \underline{A} \wedge \underline{B}\\ \underline{1} &= \top\\ \underline{A \multimap B} &= \underline{A} \to \underline{B}\\ \underline{!A} &= \underline{A}\end{aligned}$$

Show that $\Gamma \vdash A$ is provable in ILL implies that $\underline{\Gamma} \vdash \underline{A}$ is provable in the intuitionistic sequent calculus LJ.

**Question 3.** Using the following translation of intuitionistic formulas into $\mathsf{ILL}$ formulas:

$$X^G = X$$
$$(A \wedge B)^G = !A^G \otimes !B^G$$
$$\top^G = 1$$
$$(A \to B)^G = !A^G \multimap B^G$$

Show that $\Gamma \vdash A$ is provable in $\mathsf{LJ}$ implies that $!\Gamma^G \vdash A^G$ is provable in $\mathsf{ILL}$.

**Question 4.** Show that $\Gamma \vdash A$ is provable in $\mathsf{LJ}$ if and only if $!\Gamma^G \vdash A^G$ is provable in $\mathsf{ILL}$.

## The System JLL

A *fragment* $\mathsf{B}$ of a logical system $\mathsf{A}$ is a sub-system of $\mathsf{A}$ obtained by constraining formulas to live in a given sub-set $\mathcal{F}$ of the set of formulas of $\mathsf{A}$. $\mathcal{F}$ is supposed to be closed under sub-formula (*i.e.* if $A \in \mathcal{F}$ and $B$ is a sub-formula of $A$ then $B \in \mathcal{F}$). The rules of the fragment $\mathsf{B}$ are exactly the rules of $\mathsf{A}$ restricted to the chosen sub-set of formulas $\mathcal{F}$.

**Question 5.** Show that if $\mathsf{F}$ is a fragment of (propositional) $\mathsf{LL}$ and $\Gamma$ contains only formulas of $\mathsf{F}$ then $\vdash \Gamma$ is provable in $\mathsf{F}$ if and only if it is provable in $\mathsf{LL}$.

$I/O$-formulas are defined through the grammar:

$$\begin{array}{lcccccccccccc} O & ::= & X & \mid & O \otimes O & \mid & 1 & \mid & I \parr O & \mid & O \parr I & \mid & !O \\ I & ::= & X^\perp & \mid & I \parr I & \mid & \perp & \mid & O \otimes I & \mid & I \otimes O & \mid & ?I \end{array}$$

$\mathsf{JLL}$ is the fragment of $\mathsf{LL}$ obtained by using $I/O$-formulas only.

**Question 6.** Give an $\mathsf{(ME)LL}$ formula which is not an $I/O$-formula.

**Question 7.** Show that $\Gamma \vdash A$ is provable in $\mathsf{ILL}$ implies that $\vdash \overline{\Gamma}^\perp, \overline{A}$ is provable in $\mathsf{JLL}$.

**Question 8.** Show that if $\vdash \Gamma$ is provable in $\mathsf{LL}$ and contains only $I/O$-formulas then $\Gamma$ contains exactly one $O$-formula.

**Question 9.** Show that $\Gamma \vdash A$ is provable in $\mathsf{ILL}$ if and only if $\vdash \overline{\Gamma}^\perp, \overline{A}$ is provable in $\mathsf{JLL}$.

**Question 10.** Conclude that $\Gamma \vdash A$ is provable in $\mathsf{LJ}$ if and only if $\vdash ?(\overline{\Gamma^G})^\perp, \overline{A^G}$ is provable in $\mathsf{LL}$.

In particular the formula $A$ is provable in $\mathsf{LJ}$ if and only if $\overline{A^G}$ is provable in $\mathsf{LL}$ with:

$$\overline{X^G} = X$$
$$\overline{(A \wedge B)^G} = !\overline{A^G} \otimes !\overline{B^G}$$
$$\overline{\top^G} = 1$$
$$\overline{(A \to B)^G} = ?\overline{A^G}^\perp \parr \overline{B^G}$$