Subject 1: Classical Sequent Calculus

to be returned on Friday, September 25th

In the whole subject, exchange rules can be left implicit.

Formulas are given by:

$$A ::= X \mid \neg A \mid A \wedge A \mid A \vee A \mid \top \mid \bot$$

where X ranges over the elements of a given countable set of variables.

Two-sided LK

We consider the following rules for two-sided classical sequent calculus LK:

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \tau(\Delta)}$$

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta} \qquad \frac{}{\Gamma \vdash \Delta, \top} \qquad \frac{}{\Gamma, \bot \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \wedge B} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}$$

Question 1. For each sequent below, if it is provable give a proof in two-sided LK, and if it is not provable try to give a short justification.

a. $X \vee X \vdash X$

b. $X \vdash X \vee X$

c. $X \wedge Y \vdash Y \wedge X$

d. $\bot \wedge X \vdash Y$

e. $Y \vdash \bot \wedge X$

f. $(\neg X \wedge Y) \vee X \vdash Y$

g. $Y \vdash (\neg X \wedge Y) \vee X$

h. $X \wedge \neg X \vdash Y$

i. $X \vee (Y \vee Z) \vdash (X \vee Y) \vee Z$

j. $X \wedge (Y \vee Z) \vdash (X \wedge Y) \vee Z$

k. $(X \wedge Y) \vee Z \vdash X \wedge (Y \vee Z)$

l. $(X \wedge Y) \vee (Z \wedge T) \vdash (X \vee Z) \wedge (Y \vee T)$

m. $(X \vee Z) \wedge (Y \vee T) \vdash (X \wedge Y) \vee (Z \wedge T)$

n. $X \wedge (Y \vee Z) \vdash (X \wedge Y) \vee (X \wedge Z)$

o. $(X \wedge Y) \vee (X \wedge Z) \vdash X \wedge (Y \vee Z)$

p. $\neg(X \vee \neg X) \vdash \neg(\neg X \wedge X)$

q. $\vdash (\neg(X \vee X) \vee Y) \vee X$

r. $X \vee \neg(Y \wedge Z) \vdash \neg(\neg X \wedge Y) \vee \neg Z$

One-sided LK

We consider the following rules for one-sided classical sequent calculus LK:

$$\frac{}{\vdash A, \neg A} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta} \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}$$

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A} \qquad \frac{}{\vdash \Gamma, \top}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}$$

Question 2. For every sequent of Question 1, if it is provable in two-sided LK, give its one-sided translation and prove it in one-sided LK.

Question 3. If $\vdash \Gamma$ is provable in one-sided LK, prove that $\vdash \Gamma[{}^{A}/_{X}]$ is provable as well for any formula A.