Subject 2: Zoology of Linear Formulas

to be returned on Monday, October 6th

In the whole subject, we work in multiplicative exponential linear logic (MELL), and exchange rules can be left implicit.

1 Synchrony and Polarity

Synchronous Formulas

A formula S of linear logic is called *synchronous* if S is linearly equivalent to $!S$ (denoted $S \multimap\!\!\circ\; !S$), which means that both $\vdash S^\perp, !S$ and $\vdash ?(S^\perp), S$ are provable.

Question 1. Prove 1 is synchronous.

Question 2. Prove $\perp$ is not synchronous.

Question 3. Prove $!A$ is synchronous for any A.

Question 4. Prove $S_1 \otimes S_2$ is synchronous if S_1 and S_2 are both synchronous.

Question 5. If S_1 and S_2 are two synchronous formulas, show that $\vdash S_1, S_2$ cannot be provable.

Asynchronous Formulas

A formula L of linear logic is called *asynchronous* if L is linearly equivalent to $?L$ (*i.e.* both $\vdash L^\perp, ?L$ and $\vdash !(L^\perp), L$ are provable).

Question 6. Prove L is asynchronous if and only if $L^\perp$ is synchronous.

Question 7. Find an asynchronous formula L and two synchronous formulas S_1 and S_2 such that $\vdash L, S_1, S_2$ is provable.

Question 8. Prove no formula can be both synchronous and asynchronous.

Question 9. Give a formula which is neither synchronous nor asynchronous.

Question 10. Prove the following generalized weakening rule is derivable for any asynchronous formula L:

$$\frac{\vdash \Gamma}{\vdash \Gamma, L}$$

Question 11. Prove the following generalized contraction rule is derivable for any asynchronous formula L:

$$\frac{\vdash \Gamma, L, L}{\vdash \Gamma, L}$$

Question 12. Prove the following generalized promotion rule is derivable for any Λ which is a list of asynchronous formulas:

$$\frac{\vdash \Lambda, A}{\vdash \Lambda, !A}$$

Regular Formulas

A formula R of linear logic is called *regular* if R is linearly equivalent to $?!R$.

Question 13. Prove $\bot$ is regular.

Question 14. Prove $?S$ is regular for any synchronous formula S.

Question 15. Prove $?!A$ is regular for any formula A.

Question 16. Prove $R_1 \parr R_2$ is regular if R_1 and R_2 are both regular.

Polarized Formulas

Positive (P) and *negative* (N) formulas are defined through the following grammar:

$$\begin{array}{lcccccccc} P & ::= & !X & \mid & 1 & \mid & P \otimes P & \mid & !N \\ N & ::= & ?X^{\bot} & \mid & \bot & \mid & N \parr N & \mid & ?P \end{array}$$

Question 17. Prove any positive formula is synchronous.

Question 18. Prove any negative formula is asynchronous.

Question 19. Prove, if P and Q are two positive formulas, there is no polarized context Γ (*i.e.* containing only positive and negative formulas) such that $\vdash \Gamma, P, Q$ is provable in MELL.

2 Exponential Modalities

An exponential modality μ is an arbitrary (possibly empty) sequence built with the two exponential connectives ! and ? (*i.e.* an element of $\mathcal{M} = \{!, ?\}^*$). It can be considered itself as a unary connective. This leads to the notation μA for applying an exponential modality μ to a formula A.

An exponential modality μ is *smaller* than an exponential modality ν (denoted $\mu \lesssim \nu$) if, for any formula A of MELL, $\vdash (\mu A)^{\bot}, \nu A$ is provable (*i.e.* $\mu A \multimap \nu A$). Two exponential modalities μ and ν are *equivalent* (denoted $\mu \sim \nu$) if $\mu \lesssim \nu$ and $\nu \lesssim \mu$.

Question. Characterize the relation induced by $\lesssim$ on the quotient set $\mathcal{M}/\!\sim$.