Subject 2: Linear Sequent Calculus

to be returned on Tuesday, September 26th

In the whole subject, exchange rules can be left implicit.

MLL

Formulas are given by:

$$A ::= X \mid X^\perp \mid A \otimes A \mid A \parr A \mid 1 \mid \perp$$

where X ranges over the elements of a given countable set $\mathcal{V}$ of variables.
We consider the following rules for the one-sided multiplicative linear sequent calculus MLL:

$$\frac{}{\vdash A, A^\perp}\,ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\,ex$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \qquad \frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp$$

Question 1. For each sequent below, if it is provable give a proof in one-sided MLL, and if it is not provable try to give a short justification.

a. $\vdash \perp, X^\perp \parr X$

b. $\vdash X \otimes Y, Y^\perp \parr (1 \otimes X^\perp)$

c. $\vdash X \otimes X, X^\perp \otimes X^\perp$

d. $\vdash X^\perp, X \otimes (X^\perp \parr X)$

e. $\vdash Y \parr X^\perp, \perp \otimes \perp, X, Y^\perp$

f. $\vdash X \otimes (Y \parr X^\perp), X^\perp, X$

g. $\vdash X^\perp \parr (Y^\perp \otimes Z^\perp), (X \otimes Y) \parr Z$

h. $\vdash X^\perp \otimes X, (X \parr X^\perp) \otimes (X \parr X^\perp)$

i. $\vdash X \parr X^\perp, (X^\perp \otimes X) \parr (X^\perp \otimes X)$

Question 2. Prove the following facts about MLL:

a. $\vdash A \otimes B$ is provable if and only if both $\vdash A$ and $\vdash B$ are provable.

b. If $\vdash 1, A \otimes B$ is provable then either $\vdash A$ or $\vdash B$ is provable.

c. If $\vdash A \otimes B, C \otimes D$ is provable then $\vdash A$ or $\vdash B$ or $\vdash C$ or $\vdash D$ is provable.

d. Let $\sharp_1(\Gamma)$ be the number of occurrences of 1 in Γ, $\sharp_\otimes(\Gamma)$ be the number of occurrences of $\otimes$ in Γ, $\sharp_\mathcal{V}(\Gamma)$ be the number of occurrences of elements of $\mathcal{V}$ not below a $\_^\perp$ in Γ, prove that $\sharp_1(\Gamma) + \sharp_\mathcal{V}(\Gamma) = 1 + \sharp_\otimes(\Gamma)$ for all Γ such that $\vdash \Gamma$ is provable.

e. Assuming that $\vdash A$ is provable, show $\vdash A, A$ is not provable.

Question 3. Define the formulas and rules of two-sided MLL (do not forget to be careful with negation).

MALL

Formulas are given by:

$$A ::= X \mid X^\perp \mid A \otimes A \mid A \parr A \mid 1 \mid \perp \mid A \,\&\, A \mid A \oplus A \mid \top \mid 0$$

where X ranges over the elements of a given countable set $\mathcal{V}$ of variables.
We consider the following rules for the one-sided multiplicative-additive linear sequent calculus MALL:

$$\frac{}{\vdash A, A^\perp}\,ax \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\,ex$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \qquad \frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus_2 \qquad \frac{}{\vdash \Gamma, \top}\,\top$$

Question 4. For each sequent below, if it is provable give a proof in one-sided MALL, and if it is not provable try to give a short justification.

a. $\vdash X, X \oplus X^\perp$

b. $\vdash X^\perp, X \oplus X$

c. $\vdash X^\perp, X \,\&\, X$

d. $\vdash X^\perp, X \,\&\, X^\perp$

e. $\vdash X^\perp, X \,\&\, (X \oplus Y)$

f. $\vdash X, X^\perp \oplus (X^\perp \,\&\, Y^\perp)$

g. $\vdash 0, \top \parr X$

h. $\vdash X^\perp \,\&\, (Y^\perp \oplus Z^\perp), (X \oplus Y) \,\&\, (X \oplus Z)$

i. $\vdash X^\perp \parr (Y^\perp \oplus Z^\perp), (X \otimes Y) \,\&\, (X \otimes Z)$

j. $\vdash X^\perp \parr (Y^\perp \,\&\, Z^\perp), (X \otimes Y) \oplus (X \otimes Z)$

k. $\vdash (X^\perp \parr Y^\perp) \oplus (X^\perp \parr Z^\perp), X \otimes (Y \,\&\, Z)$

l. $\vdash (X^\perp \parr Y^\perp) \,\&\, (X^\perp \parr Z^\perp), X \otimes (Y \oplus Z)$