Subject 3: Computing with Proof-Nets

to be returned on Monday, October 20th

Questions marked with (*) might be a bit more involved.
We only use the terminology *proof-net* for a proof-structure which satisfies the Danos-Regnier correctness criterion (acyclicity and connectedness in the multiplicative case, and acyclicity only in the multiplicative exponential case).

Multiplicative Booleans

In this part we work with proof-nets for multiplicative linear logic without units, thus built upon the following formulas:

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid A \parr A$$

X ranges over the elements of a given set of variables. X and $X^{\perp}$ are called *atomic formulas.* A proof-net is *atomic* if the conclusions of all its axiom nodes are labelled with atomic formulas.

Question 1. Prove it is possible to transform any proof-net into an atomic one without changing the labels of its conclusions.

We consider the formula $\mathbf{B} = (X^{\perp} \parr X^{\perp}) \parr (X \otimes X)$.

Question 2. Give all the cut-free proof-nets with a unique conclusion labelled $\mathbf{B}$.

Among the two atomic proof-nets of Question 2, only one can be obtained through the transformation of Question 1. We call it TRUE. The other atomic one is called FALSE. The set of Booleans is $\mathbb{B} = \{\mathsf{true}, \mathsf{false}\}$ and we define $\overline{\mathsf{true}} = \text{TRUE}$ and $\overline{\mathsf{false}} = \text{FALSE}$.
A function f from $\mathbb{B}$ to $\mathbb{B}$ is said to be *represented* by the proof-net $\mathcal{R}$ with two conclusions $\mathbf{B}^{\perp}$ and $\mathbf{B}$ if the normal form of the proof-net $\mathcal{R}_b$ (obtained by putting a cut node between the conclusion $\mathbf{B}$ of $\overline{b}$ and the conclusion $\mathbf{B}^{\perp}$ of $\mathcal{R}$) is $\overline{f(b)}$, for any $b \in \mathbb{B}$.

Question 3. Give a proof-net representing the negation function $\mathbb{B} \to \mathbb{B}$:

$$\mathsf{true} \mapsto \mathsf{false}$$
$$\mathsf{false} \mapsto \mathsf{true}$$

Question 4. Give all the cut-free proof-nets with two conclusions: $\mathbf{B}^{\perp}$ and $\mathbf{B}$.

Question 5. Give a function from $\mathbb{B}$ to $\mathbb{B}$ which cannot be represented by a proof-net.

Let G be a formula, a function f from $\mathbb{B}^n$ ($n \geq 0$) to $\mathbb{B}$ is said to be *G-represented* by the proof-net $\mathcal{R}$ with n conclusions $\mathbf{B}^{\perp}$ and a conclusion $\mathbf{B} \otimes G$ if the normal form of the proof-net $\mathcal{R}_{\vec{b}}$ (obtained by putting n cut nodes between the conclusion $\mathbf{B}$ of each $\overline{b_i}$ ($b_i \in \vec{b}$) and the ith conclusion $\mathbf{B}^{\perp}$ of $\mathcal{R}$) has a $\otimes$ node above its unique conclusion with $\overline{f(\vec{b})}$ above its left premise, for any $\vec{b} \in \mathbb{B}^n$.

Question 6. Prove any function from $\mathbb{B}$ to $\mathbb{B}$ can be **B**-represented by some proof-net.

If A is a formula, we define $\mathbf{B}[A] = \mathbf{B}[{}^{A}/_{X}] = (A^{\perp} \parr A^{\perp}) \parr (A \otimes A)$ (in particular $\mathbf{B}[X] = \mathbf{B}$).

Question 7. If $\mathcal{R}$ is a proof-net with conclusions $C_1, \ldots, C_n$ and A is a formula, define a proof-net $\mathcal{R}[{}^{A}/_{X}]$ with conclusions $C_1[{}^{A}/_{X}], \ldots, C_n[{}^{A}/_{X}]$.

We define $\textsc{true}[A] = \textsc{true}[{}^{A}/_{X}]$ and $\textsc{false}[A] = \textsc{false}[{}^{A}/_{X}]$ with conclusion $\mathbf{B}[A]$.
Let G and $A_1, \ldots, A_n$ be formulas, a function f from $\mathbb{B}^n$ ($n \geq 0$) to $\mathbb{B}$ is said to be *G-represented up to* $(A_1, \cdots, A_n)$ by the proof-net $\mathcal{R}$ with conclusions $\mathbf{B}[A_1]^{\perp}, \ldots, \mathbf{B}[A_n]^{\perp}$ and $\mathbf{B} \otimes G$ if the normal form of the proof-net $\mathcal{R}_{\vec{b}}$ (obtained by putting n cut nodes between the conclusion $\mathbf{B}[A_i]$ of each $\overline{b_i}[A_i]$ and the conclusion $\mathbf{B}[A_i]^{\perp}$ of $\mathcal{R}$, with $b_i \in \vec{b}$) has a $\otimes$ node above its unique conclusion with $\overline{f(\vec{b})}$ above its left premise, for any $\vec{b} \in \mathbb{B}^n$.

Question 8. Give a proof-net **B**-representing up to $(\mathbf{B}, X, X)$ the `if` function $\mathbb{B}^3 \to \mathbb{B}$:

$$\mathsf{true}, b_1, b_2 \mapsto b_1$$
$$\mathsf{false}, b_1, b_2 \mapsto b_2$$

Question 9. If $f : \mathbb{B}^{n+1} \to \mathbb{B}$ is G-represented up to $(A_0, \cdots, A_n)$ by $\mathcal{R}$ and $g : \mathbb{B}^m \to \mathbb{B}$ is H-represented up to $(C_1, \cdots, C_m)$ by $\mathcal{S}$, explain how to represent the composition:

$$b_1, \cdots, b_m, b_{m+1}, \cdots, b_{m+n} \mapsto f(g(b_1, \cdots, b_m), b_{m+1}, \cdots, b_{m+n})$$

Question 10. (*) Prove any function from $\mathbb{B}^2$ to $\mathbb{B}$ can be $(\mathbf{B}\otimes\mathbf{B}\otimes\mathbf{B}\otimes\mathbf{B}[\mathbf{B}]\otimes\mathbf{B}[\mathbf{B}])$-represented up to $(\mathbf{B}[\mathbf{B}] \otimes \mathbf{B}[\mathbf{B}], \mathbf{B})$ by some proof-net.

Exponential Booleans

We now move to proof-nets for multiplicative exponential linear logic without units, thus built upon the following formulas:

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid A \parr A \mid !A \mid ?A$$

We consider the formula $\mathbf{C} = !(?X^{\perp} \parr (?X^{\perp} \parr X))$.

Question 11. Prove there exist exactly two cut-free proof-nets with a unique conclusion **C** (up to the Rétoré equivalence).

As in the previous part, we obtain a representation of Booleans by defining $\widetilde{\mathsf{true}}$ and $\widetilde{\mathsf{false}}$ to be these two proof-nets.

Question 12. Give a proof-net with two conclusions: $?(\mathbf{B}[!X]^{\perp})$ and **C**.

Question 13. Give a proof-net with two conclusions: $\mathbf{C}^{\perp}$ and $!\mathbf{B}[!X]$.

Question 14. Compute the normal form of the proof-net with conclusions $?(\mathbf{B}[!X]^{\perp})$ and $!\mathbf{B}[!X]$ obtained by adding a cut node between the conclusions **C** and $\mathbf{C}^{\perp}$ of the proof-nets of the previous two questions. *Give some comments.*

A function f from $\mathbb{B}^n$ ($n \geq 0$) to $\mathbb{B}$ is said to be *e-represented* by the proof-net $\mathcal{R}$ with n conclusions $\mathbf{C}^{\perp}$ and a conclusion **C** if the normal form of the proof-net $\mathcal{R}_{\vec{b}}$ (obtained by putting n cut nodes between the conclusion **C** of each $\widetilde{b_i}$ and the ith conclusion $\mathbf{C}^{\perp}$ of $\mathcal{R}$, with $b_i \in \vec{b}$) is $\widetilde{f(\vec{b})}$ (up to the Rétoré equivalence), for any $\vec{b} \in \mathbb{B}^n$.

Question 15. (*) Prove any function from $\mathbb{B}^n$ ($n \geq 0$) to $\mathbb{B}$ can be e-represented by some proof-net.