Non-Wellfounded Derivations for Intersection Subtyping with Fixpoints

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Abstract

Subtyping is a key ingredient of many intersection type systems. In the case of the BCD system, B. Pierce gave a transitivity-free presentation of subtyping. This provides better structural properties for the analysis of this relation and leads to a simple decision algorithm.

We generalize this transitivity-free approach to a general class of extensions of BCD allowing to impose some pre-order as well as some fixpoint equations on atoms. This includes in particular the case of various intersection type systems compatible with η-equality (Scott, Park, etc.).

Proving the equivalence between the transitivity-free systems and their BCD-style presentation is addressed by means of cut-elimination techniques from proof theory. Due to the presence of fixpoints, we are led to introduce non-wellfounded derivations. In the context of the structural analysis of intersection subtyping, this happens to be the first use of infinitary derivations.

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Supplementary Material `Rocq` formalization of results of the paper:
Software (Source Code): `https://github.com/olaure01/islogic/tree/FSCD2026`

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1 Introduction

Intersection type systems are a family of type assignment systems for the λ-calculus that extend the simply-typed λ-calculus with a new intersection type constructor $\cap$. Two main consequences can be stressed. First, they make more λ-terms typable, which leads to various characterizations of *operational* properties of λ-terms. Second, they allow to extend subject β-reduction to subject β-expansion (and similarly for η), so that typing becomes invariant under β-conversion. This opens the possibility to use the set of types associated with a given λ-term as its interpretation in a *denotational* model. The induced models are called *filter models*. We refer to [3, Part III] for a more detailed presentation (other natural references for the present work are [2, 9, 1, 8]).

A key intersection type system was introduced in 1983 by Barendregt, Coppo and Dezani-Ciancaglini [2] (now known under the name BCD, see Table 1). It has been used as a core system for many extensions. An important feature of these systems is the presence of a *subtyping relation*, together with a subsumption rule. It is then possible to make the subtyping relation vary (without modifying the typing rules themselves) to explore a wide range of systems. Among them, one usually restricts to those compatible with β-conversion (and in

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many cases η-conversion). A sufficient condition on subtyping, *the β-condition*, has been identified which guarantees that typing will be compatible with β-conversion. In practice, this allows studies to focus on intersection subtyping, somehow ignoring the typing part as long as, in the end, the β-condition is satisfied. Through this approach intersection subtyping becomes the main object of study and the β-condition emerges as a crucial property (see for example [1]).

Here starts the structural analysis of intersection subtyping (denoted $\leq$). From an algorithmic point of view as well as for analysing the structure of subtyping derivations, the presence of an explicit *transitivity rule* in the presentation of BCD subtyping complicates matters. From a proof-theoretical point of view, this can be analysed as a lack of a sub-formula property. The standard approach from sequent calculi is to give presentations of the systems for which the cut rule (the analogue of transitivity) is an admissible rule, not a rule of the system. In the case of BCD subtyping, such a presentation can be found in Pierce's work [13]. Our goal is then to extend his ideas to a larger variety of intersection subtyping systems in a uniform way (in the spirit of [1]).

A weakness of the BCD typing system is that it does not satisfy subject η-expansion, meaning that the induced filter model is not a model of $\beta\eta$-conversion. Extending BCD subtyping to obtain systems compatible with η requires having enough axioms so that any atom becomes equivalent to an intersection of arrow types. Many such proposals can be found in the literature [14, 12, 5, 10, 9, 15]. They come as extensions of BCD subtyping relying on two main parameters: a pre-order relation on atoms (already taken into account in Pierce's system), and equations of the form $X \sim A$ (*i.e.* $X \leq A$ and $A \leq X$) which can be understood as defining the atom X by a given type A. As X may occur in A, this becomes a fixpoint equation.

The challenge we address is the extension of Pierce's work with fixpoint equations in such a way that the obtained system preserves good structural properties (the admissibility of transitivity for example). We consider a setting where each atom X is associated with a "definition" δX with the property $X \sim \delta X$. One way sequent calculi manage to get admissibility of cut (and hence the sub-formula property) is by turning axioms into rules which (when read bottom-up) decompose the connectives. In the case of fixpoints, a standard principle is unfolding which replaces a fixpoint type $\mu X.F$ by its unfolding $F[{}^{\mu X.F}/_{X}]$. In our setting, this corresponds to replacing X by δX. Such a rule naturally breaks the sub-formula property, but in a somehow controlled way since the Fischer-Ladner closure describes a finite set of generated types. Now comes the problem of cut (and then transitivity) admissibility. In the context of fixpoint logics, most existing approaches currently rely on non-finitary proof systems where proofs either have infinitely branching rules or infinite branches (see for example [7]). In this second category, proofs are called *non-wellfounded* proofs.

We introduce an intersection subtyping system which extends Pierce's system with unfolding rules and allows for non-wellfounded derivations. As far as we know, this is the first use of such derivations in the structural analysis of intersection subtyping (for uses in an intersection-free setting see [6] for example). The system is parameterized by the subtyping relation on atoms and a function δ from atoms to types (leading to the fixpoint equations $X \sim \delta X$). The transitivity rule is proved to be admissible, so that provability in our system coincides with provability in the corresponding extension of BCD subtyping. This requires a compatibility property between subtyping on atoms and δ. Thanks to the flexibility of our approach, we are able to rely on a *safety* condition which strictly subsumes the "strong beta condition" of [1]. Although the sub-formula property does not hold in its standard form (because of unfoldings) and despite the use of non-wellfounded derivations, the structure of

$$\frac{}{A \leq A} \qquad \frac{A \leq B \quad B \leq C}{A \leq C} \qquad \frac{}{A \leq \Omega}$$

$$\frac{}{A \cap B \leq A} \qquad \frac{}{A \cap B \leq B} \qquad \frac{}{A \leq A \cap A} \qquad \frac{A \leq C \quad B \leq D}{A \cap B \leq C \cap D}$$

$$\frac{C \leq A \quad B \leq D}{A \to B \leq C \to D} \qquad \frac{}{(C \to A) \cap (C \to B) \leq C \to (A \cap B)} \qquad \frac{}{\Omega \leq \Omega \to \Omega}$$

Table 1 BCD Subtyping

the obtained system is good enough to be able to prove that any safe instance validates the β-condition by analysing a finite prefix of each derivation.

Contents

After recalling the definition of intersection type systems (with the BCD typing system as a key example) with the β- and η-conditions in Section 2, we present Pierce's transitivity-free system which takes a primitive order relation on atoms into account in Section 3. More precisely we define a slight variant $\mathsf{IS}_{\prec}$.

Section 4 introduces fixpoint equations in BCD subtyping (system $\mathsf{BCD}^{\delta}_{\prec}$) and the non-wellfounded system $\mathsf{IS}^{\delta}_{\prec}$ which is our key contribution. The major technical result is cut admissibility in Section 4.3 from which we derive the equivalence between $\mathsf{BCD}^{\delta}_{\prec}$ and $\mathsf{IS}^{\delta}_{\prec}$ (Section 4.4). All safe instances validate the β-condition and Section 4.5 provides a comparison with [1]. The expressiveness of our approach is illustrated through various instances in Section 5.

Core results of the paper have been formalized in `Rocq` (Section 6).

2 Intersection Types

We briefly recall the key ingredients of intersection type systems (more details can be found in [3, Part III]). Some definitions are simplified by relying on the fact that we focus on extensions of the BCD intersection type system [2] only.

2.1 Intersection Type Systems

Given a set $\mathcal{A}$ of **atoms** denoted X, Y, Z, etc, **types** of intersection type systems are given by:

$$A ::= X \mid \Omega \mid A \cap A \mid A \to A$$

Terms are usual λ-terms and typing judgements are of the form $\Gamma \vdash t : A$ where Γ is a mapping (with finite support) from term variables to types, t is a λ-term and A is a type.

We assume given a pre-order relation $\leq$ on types called the **intersection subtyping** relation which satisfies the properties of Table 1 (meaning that each rule is admissible, not that these are the only rules to consider). In particular this endows types with a bounded meet-semi(pre)lattice structure such that $\cap$ defines (binary) meets and Ω is the top element. We use the notation $A \sim B$ if both $A \leq B$ and $B \leq A$ hold (this is the equivalence relation induced by $\leq$).

$$\frac{}{\Gamma, x : A \vdash x : A}\ var \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ abs \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B}\ app$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash t : B}{\Gamma \vdash t : A \cap B}\ inter \qquad \frac{}{\Gamma \vdash t : \Omega}\ omg \qquad \frac{\Gamma \vdash t : A \quad A \leq B}{\Gamma \vdash t : B}\ sub$$

Table 2 Intersection Typing System

Up to $\sim$, the constructor $\cap$ is then associative, commutative and idempotent, with Ω as unit. If I is a finite set $\{i_1, \dots, i_n\}$ (possibly empty), we use the notation $\bigcap_{i \in I} A_i$ to represent the equivalence class of $A_{i_1} \cap (A_{i_2} \cap \cdots (A_{i_n} \cap \Omega))$ with respect to associativity, commutativity and unitality (we do not take idempotency into account here). This will be used to state results and properties which do not depend on a specific choice of an element of this equivalence class (see for example β- and η-conditions below).

▶ **Example 1.** From Table 1, one gets $\Omega \sim \Omega \to \Omega$ (since both inequalities are provided). It is also possible to deduce $(C \to A) \cap (C \to B) \sim C \to (A \cap B)$.

The typing rules are presented on Table 2. Note that the first three rules are exactly those of the simply typed λ-calculus, and that the subsumption rule (*sub*) is the only one depending on $\leq$.

It is known from the literature (see [3, 11] for example), that **subject β-expansion** and **subject η-reduction** always hold:

$$\begin{array}{llll} \text{if } t\ \beta\text{-reduces to } u \text{ then} & \Gamma \vdash u : A & \Longrightarrow & \Gamma \vdash t : A \\ \text{if } t\ \eta\text{-reduces to } u \text{ then} & \Gamma \vdash t : A & \Longrightarrow & \Gamma \vdash u : A \end{array}$$

Additional assumptions on subtyping lead to subject β-reduction and/or subject η-expansion, thus yielding models of the β-conversion and/or η-conversion. Through these assumptions, it is thus possible to focus on the properties of the subtyping relation to get properties of the typing system without coming back to the typing rules themselves.

2.2 The β-Condition

The **β-condition** is the following property:

$$\bigcap_{i \in I}(A_i \to B_i) \leq A \to B \quad \Longrightarrow \quad \exists J \subseteq I, \quad \left(A \leq \bigcap_{j \in J} A_j\right) \quad \wedge \quad \left(\bigcap_{j \in J} B_j \leq B\right) \qquad (\beta)$$

Note that both I and J may be empty sets in (β). The exact formulation of this condition (also called *β-soundness*) may vary in the literature. We rely here on the fact that we are only interested in theories validating the properties of Table 1.

▶ **Proposition 2** (Subject β-Reduction)**.** *The β-condition entails subject β-reduction.*

Proof. See for example [3]. ◀

Due to the presence of a transitivity rule in Table 1 (and thus the lack of a "sub-formula property"), proofs of (β) cannot be usually obtained by direct induction on a derivation of $\bigcap_{i \in I}(A_i \to B_i) \leq A \to B$, and the induction hypothesis then needs to be strengthened.

As the β-condition is a key property, it appears to be unfortunate that a detour is needed to prove it, due to the way the subtyping relation is presented. Indeed, from the point of view of proof theory, deduction systems are typically presented in a way so that cuts (the analogue of transitivity) can be removed from the system, ensuring the sub-formula property and making structural induction on statements simpler. Such a presentation of the rules of Table 1 has been introduced by B. Pierce [13] (see Section 3).

2.3 The η-Condition

The **η-condition** is the following property:

$$\forall X \in \mathcal{A}, \quad \exists (A_i)_{i\in I}, \exists (B_i)_{i\in I}, \quad X \sim \bigcap_{i\in I}(A_i \to B_i) \tag{η}$$

Note that I may be empty in (η). Again, the exact formulation of this condition (also called *η-soundness*) may vary in the literature. We rely here on the fact that we are only interested in theories validating the properties of Table 1. In particular Table 1 entails $\Omega \sim \Omega \to \Omega$ (see Example 1), so that the case of an empty I would give $X \sim \Omega \sim \Omega \to \Omega$, and any atom is then equivalent to a *non-empty* intersection of arrows (this includes "singleton" intersections).

▶ **Lemma 3** (Strong η-Condition). *If the η-condition holds then:*

$$\forall C, \quad \exists (A_i)_{i\in I}, \exists (B_i)_{i\in I}, \quad C \sim \bigcap_{i\in I}(A_i \to B_i)$$

Proof. By induction on the type C:
- if $C \in \mathcal{A}$, we apply the η-condition;
- if $C = \Omega$, we are done with an empty indexing set;
- if $C = C_1 \cap C_2$, we apply the induction hypothesis to C_1 and C_2 and consider the union of the indexing sets (Table 1 is used to guarantee the compatibility of $\sim$ with $\cap$);
- if $C = C_1 \to C_2$, we are done with a singleton indexing set. ◀

▶ **Proposition 4** (Subject η-Expansion). *The η-condition holds iff subject η-expansion holds.*

Proof. See for example [3].

As an intuition of how the η-condition is used: in order to move from a typing derivation of $\Gamma \vdash t : X$ to a typing derivation of $\Gamma \vdash \lambda x.(t\,x) : X$, we need to apply t to x and thus that X is smaller than an arrow type. ◀

The β- and η-conditions provide modularity in the study of (BCD style) intersection type systems: one can focus on the subtyping part independently of the typing part, and use Propositions 2 and 4 to get properties of typing from subtyping. From now on, we will thus focus on the study of the subtyping relation. Our goal is to provide simple presentations of the BCD subtyping relation described on Table 1 (and of its extensions) so that, in particular, the β-condition can be easily derived.

2.4 BCD Subtyping and its Extensions

Given a set of atoms $\mathcal{A}$, BCD($\mathcal{A}$) is the subtyping system obtained from Table 1. All the subtyping systems we consider are extensions of BCD($\mathcal{A}$) (for some appropriate $\mathcal{A}$) with additional axioms. Among the useful sets of atoms, we consider the two-element set

$\mathbb{B} = \{\varphi, \psi\}$. Here are some systems from the literature we will be able to address (see Section 5):

Name	Reference	Atoms		Axioms	(β)	(η)
BCD	[2]	$\mathcal{A}$			✓	
Scott	[14]	$\mathcal{A}$		$X \sim \Omega \to X$	✓	✓
Park	[12]	$\mathcal{A}$		$X \sim X \to X$	✓	✓
CDZ	[5]	$\mathbb{B}$	$\varphi \leq \psi$	$\varphi \sim \psi \to \varphi \quad \psi \sim \varphi \to \psi$	✓	✓
HR	[10]	$\mathbb{B}$	$\varphi \leq \psi$	$\varphi \sim \psi \to \varphi \quad \psi \sim (\varphi \to \varphi) \cap (\psi \to \psi)$	✓	✓
DHM	[9]	$\mathbb{B}$	$\varphi \leq \psi$	$\varphi \sim \Omega \to \varphi \quad \psi \sim \varphi \to \psi$	✓	✓
TLCA	[15]	$\mathbb{B}$		$\varphi \sim \psi \to \varphi \quad \psi \sim (\psi \to \psi) \cap (\varphi \to \psi)$	✓	✓

For example, the CDZ line in the table above has to be read: "the CDZ subtyping system is described in [5], it is obtained from BCD($\mathbb{B}$) by adding the following new axioms:

$$\frac{}{\varphi \leq \psi} \qquad \frac{}{\varphi \leq \psi \to \varphi} \quad \frac{}{\psi \to \varphi \leq \varphi} \qquad \frac{}{\psi \leq \varphi \to \psi} \quad \frac{}{\varphi \to \psi \leq \psi}$$

It satisfies the β- and η-conditions".

Note that the Scott and Park systems are usually presented with a single atom, but considering the more general case of an arbitrary set $\mathcal{A}$ of atoms makes no difference here.

In the spirit of the "Strong beta condition" [1, Definition 5] (see Definition 18 and Section 4.5), it is important to remark that the added axioms all correspond either to an ordering of two atoms: $X \leq Y$, or to an equation of the shape: $X \sim A$. Moreover, in the extensions mentioned above, such an A is always a (possibly singleton) intersection of arrows.

3 The $\mathsf{IS}_{\prec}$ Subtyping System

We are looking for a presentation of the BCD subtyping relation satisfying the sub-formula property. This has been introduced by B. Pierce [13] (see also [4]). More precisely, we are interested in the particular case of Pierce's system with no type variable and no bound constraint: Pierce's types are built on top of both a set of type variables and a set of primitive types, our atoms correspond to these primitive types only, not to the variables. Given a relation $\prec$ on atoms (with $\prec^*$ denoting its reflexive-transitive closure), Pierce's system in sequent-calculus form is presented on Table 3. Sequents are of the form $A \trianglelefteq B_1, \dots, B_n \vdash B$ with three parts: a type A, a list of types $B_1, \dots, B_n$, $n \geq 0$, and a type B. Γ and Δ denote lists of types (see Proposition 5 for understanding how these sequents are related with BCD subtyping).

The relation $\prec$ provides a parameter for describing a primitive order relation on atoms. The reason why we do not require $\prec$ to be reflexive and transitive is explained in Section 5.

We are going to work with a slightly extended system, which we call $\mathsf{IS}_{\prec}$. This is not essential but it makes a few proofs slightly simpler. Sequents of $\mathsf{IS}_{\prec}$ are denoted $A \trianglelefteq \Gamma \vdash_{\prec} B$. The rules of $\mathsf{IS}_{\prec}$ are on Table 4. From the proof-theoretical point of view, Pierce's system is the restriction of $\mathsf{IS}_{\prec}$ which applies invertible rules as soon as possible in bottom-up proof constructions. This makes Pierce's system more suited for proof-search algorithms.

The system $\mathsf{BCD}_{\prec}$ is the extension of BCD subtyping (Table 1) with the axiom:

$$\frac{X \prec Y}{X \leq Y}$$

It is easy to see from Table 1 that $X \prec^* Y$ entails $X \leq Y$ in $\mathsf{BCD}_{\prec}$. One finds back the usual BCD subtyping system when $\prec$ is the empty relation.

$$X \prec^* Y \quad \frac{}{X \trianglelefteq\ \vdash Y}\ {\prec^*} \qquad\qquad \frac{}{C \trianglelefteq \Gamma \vdash \Omega}\ \Omega R$$

$$\frac{C \trianglelefteq \Gamma \vdash A \qquad C \trianglelefteq \Gamma \vdash B}{C \trianglelefteq \Gamma \vdash A \cap B}\ \cap R \qquad \frac{A \trianglelefteq \Gamma \vdash X}{A \cap B \trianglelefteq \Gamma \vdash X}\ \cap L_1 \qquad \frac{B \trianglelefteq \Gamma \vdash X}{A \cap B \trianglelefteq \Gamma \vdash X}\ \cap L_2$$

$$\frac{C \trianglelefteq \Gamma, A \vdash B}{C \trianglelefteq \Gamma \vdash A \to B}\ {\to}R \qquad\qquad \frac{C \trianglelefteq\ \vdash A \qquad B \trianglelefteq \Gamma \vdash X}{A \to B \trianglelefteq C, \Gamma \vdash X}\ {\to}L$$

Table 3 Pierce's Deduction System

$$X \prec^* Y \quad \frac{}{X \trianglelefteq\ \vdash_{\prec} Y}\ {\prec^*} \qquad\qquad \frac{}{C \trianglelefteq \Gamma \vdash_{\prec} \Omega}\ \Omega R$$

$$\frac{C \trianglelefteq \Gamma \vdash_{\prec} A \qquad C \trianglelefteq \Gamma \vdash_{\prec} B}{C \trianglelefteq \Gamma \vdash_{\prec} A \cap B}\ \cap R \qquad \frac{A \trianglelefteq \Gamma \vdash_{\prec} C}{A \cap B \trianglelefteq \Gamma \vdash_{\prec} C}\ \cap L_1 \qquad \frac{B \trianglelefteq \Gamma \vdash_{\prec} C}{A \cap B \trianglelefteq \Gamma \vdash_{\prec} C}\ \cap L_2$$

$$\frac{C \trianglelefteq \Gamma, A \vdash_{\prec} B}{C \trianglelefteq \Gamma \vdash_{\prec} A \to B}\ {\to}R \qquad\qquad \frac{C \trianglelefteq\ \vdash_{\prec} A \qquad B \trianglelefteq \Gamma \vdash_{\prec} D}{A \to B \trianglelefteq C, \Gamma \vdash_{\prec} D}\ {\to}L$$

Table 4 $\mathsf{IS}_{\prec}$ Deduction System

▶ **Proposition 5.** *$A \leq B_1 \to \cdots \to B_n \to B$ is derivable in $\mathsf{BCD}_{\prec}$ iff $A \trianglelefteq B_1, \ldots, B_n \vdash B$ is derivable in Pierce's system iff $A \trianglelefteq B_1, \ldots, B_n \vdash_{\prec} B$ is derivable in $\mathsf{IS}_{\prec}$.*

Proof. Thanks to [13, 4], we know that $A \leq B$ in $\mathsf{BCD}_{\prec}$ iff $A \trianglelefteq\ \vdash B$ in Pierce's system. By induction on n, we get that $A \trianglelefteq\ \vdash B_1 \to \cdots \to B_n \to B$ in Pierce's system iff $A \trianglelefteq B_1, \ldots, B_n \vdash B$ in Pierce's system. Each rule of Pierce's system is also a valid rule of $\mathsf{IS}_{\prec}$ thus $A \trianglelefteq B_1, \ldots, B_n \vdash B$ in Pierce's system entails $A \trianglelefteq B_1, \ldots, B_n \vdash_{\prec} B$ in $\mathsf{IS}_{\prec}$.

Finally, the proof of [13, Theorem 28] given for Pierce's system also shows that $A \trianglelefteq B_1, \ldots, B_n \vdash_{\prec} B$ in $\mathsf{IS}_{\prec}$ entails $A \leq B_1 \to \cdots \to B_n \to B$ in $\mathsf{BCD}_{\prec}$: see Appendix A.1. ◀

A crucial property of Pierce's system and $\mathsf{IS}_{\prec}$ is the *sub-formula property* which makes proofs by induction on derivations much simpler. At the same time, they are expressive enough to get various admissibility properties. From Proposition 5, we get the admissibility of the (ax) rule and of two cut rules for these two systems:

$$\frac{}{A \trianglelefteq\ \vdash A}\ ax \qquad \frac{A \trianglelefteq \Gamma \vdash B \qquad B \trianglelefteq \Delta \vdash C}{A \trianglelefteq \Gamma, \Delta \vdash C}\ tcut \qquad \frac{A \trianglelefteq\ \vdash B \qquad C \trianglelefteq \Gamma, B, \Delta \vdash D}{C \trianglelefteq \Gamma, A, \Delta \vdash D}\ scut$$

The name ($tcut$) refers to *transitivity* cut (as one gets transitivity when Γ and Δ are empty). The name ($scut$) refers to *substitution* cut (as B is substituted by A).

We can give a direct proof of the β-condition for $\mathsf{IS}_{\prec}$. We just need a slight generalization with contexts for the induction to go through.

▶ **Lemma 6** (β-Condition for $\mathsf{IS}_{\prec}$)**.** *The following property holds in $\mathsf{IS}_{\prec}$:*

$$\left(\bigcap_{i \in I} A_i \to B_i\right) \trianglelefteq A, \Gamma \vdash_{\prec} B \quad \Longrightarrow \quad \exists J \subseteq I, \quad \left(A \trianglelefteq\ \vdash_{\prec} \bigcap_{j \in J} A_j\right) \wedge \left(\bigcap_{j \in J} B_j \trianglelefteq \Gamma \vdash_{\prec} B\right)$$

The β-condition from Section 2.2 corresponds to the particular case of an empty Γ.

Proof. By induction on the derivation of $(\bigcap_{i\in I} A_i \to B_i) \trianglelefteq A, \Gamma \vdash_{\prec} B$, we look at each possible last rule: see Appendix A.2. ◀

4 Fixpoint Equations

In this section, we propose the intersection subtyping system $\mathsf{BCD}^{\delta}_{\prec}$, parameterized by a relation $\prec$ on atoms and a function δ from atoms to types. We also provide the corresponding sequent system $\mathsf{IS}^{\delta}_{\prec}$ in the style of $\mathsf{IS}_{\prec}$. It is very important to notice that, with the latter new system, we are now moving from the traditional setting of finite derivations to the case of non-wellfounded systems.

The function δ is our tool to introduce the fixpoint equations $X \sim \delta X$ required to represent most of the systems of Section 2.4.

Let us consider two key properties on the parameters $\prec$ and δ.

▶ **Definition 7** (Safety and η-Safety). *We say that the pair $(\prec, \delta)$ is* ***safe*** *if for each pair of atoms X and Y such that $X \prec Y$, we have $\delta X \leq \delta Y$ in* $\mathsf{BCD}_{\prec}$ *(or equivalently, thanks to Proposition 5, if we have an* $\mathsf{IS}_{\prec}$*-derivation of $\delta X \trianglelefteq\ \vdash_{\prec} \delta Y$).*

*We say that δ is η-**safe** if δX is a (possibly empty) intersection of arrows for every X.*

In the following, we assume that $(\prec, \delta)$ is *always safe*. In particular for each pair of atoms satisfying $X \prec^* Y$, we have $\delta X \leq \delta Y$ in $\mathsf{BCD}_{\prec}$ (thanks to transitivity) and then an $\mathsf{IS}_{\prec}$-derivation $\tau_{X,Y}$ of $\delta X \trianglelefteq\ \vdash_{\prec} \delta Y$ (by Proposition 5).

4.1 $\mathsf{BCD}^{\delta}_{\prec}$

The $\mathsf{BCD}^{\delta}_{\prec}$ **subtyping system** is obtained from $\mathsf{BCD}_{\prec}$ (Section 3) by adding the following axioms for each atom X:

$$\frac{}{X \leq \delta X} \qquad \frac{}{\delta X \leq X}$$

so that we get $X \sim \delta X$ in $\mathsf{BCD}^{\delta}_{\prec}$.

4.2 $\mathsf{IS}^{\delta}_{\prec}$

We now define the *non-wellfounded* system $\mathsf{IS}^{\delta}_{\prec}$. Sequents of $\mathsf{IS}^{\delta}_{\prec}$ are denoted $A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B$. The $\mathsf{IS}^{\delta}_{\prec}$ system includes all the rules of $\mathsf{IS}_{\prec}$, except for the $(\prec^*)$ rule, which is replaced by the following **checkpoint** rule, and the two following **unfolding** rules are added for each atom X:

$$X \prec^* Y \ \frac{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}\ CP \qquad \frac{\delta X \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B}{X \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B}\ \mathcal{A}L \qquad \frac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} \delta X}{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} X}\ \mathcal{A}R$$

▶ Remark. The rule $(\prec^*)$ of $\mathsf{IS}_{\prec}$ is not included in $\mathsf{IS}^{\delta}_{\prec}$ but, as we shall see, it can actually be shown to be admissible.

▶ **Definition 8** ($\mathsf{IS}^{\delta}_{\prec}$-derivations). *An* $\mathsf{IS}^{\delta}_{\prec}$***-derivation*** *is a possibly* infinite *tree built with the rules of* $\mathsf{IS}^{\delta}_{\prec}$ *and satisfying the following* ***validity*** *condition:* on each infinite branch, there are infinitely many occurrences of the checkpoint rule, and moreover there are exactly one $(\mathcal{A}L)$ rule and exactly one $(\mathcal{A}R)$ rule between any two consecutive checkpoints of any branch.

The ***prefix*** *of an* $\mathsf{IS}^{\delta}_{\prec}$*-derivation is defined as its bottom* partial *sub-derivation containing all the prefixes of its branches going up to its bottom-most checkpoints.*

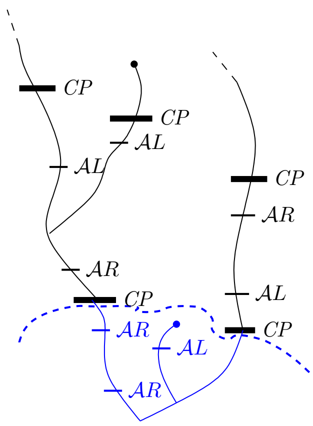

Figure 1 Structure of $\mathsf{IS}^{\delta}_{\prec}$-derivations (with prefix in blue below the dashed line).

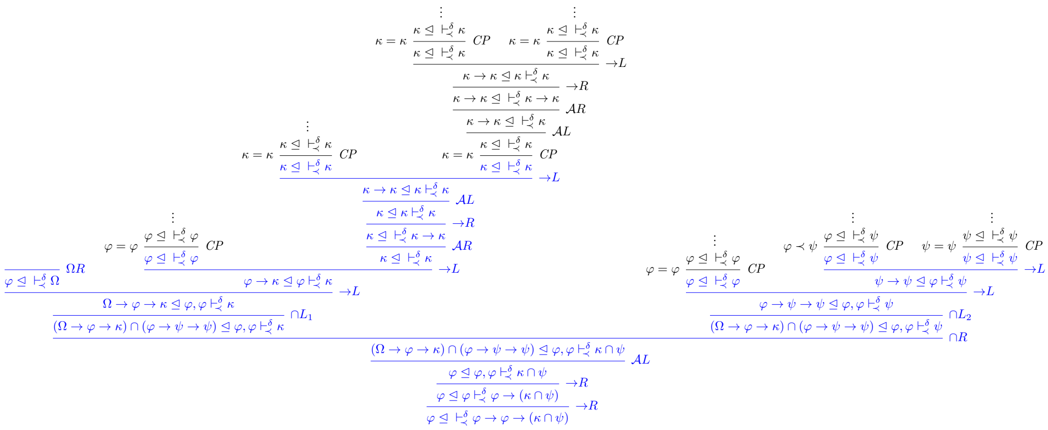

Figure 2 An $\mathsf{IS}^{\delta}_{\prec}$-derivation (with its prefix colored in blue). Here, we have three atoms φ, ψ and κ with $\varphi \prec \psi$, $\delta\varphi := (\Omega \to \varphi \to \kappa) \cap (\varphi \to \psi \to \psi)$, $\delta\psi := \varphi \to \varphi \to (\kappa \cap \psi)$ and $\delta\kappa := \kappa \to \kappa$.

The validity condition says nothing about rules in the prefix. It involves only the parts of the derivation that are above at least one checkpoint rule. See Figure 1 for an illustration and Figure 2 for a concrete example.

The validity condition in the definition of $\mathsf{IS}^{\delta}_{\prec}$-derivations requires that exactly one $(\mathcal{A}L)$ rule and one $(\mathcal{A}R)$ rule occur between two consecutive checkpoints. In the following, we show that these two rule applications can actually be organized in a canonical way, that is, they can be applied right above the checkpoints, without affecting provability of the conclusion.

▶ **Definition 9** (Prefix-δ-Free Derivation)**.** *An $\mathsf{IS}^{\delta}_{\prec}$-derivation is* ***prefix-δ-free*** *if its prefix does not contain any unfolding rule.*

▶ **Lemma 10.** *If $X \trianglelefteq\ \vdash^{\delta}_{\prec} Y$ appears as the premise of a checkpoint in an $\mathsf{IS}^{\delta}_{\prec}$-derivation, then $\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y$ has a prefix-δ-free $\mathsf{IS}^{\delta}_{\prec}$-derivation.*

Proof. Suppose that $X \trianglelefteq\ \vdash^{\delta}_{\prec} Y$ appears as a premise of a checkpoint in an $\mathsf{IS}^{\delta}_{\prec}$-derivation. Then its corresponding sub-derivation π must end with either an $(\mathcal{A}L)$ rule or an $(\mathcal{A}R)$ rule. Suppose that π ends with an $(\mathcal{A}L)$ rule whose premise π' has conclusion $\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} Y$ with each branch of π' containing no $(\mathcal{A}L)$ rule and exactly one $(\mathcal{A}R)$ rule until the bottom-most checkpoints. Consider now the prefix of π'. We prove by induction that for each sequent of the form $A \trianglelefteq\ \vdash^{\delta}_{\prec} Y$ in the prefix, there is a prefix-δ-free $\mathsf{IS}^{\delta}_{\prec}$-derivation of $A \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y$. For such a sequent, we consider its corresponding sub-derivation of π. First note that this sub-derivation cannot end with an $(\to L)$ rule. Moreover, it cannot end with a checkpoint, otherwise the validity condition would not be satisfied. If it ends with an $(\mathcal{A}R)$ rule, then we can conclude immediately. If it ends with an $(\cap L)$ rule, then we can conclude by the induction hypothesis.

Suppose now that π ends with an $(\mathcal{A}R)$ rule whose premise π' has conclusion $X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y$ with each branch of π' containing no $(\mathcal{A}R)$ rule and exactly one $(\mathcal{A}L)$ rule until the bottom-most checkpoints. Consider now the prefix of π'. We prove by induction that for each sequent of the form $X \trianglelefteq \Gamma \vdash^{\delta}_{\prec} A$ in the prefix, there is a prefix-δ-free $\mathsf{IS}^{\delta}_{\prec}$-derivation of $\delta X \trianglelefteq \Gamma \vdash^{\delta}_{\prec} A$. For such a sequent, we consider its corresponding sub-derivation of π. As in the previous case, this sub-derivation cannot end with a checkpoint (and a (ΩR) rule neither by validity). If it ends with an $(\mathcal{A}L)$ rule, then we can conclude immediately. If it ends with an $(\cap R)$ rule or an $(\to R)$ rule, then we can conclude by the induction hypothesis. ◀

In particular the premise of a checkpoint can always be reorganized:

$$X \prec^* Y \ \dfrac{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}\ CP \qquad \mapsto \qquad X \prec^* Y \ \dfrac{\dfrac{\overset{\text{Lemma 10}}{\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y}}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}\ \mathcal{A}R, \mathcal{A}L}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}\ CP$$

Thanks to the family of derivations $(\tau_{X,Y})_{X \prec^* Y}$ (see comment after Definition 7), we can map any $\mathsf{IS}_{\prec}$-derivation ρ with conclusion $A \trianglelefteq \Gamma \vdash_{\prec} B$ into a prefix-δ-free $\mathsf{IS}^{\delta}_{\prec}$-derivation $\overline{\rho}$ with conclusion $A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B$. This is done corecursively. We start from ρ and look at its $(\prec^*)$ leaves which we transform in the following way (all the other rules already belong to $\mathsf{IS}^{\delta}_{\prec}$):

$$X \prec^* Y \ \dfrac{}{X \trianglelefteq\ \vdash_{\prec} Y}\ \prec^* \qquad \mapsto \qquad X \prec^* Y \ \dfrac{\dfrac{\overset{\overline{\tau_{X,Y}}}{\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y}}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}\ \mathcal{A}R, \mathcal{A}L}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}\ CP$$

Corecursion is required here to build $\overline{\tau_{X,Y}}$. The process is productive since we introduce one checkpoint and two unfolding rules (plus the non-$(\prec^*)$ rules from $\tau_{X,Y}$) at each step.

▶ **Proposition 11.** *Given an $\mathsf{IS}_{\prec}$-derivation ρ with conclusion $A \trianglelefteq \Gamma \vdash_{\prec} B$, $\overline{\rho}$ is a prefix-δ-free $\mathsf{IS}^{\delta}_{\prec}$-derivation of $A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B$.*

Proof. By construction, every infinite branch of $\overline{\rho}$ contains infinitely many checkpoints and between any two consecutive checkpoints in a branch there are exactly one $(\mathcal{A}L)$ rule and one $(\mathcal{A}R)$ rule. Moreover, the prefix of $\overline{\rho}$ is exactly ρ, except for its $(\prec^*)$ leaves, and contains no $(\mathcal{A}L)$ rule and no $(\mathcal{A}R)$ rule. ◀

As an immediate consequence, we get the admissibility of the (ax) rule in $\mathsf{IS}^{\delta}_{\prec}$.

Conversely, we can map any *prefix-δ-free* $\mathsf{IS}^{\delta}_{\prec}$-derivation π of $A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B$ into an $\mathsf{IS}_{\prec}$-derivation $\underline{\pi}$ of $A \trianglelefteq \Gamma \vdash_{\prec} B$ by transforming its lowest checkpoints by:

$$X \prec^* Y \ \dfrac{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}\ CP \qquad \mapsto \qquad X \prec^* Y \ \dfrac{}{X \trianglelefteq\ \vdash_{\prec} Y}\ \prec^*$$

This mostly corresponds to extracting the prefix of the derivation π. It happens to provide an $\mathsf{IS}_{\prec}$-derivation in the case where there is no unfolding rule in the prefix of π.

Note that we have $\underline{(\overline{\rho})} = \rho$ for any $\mathsf{IS}_{\prec}$-derivation ρ. On the contrary, many $\mathsf{IS}^{\delta}_{\prec}$-derivations are not in the image of $\overline{(\cdot)}$.

4.3 Cuts Admissibility in $\mathsf{IS}^{\delta}_{\prec}$

We now show that appropriate cuts are admissible in $\mathsf{IS}^{\delta}_{\prec}$, which eventually leads to the equivalence between $\mathsf{BCD}^{\delta}_{\prec}$ and $\mathsf{IS}^{\delta}_{\prec}$.

▶ **Lemma 12.** *Suppose that $\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y$ and $\delta Y \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Z$ have prefix-δ-free $\mathsf{IS}^{\delta}_{\prec}$-derivations. Then $\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Z$ has a prefix-δ-free $\mathsf{IS}^{\delta}_{\prec}$-derivation.*

Proof. Let π_1 (resp. π_2) be a prefix-δ-free $\mathsf{IS}^{\delta}_{\prec}$-derivation of $\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y$ (resp. $\delta Y \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Z$). Consider their corresponding $\mathsf{IS}_{\prec}$-derivations $\underline{\pi_1}$ and $\underline{\pi_2}$. As pointed out in Section 3, the $(tcut)$ rule is admissible in $\mathsf{IS}_{\prec}$, yielding an $\mathsf{IS}_{\prec}$-derivation ρ of $\delta X \trianglelefteq\ \vdash_{\prec} \delta Z$. We then get a prefix-$\delta$-free $\mathsf{IS}^{\delta}_{\prec}$-derivation $\overline{\rho}$ of $\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Z$ by Proposition 11. ◀

In order to prove the admissibility of cuts in $\mathsf{IS}^{\delta}_{\prec}$, we first define the following two measures, called *prefix weight* and *fixpoint weight*, on $\mathsf{IS}^{\delta}_{\prec}$-derivations.

▶ **Definition 13** (Prefix and Fixpoint Weight)**.** *The **prefix weight** $pw(\pi)$ and **fixpoint weight** $fw(\pi)$ of an $\mathsf{IS}^{\delta}_{\prec}$-derivation π are defined as follows:*

- *if π ends with a checkpoint or π contains an (ΩR) rule only, $pw(\pi) = 1$ and $fw(\pi) = 0$;*
- *if π is obtained from π' by means of an $(\mathcal{A}L)$ rule or an $(\mathcal{A}R)$ rule, $pw(\pi) = pw(\pi') + 1$ and $fw(\pi) = fw(\pi') + 1$;*
- *if π is obtained from π_1 and π_2 by means of a $(\cap R)$ rule, $pw(\pi) = \max(pw(\pi_1), pw(\pi_2)) + 1$ and $fw(\pi) = \max(fw(\pi_1), fw(\pi_2))$;*
- *if π is obtained from π' by means of a $(\cap L_i)$ rule or a $(\rightarrow R)$ rule, $pw(\pi) = pw(\pi') + 1$ and $fw(\pi) = fw(\pi')$;*
- *if π is obtained from π_1 and π_2 by means of an $(\rightarrow L)$ rule, $pw(\pi) = pw(\pi_1) + pw(\pi_2) + 1$ and $fw(\pi) = fw(\pi_1) + fw(\pi_2)$.*

▶ Remark. The prefix weight is a variant of both the size and the height of the prefix: note the use of max in the $(\cap R)$ case and of sum in the $(\rightarrow L)$ case. The fixpoint weight of a prefix-δ-free derivation is 0. Both the prefix weight and the fixpoint weight depend on the prefix of a derivation only (which is a finite object).

▶ **Theorem 14** (Cuts admissibility in $\mathsf{IS}^{\delta}_{\prec}$)**.** *The following two cut rules are admissible in the $\mathsf{IS}^{\delta}_{\prec}$ system:*

$$\dfrac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B \qquad B \trianglelefteq \Delta \vdash^{\delta}_{\prec} C}{A \trianglelefteq \Gamma, \Delta \vdash^{\delta}_{\prec} C}\ tcut \qquad\qquad \dfrac{A \trianglelefteq\ \vdash^{\delta}_{\prec} B \qquad C \trianglelefteq \Gamma, B, \Delta \vdash^{\delta}_{\prec} D}{C \trianglelefteq \Gamma, A, \Delta \vdash^{\delta}_{\prec} D}\ scut$$

Proof. We prove both admissibilities simultaneously by induction on the triple (f, s, w) where f (resp. w) is the sum of the fixpoint weights (resp. prefix weights) of the two input derivations π_1 and π_2 (*i.e.* the derivations π_1 of $A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B$ and π_2 of $B \trianglelefteq \Delta \vdash^{\delta}_{\prec} C$, or π_1 of $A \trianglelefteq\ \vdash^{\delta}_{\prec} B$ and π_2 of $C \trianglelefteq \Gamma, B, \Delta \vdash^{\delta}_{\prec} D$), and s is the size (*i.e.* number of connectives) of the cut type B. We moreover show in the induction that the fixpoint weight of the output derivation is bounded by this sum $f = fw(\pi_1) + fw(\pi_2)$.

We consider all the possible cases of rules concluding the left and the right input derivations. We list below most key cases, along with some representative commutative cases.

The remaining cases are omitted. See the comments below about the calls to the induction hypothesis and the fixpoint weight of the output.

$$\dfrac{\dfrac{C \trianglelefteq \Gamma, A \vdash^{\delta}_{\prec} B}{C \trianglelefteq \Gamma \vdash^{\delta}_{\prec} A \to B}{\to}R \qquad \dfrac{D \trianglelefteq\ \vdash^{\delta}_{\prec} A \quad B \trianglelefteq \Delta \vdash^{\delta}_{\prec} E}{A \to B \trianglelefteq D, \Delta \vdash^{\delta}_{\prec} E}{\to}L}{C \trianglelefteq \Gamma, D, \Delta \vdash^{\delta}_{\prec} E}\,tcut \quad \rightsquigarrow \quad \dfrac{\dfrac{D \trianglelefteq\ \vdash^{\delta}_{\prec} A \quad C \trianglelefteq \Gamma, A \vdash^{\delta}_{\prec} B}{C \trianglelefteq \Gamma, D \vdash^{\delta}_{\prec} B}\,scut \qquad B \trianglelefteq \Delta \vdash^{\delta}_{\prec} E}{C \trianglelefteq \Gamma, D, \Delta \vdash^{\delta}_{\prec} E}\,tcut$$

$$\dfrac{\dfrac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} \delta X}{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} X}\mathcal{A}R \qquad \dfrac{\delta X \trianglelefteq \Delta \vdash^{\delta}_{\prec} B}{X \trianglelefteq \Delta \vdash^{\delta}_{\prec} B}\mathcal{A}L}{A \trianglelefteq \Gamma, \Delta \vdash^{\delta}_{\prec} B}\,tcut \quad \rightsquigarrow \quad \dfrac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} \delta X \quad \delta X \trianglelefteq \Delta \vdash^{\delta}_{\prec} B}{A \trianglelefteq \Gamma, \Delta \vdash^{\delta}_{\prec} B}\,tcut$$

$$\dfrac{X \prec^* Y\ \dfrac{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}CP \qquad Y \prec^* Z\ \dfrac{Y \trianglelefteq\ \vdash^{\delta}_{\prec} Z}{Y \trianglelefteq\ \vdash^{\delta}_{\prec} Z}CP}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Z}\,tcut \quad \rightsquigarrow \quad X \prec^* Z\ \dfrac{\dfrac{\overset{\text{Lemmas 10 and 12}}{\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Z}}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Z}\mathcal{A}R, \mathcal{A}L}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Z}CP$$

$$\dfrac{\dfrac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} \delta X}{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} X}\mathcal{A}R \qquad X \prec^* Y\ \dfrac{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}CP}{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} Y}\,tcut \quad \rightsquigarrow \quad \dfrac{\dfrac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} \delta X \quad \overset{\text{Lemma 10}}{\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y}}{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} \delta Y}\,tcut}{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} Y}\mathcal{A}R$$

$$\dfrac{X \prec^* Y\ \dfrac{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}{X \trianglelefteq\ \vdash^{\delta}_{\prec} Y}CP \qquad \dfrac{\delta Y \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B}{Y \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B}\mathcal{A}L}{X \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B}\,tcut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\text{Lemma 10}}{\delta X \trianglelefteq\ \vdash^{\delta}_{\prec} \delta Y} \quad \delta Y \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B}{\delta X \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B}\,tcut}{X \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B}\mathcal{A}L$$

$$\dfrac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B \qquad \dfrac{B \trianglelefteq \Delta \vdash^{\delta}_{\prec} C \quad B \trianglelefteq \Delta \vdash^{\delta}_{\prec} D}{B \trianglelefteq \Delta \vdash^{\delta}_{\prec} C \cap D}\cap R}{A \trianglelefteq \Gamma, \Delta \vdash^{\delta}_{\prec} C \cap D}\,tcut \quad \rightsquigarrow \quad \dfrac{\dfrac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B \quad B \trianglelefteq \Delta \vdash^{\delta}_{\prec} C}{A \trianglelefteq \Gamma, \Delta \vdash^{\delta}_{\prec} C}\,tcut \qquad \dfrac{A \trianglelefteq \Gamma \vdash^{\delta}_{\prec} B \quad B \trianglelefteq \Delta \vdash^{\delta}_{\prec} D}{A \trianglelefteq \Gamma, \Delta \vdash^{\delta}_{\prec} D}\,tcut}{A \trianglelefteq \Gamma, \Delta \vdash^{\delta}_{\prec} C \cap D}\cap R$$

$$\dfrac{A \trianglelefteq\ \vdash^{\delta}_{\prec} B \qquad \dfrac{B \trianglelefteq\ \vdash^{\delta}_{\prec} C \quad D \trianglelefteq \Gamma \vdash^{\delta}_{\prec} E}{C \to D \trianglelefteq B, \Gamma \vdash^{\delta}_{\prec} E}{\to}L}{C \to D \trianglelefteq A, \Gamma \vdash^{\delta}_{\prec} E}\,scut \quad \rightsquigarrow \quad \dfrac{\dfrac{A \trianglelefteq\ \vdash^{\delta}_{\prec} B \quad B \trianglelefteq\ \vdash^{\delta}_{\prec} C}{A \trianglelefteq\ \vdash^{\delta}_{\prec} C}\,tcut \qquad D \trianglelefteq \Gamma \vdash^{\delta}_{\prec} E}{C \to D \trianglelefteq A, \Gamma \vdash^{\delta}_{\prec} E}{\to}L$$

In the $({\to}L/{\to}R)$ case, we first apply the induction hypothesis to the upper cut ($scut$). Then we apply the induction hypothesis to the lower cut ($tcut$) thanks to the bound on the fixpoint weight of the derivation obtained by eliminating the upper cut. In both cases the size s of the cut type strictly decreases.

In the cases involving the $(\mathcal{A}L)$ and $(\mathcal{A}R)$ rules, the first component f strictly decreases in the calls to the induction hypothesis. This relies on the fact that the derivations obtained through Lemma 10 are prefix-δ-free. In the commutative cases, the first two components of the triple remain unchanged and the third one strictly decreases. The use of the max in the definitions of the fixpoint weight and of the prefix weight is crucial in the $(*/\cap R)$ case.

Regarding validity, note that we act on the prefix only, except in the (CP/CP) case. In this case the generated derivation also preserves the validity condition. ◀

4.4 Equivalence between $\mathsf{IS}^{\delta}_{\prec}$ and $\mathsf{BCD}^{\delta}_{\prec}$

In this section, we establish the equivalence between the two systems, $\mathsf{IS}^{\delta}_{\prec}$ and $\mathsf{BCD}^{\delta}_{\prec}$.

▶ **Theorem 15.** *If we have $A \le B$ in $\mathsf{BCD}^{\delta}_{\prec}$ then $A \trianglelefteq\ \vdash^{\delta}_{\prec} B$ has an $\mathsf{IS}^{\delta}_{\prec}$-derivation.*

Proof. We proceed by induction on the definition of $A \le B$. Axioms from BCD can be mapped into $\mathsf{IS}_{\prec}$-derivations by Proposition 5 and then into $\mathsf{IS}^{\delta}_{\prec}$-derivations by Proposition 11.

The other axioms (dealing with δ and $\prec^*$) can be translated by:

$$\dfrac{\dfrac{\overline{\tau_{X,X}}}{\delta X \trianglelefteq \vdash^\delta_\prec \delta X}}{X \trianglelefteq \vdash^\delta_\prec \delta X}\ \mathcal{A}L \qquad \dfrac{\dfrac{\overline{\tau_{X,X}}}{\delta X \trianglelefteq \vdash^\delta_\prec \delta X}}{\delta X \trianglelefteq \vdash^\delta_\prec X}\ \mathcal{A}R \qquad \dfrac{\dfrac{\overline{\tau_{X,Y}}}{\delta X \trianglelefteq \vdash^\delta_\prec \delta Y}}{X \trianglelefteq \vdash^\delta_\prec Y}\ \mathcal{A}R, \mathcal{A}L$$

Regarding rules, we have:

$$\dfrac{A \trianglelefteq \vdash^\delta_\prec B \qquad B \trianglelefteq \vdash^\delta_\prec C}{A \trianglelefteq \vdash^\delta_\prec C}\ \text{Theorem 14}$$

$$\dfrac{\dfrac{A \trianglelefteq \vdash^\delta_\prec C}{A \cap B \trianglelefteq \vdash^\delta_\prec C}\ \cap L_1 \qquad \dfrac{B \trianglelefteq \vdash^\delta_\prec D}{A \cap B \trianglelefteq \vdash^\delta_\prec D}\ \cap L_2}{A \cap B \trianglelefteq \vdash^\delta_\prec C \cap D}\ \cap R \qquad \dfrac{\dfrac{C \trianglelefteq \vdash^\delta_\prec A \qquad B \trianglelefteq \vdash^\delta_\prec D}{A \to B \trianglelefteq C \vdash^\delta_\prec D}\ \to L}{A \to B \trianglelefteq \vdash^\delta_\prec C \to D}\ \to R$$

◀

The $\mathsf{IS}^\delta_\prec$-derivations obtained from $\mathsf{BCD}^\delta_\prec$ by means of Theorem 15 are of a rather particular form, repeatedly using the $\tau_{X,Y}$ derivations through $\overline{\tau_{X,Y}}$. As a consequence they can be represented as circular derivations when derivations use a finite number of atoms only. In general an $\mathsf{IS}^\delta_\prec$-derivation does not have to be regular in this way, and Theorem 14 applies to this more general setting as well as Theorem 16 below.

▶ **Theorem 16.** *If $A \trianglelefteq \vdash^\delta_\prec B$ has an $\mathsf{IS}^\delta_\prec$-derivation then we have $A \leq B$ in $\mathsf{BCD}^\delta_\prec$.*

Proof. We follow the same path as for Proposition 5, see Appendix A.3. ◀

4.5 β- and η-Conditions

We show that the β-condition is satisfied in $\mathsf{IS}^\delta_\prec$ and $\mathsf{BCD}^\delta_\prec$ for any safe pair $(\prec, \delta)$.

▶ **Lemma 17** (β-Condition for $\mathsf{IS}^\delta_\prec$)**.** *The following property holds in $\mathsf{IS}^\delta_\prec$:*

$$\left(\bigcap_{i\in I} A_i \to B_i\right) \trianglelefteq A, \Gamma \vdash^\delta_\prec B \quad\Longrightarrow\quad \exists J \subseteq I,\ \left(A \trianglelefteq \vdash^\delta_\prec \bigcap_{j\in J} A_j\right) \wedge \left(\bigcap_{j\in J} B_j \trianglelefteq \Gamma \vdash^\delta_\prec B\right)$$

Proof. By induction on the prefix of the derivation of $(\bigcap_{i\in I} A_i \to B_i) \trianglelefteq A, \Gamma \vdash^\delta_\prec B$, we look at the last rule. All cases coming from $\mathsf{IS}_\prec$ are the same as for Lemma 6 (see Appendix A.2). We add the new case $(\mathcal{A}R)$ (both $(\mathcal{A}L)$ and (CP) are not possible):

If the last rule is $(\mathcal{A}R)$, we have $B = X$ with $(\bigcap_{i\in I} A_i \to B_i) \trianglelefteq A, \Gamma \vdash^\delta_\prec \delta X$. By induction hypothesis, we have $J \subseteq I$ allowing us to build:

$$A \trianglelefteq \vdash^\delta_\prec \textstyle\bigcap_{j\in J} A_j \qquad\qquad \dfrac{\bigcap_{j\in J} B_j \trianglelefteq \Gamma \vdash^\delta_\prec \delta X}{\bigcap_{j\in J} B_j \trianglelefteq \Gamma \vdash^\delta_\prec X}\ \mathcal{A}R$$

◀

The β-condition for $\mathsf{BCD}^\delta_\prec$ then comes from Theorems 15 and 16. Thanks to the absence of the transitivity rule, it is much easier to get the result through $\mathsf{IS}^\delta_\prec$ rather than with a direct approach in $\mathsf{BCD}^\delta_\prec$.

The η-condition is an immediate consequence of η-safety since $X \sim \delta X$ in $\mathsf{BCD}^\delta_\prec$.

This idea of considering a family of extensions of the BCD subtyping system and of providing a sufficient condition to derive uniformly the β-condition for the whole family is not new. Such an approach was developed in [1], through the "strong beta condition" (Definition 18). Let us compare with our work.

▶ **Definition 18** (Strong β System [1, Definition 5]). ***Strong β systems*** *are extensions of* BCD *which are obtained by adding:*

- *a set $\mathcal{V}$ of axioms of the form $X \leq Y$;*
- *a set $\mathcal{E}$ of equations (*i.e. *pairs of axioms), all of the form $X \sim \bigcap_{i\in I} T_i \to Z_i$ (with $T_i \in \mathcal{A} \cup \{\Omega\}$), with $\mathcal{E}$ containing exactly one such equation for each atom X;*

such that, given two atoms X and Y (with $X \sim \bigcap_{i\in I} T_i \to X'_i \in \mathcal{E}$ and $Y \sim \bigcap_{j\in J} S_j \to Y'_j \in \mathcal{E}$), $X \leq Y \in \mathcal{V}$ if and only if for each $j \in J$, there exists $i \in I$ such that $S_j \leq T_i$ is valid in $\mathsf{BCD} \cup \mathcal{V}$ *and $X'_i \leq Y'_j \in \mathcal{V}$.*

In [1, Definition 5], the "$S_j \leq T_i$ valid in $\mathsf{BCD} \cup \mathcal{V}$" clause is "$S_j \leq T_i \in \mathcal{V}$" but this looks problematic in the case where S_j or T_i is Ω since $\mathcal{V}$ contains only axioms about atoms.

▶ **Lemma 19.** *Strong β systems validate the β- and η-conditions.*

Proof. The case of the β-condition is [1, Lemma 6]. Note that they need to introduce a tricky induction hypothesis which relies on the specific form of elements of $\mathcal{E}$.

The case of η is immediate since $\mathcal{E}$ contains an appropriate equation for each atom. ◀

Our constructions subsume this strong β condition, mostly by generalizing the form of equations to arbitrary $X \sim A$ (rather than constraining A to be an intersection of arrows between elements of $\mathcal{A} \cup \{\Omega\}$ and $\mathcal{A}$). Note also that we only need one of the two implications from the last clause of Definition 18.

▶ **Proposition 20.** *If $\mathcal{S}$ is a strong β system, there exists a safe and η-safe pair $(\prec, \delta)$ such that $\mathcal{S}$ is* $\mathsf{BCD}^{\delta}_{\prec}$.

Proof. Given $\mathcal{V}$ and $\mathcal{E}$, we define $\prec$ by: $X \prec Y$ if $X \leq Y \in \mathcal{V}$, and for each $X \in \mathcal{A}$, δX is the unique type such that $X \sim \delta X \in \mathcal{E}$. With these values of $\prec$ and δ, the definition of $\mathsf{BCD}^{\delta}_{\prec}$ makes it coincide with $\mathcal{S} = \mathsf{BCD} \cup \mathcal{V} \cup \mathcal{E}$, and $\mathsf{BCD}_{\prec}$ coincides with $\mathsf{BCD} \cup \mathcal{V}$.

δ is η-safe thanks to the constraint on the form of the elements of $\mathcal{E}$. Let us now prove that $(\prec, \delta)$ is safe. Assume $X \prec Y$ (*i.e.* $X \leq Y \in \mathcal{V}$), $X \sim \bigcap_{i\in I} T_i \to X'_i \in \mathcal{E}$ and $Y \sim \bigcap_{j\in J} S_j \to Y'_j \in \mathcal{E}$. For each $j \in J$, there exists $i \in I$ such that $S_j \leq T_i$ is valid in $\mathsf{BCD} \cup \mathcal{V}$ (*i.e.* in $\mathsf{BCD}_{\prec}$) and $X'_i \leq Y'_j \in \mathcal{V}$ (*i.e.* $X'_i \prec Y'_j$). We can build the following $\mathsf{IS}_{\prec}$-derivation:

$$\dfrac{\cdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{Proposition 5}\\ S_j \trianglelefteq\ \vdash_{\prec} T_i\end{array} \qquad \dfrac{X'_i \prec Y'_j}{X'_i \trianglelefteq\ \vdash_{\prec} Y'_j}\prec^*}{T_i \to X'_i \trianglelefteq S_j \vdash_{\prec} Y'_j}\to L}{T_i \to X'_i \trianglelefteq\ \vdash_{\prec} S_j \to Y'_j}\to R}{\bigcap_{i\in I} T_i \to X'_i \trianglelefteq\ \vdash_{\prec} S_j \to Y'_j}\cap L \qquad \cdots}{\bigcap_{i\in I} T_i \to X'_i \trianglelefteq\ \vdash_{\prec} \bigcap_{j\in J} S_j \to Y'_j}\cap R$$

We conclude with Proposition 5. ◀

5 Instances

We have made the choice of fixing $\prec$ as an arbitrary relation (not necessarily a pre-order) while using directly $\prec^*$ may seem more natural (see Tables 3 and 4 or the definition of the checkpoint rule in $\mathsf{IS}^{\delta}_{\prec}$). The key point is the definition of safety (Definition 7):

$$X \prec Y \quad\Longrightarrow\quad \delta X \leq \delta Y \text{ in } \mathsf{BCD}_{\prec}$$

We already remarked that it is equivalent to: $X \prec^* Y \implies \delta X \leq \delta Y$ in $\mathsf{BCD}_\prec$. Our definition of safety having a more restricted assumption, it makes easier to prove that a pair $(\prec, \delta)$ is safe in concrete instances. This is what we look at now.

Remember that, in $\mathsf{BCD}^\delta_\prec$, we have $X \sim \delta X$, as well as $X \prec Y$ entails $X \leq Y$. The systems described in Section 2.4 then all appear as safe instances of $\mathsf{BCD}^\delta_\prec$ through the following definitions of $\prec$ and δ:

Name	Reference	Atoms	$\prec$	δ	
BCD	[2]	$\mathcal{A}$	$\emptyset$	$\delta X := X$	
Scott	[14]	$\mathcal{A}$	$\emptyset$	$\delta X := \Omega \to X$	
Park	[12]	$\mathcal{A}$	$\emptyset$	$\delta X := X \to X$	
CDZ	[5]	$\mathbb{B}$	$\varphi \prec \psi$	$\delta\varphi := \psi \to \varphi$	$\delta\psi := \varphi \to \psi$
HR	[10]	$\mathbb{B}$	$\varphi \prec \psi$	$\delta\varphi := \psi \to \varphi$	$\delta\psi := (\varphi \to \varphi) \cap (\psi \to \psi)$
DHM	[9]	$\mathbb{B}$	$\varphi \prec \psi$	$\delta\varphi := \Omega \to \varphi$	$\delta\psi := \varphi \to \psi$
TLCA	[15]	$\mathbb{B}$	$\emptyset$	$\delta\varphi := \psi \to \varphi$	$\delta\psi := (\psi \to \psi) \cap (\varphi \to \psi)$

Safety is immediate for systems with an empty relation $\prec$. The safety of CDZ, HR and DHM comes from the simple appropriate $\mathsf{IS}_\prec$-derivations which rely on $\varphi \prec \psi$. Here is the case of HR (see Appendix B for the other ones):

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \prec \psi}{\varphi \trianglelefteq\ \vdash_\prec \psi}\prec^* \qquad \dfrac{\varphi = \varphi}{\varphi \trianglelefteq\ \vdash_\prec \varphi}\prec^*}{\psi \to \varphi \trianglelefteq \varphi \vdash_\prec \varphi}\to L}{\psi \to \varphi \trianglelefteq\ \vdash_\prec \varphi \to \varphi}\to R \qquad \dfrac{\dfrac{\dfrac{\psi = \psi}{\psi \trianglelefteq\ \vdash_\prec \psi}\prec^* \qquad \dfrac{\varphi \prec \psi}{\varphi \trianglelefteq\ \vdash_\prec \psi}\prec^*}{\psi \to \varphi \trianglelefteq \psi \vdash_\prec \psi}\to L}{\psi \to \varphi \trianglelefteq\ \vdash_\prec \psi \to \psi}\to R}{\psi \to \varphi \trianglelefteq\ \vdash_\prec (\varphi \to \varphi) \cap (\psi \to \psi)}\cap R$$

Also, except BCD, all these systems are immediately η-safe.

These systems were mentioned in [1] (except TLCA) and covered by the strong β condition (except BCD since $\delta X = X$ is not an intersection of arrows). We can now consider alternative proposals which do not all fit in the setting of [1]. These are just examples to show what is possible in our setting, giving a larger spectrum of systems for which we can immediately conclude that the β-condition holds (and thus the associated typing system is invariant under β-conversion).

Among the sets of atoms, we also consider the three-elements set $\mathbb{T} = \{\varphi, \psi, \kappa\}$.

Atoms	$\prec$	δ		
$\mathbb{T}$	$\varphi \prec \psi$	$\delta\varphi := \Omega \to \varphi$	$\delta\psi := \Omega \to \psi$	$\delta\kappa := \varphi \to \psi$
$\mathbb{T}$	$\varphi \prec \psi$	$\delta\varphi := \kappa \to \varphi$	$\delta\psi := \kappa \to \psi$	$\delta\kappa := \kappa \to \kappa$
$\mathbb{B}$	$\varphi \prec \psi$	$\delta\varphi := (\varphi \to \varphi \to \varphi) \cap (\varphi \to \varphi \to \psi)$	$\delta\psi := \varphi \to \varphi \to (\varphi \cap \psi)$	
$\mathbb{T}$	$\varphi \prec \psi$	$\delta\varphi := (\varphi \to \varphi \to \kappa) \cap (\varphi \to \psi \to \psi)$ $\delta\kappa := \kappa \to \kappa$	$\delta\psi := \varphi \to \varphi \to (\kappa \cap \psi)$	
$\vdots$	$\vdots$	$\vdots$		

The validity of each of these systems requires to provide an $\mathsf{IS}_\prec$-derivation (or $\mathsf{BCD}_\prec$ derivation) of $\delta\varphi \trianglelefteq\ \vdash_\prec \delta\psi$. Here is the case of the last example, see Appendix B for the other ones:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi = \varphi}{\varphi \trianglelefteq\ \vdash_\prec \varphi}\prec^* \qquad \dfrac{\dfrac{\varphi = \varphi}{\varphi \trianglelefteq\ \vdash_\prec \varphi}\prec^* \qquad \dfrac{\kappa = \kappa}{\kappa \trianglelefteq\ \vdash_\prec \kappa}\prec^*}{\varphi \to \kappa \trianglelefteq \varphi \vdash_\prec \kappa}\to L}{\varphi \to \varphi \to \kappa \trianglelefteq \varphi, \varphi \vdash_\prec \kappa}\to L}{(\varphi \to \varphi \to \kappa) \cap (\varphi \to \psi \to \psi) \trianglelefteq \varphi, \varphi \vdash_\prec \kappa}\cap L_1 \qquad \dfrac{\dfrac{\dfrac{\varphi = \varphi}{\varphi \trianglelefteq\ \vdash_\prec \varphi}\prec^* \qquad \dfrac{\dfrac{\varphi \prec \psi}{\varphi \trianglelefteq\ \vdash_\prec \psi}\prec^* \qquad \dfrac{\psi = \psi}{\psi \trianglelefteq\ \vdash_\prec \psi}\prec^*}{\psi \to \psi \trianglelefteq \varphi \vdash_\prec \psi}\to L}{\varphi \to \psi \to \psi \trianglelefteq \varphi, \varphi \vdash_\prec \psi}\to L}{(\varphi \to \varphi \to \kappa) \cap (\varphi \to \psi \to \psi) \trianglelefteq \varphi, \varphi \vdash_\prec \psi}\cap L_2}{\dfrac{\dfrac{(\varphi \to \varphi \to \kappa) \cap (\varphi \to \psi \to \psi) \trianglelefteq \varphi, \varphi \vdash_\prec \kappa \cap \psi}{(\varphi \to \varphi \to \kappa) \cap (\varphi \to \psi \to \psi) \trianglelefteq \varphi \vdash_\prec \varphi \to (\kappa \cap \psi)}\to R}{(\varphi \to \varphi \to \kappa) \cap (\varphi \to \psi \to \psi) \trianglelefteq\ \vdash_\prec \varphi \to \varphi \to (\kappa \cap \psi)}\to R}\cap R$$

6 Rocq Formalization

The key results of the paper are formalized in Rocq:

https://github.com/olaure01/islogic/tree/FSCD2026

Because we want a self-contained development not relying on results from [13], we prove cuts admissibility for $\mathsf{IS}_{\prec}$ and for $\mathsf{IS}^{\delta}_{\prec}$ at the same time. As an intermediate step we generalize the checkpoint rule to:

$$X_0 \prec^* \cdots \prec^* X_n \; \frac{X_0 \trianglelefteq \vdash^{\delta}_{\prec} X_1 \qquad \cdots \qquad X_{n-1} \trianglelefteq \vdash^{\delta}_{\prec} X_n}{X_0 \trianglelefteq \vdash^{\delta}_{\prec} X_n} \; CP^*$$

Non-wellfounded objects are defined as `CoInductive` types (*e.g.* $\mathsf{IS}^{\delta}_{\prec}$) over `Inductive` types (*e.g.* $\mathsf{IS}_{\prec}$ or prefixes). The use of corecursion in Rocq went into limitations of the guard condition for coinductive/inductive types. We had to bypass the guard condition check through the use of `Admitted` to conclude our proofs in four places (where no goal remains but `Qed` fails due to the guard condition being too restrictive).

7 Conclusion

We have used techniques from proof theory to build transitivity-free presentations of (many) intersection subtyping systems with fixpoint equations on atoms in the spirit of Pierce's work [13]. This relies on cut elimination for non-wellfounded derivations with unfolding rules. Thanks to the generality and robustness of the approach, we have been able to generalize the "strong beta condition" of [1] to give a generic proof of the β-condition for subtyping. It is now possible to cover a larger set of intersection subtyping systems.

The present work is mostly syntactic and a key direction for future work is the study of the induced *filter models*. We have identified simple and generic conditions for an intersection subtyping relation to generate a filter model of β-conversion (and for η-conversion as well). Designing subtyping systems which are meaningful on the semantic side and understanding the expressiveness of our framework are important goals.

Since we have used standard tools and techniques from proof theory, it should be possible to extend smoothly the set of connectives under consideration. Among them, universal quantification ($\forall$) is a very natural candidate and would allow to deal with polymorphic subtyping.

The representation of non-wellfounded derivations as coinductive/inductive types in Rocq deserves some attention with the goal of providing natural translations of the paper proofs to the machine thanks to an appropriate guard condition.

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A Additional Proofs

A.1 Second Half of Proposition 5

Proof. The proof of [13, Theorem 28] given for Pierce's system also shows that $A \trianglelefteq B_1, \dots, B_n \vdash_\prec B$ in $\mathsf{IS}_\prec$ entails $A \leq B_1 \to \cdots \to B_n \to B$ in $\mathsf{BCD}_\prec$. The proof goes by induction on the derivation π of $A \trianglelefteq B_1, \dots, B_n \vdash_\prec B$, by considering each possible last rule.

In the case of a $(\prec^*)$ rule introducing $X \trianglelefteq\ \vdash_\prec Y$, we decompose $X \prec^* Y$ into $X = X_0 \prec$

$\dots \prec X_n = Y$ $(n \geq 0)$. If $n = 0$, we have $X \leq X$ in $\mathsf{BCD}_{\prec}$. If $n > 0$, we build:

$$\dfrac{X_0 \prec X_1 \quad \dfrac{}{X_0 \leq X_1} \qquad \dots \qquad X_{n-1} \prec X_n \quad \dfrac{}{X_{n-1} \leq X_n}}{X_0 \leq X_n}$$

For the (ΩR) rule, we build the following derivation by induction on n:

$$\dfrac{\dfrac{}{C \leq \Omega} \quad \dfrac{\dfrac{}{\Omega \leq \Omega \to \Omega} \quad \dfrac{\dfrac{}{B_1 \leq \Omega} \quad \dfrac{\dfrac{}{\Omega \leq \Omega \to \Omega} \quad \dfrac{\dfrac{}{B_2 \leq \Omega} \quad \dfrac{\dfrac{}{\Omega \leq \Omega}}{\Omega \leq B_3 \to \dots \to B_n \to \Omega}\vdots}{\Omega \to \Omega \leq B_2 \to \dots \to B_n \to \Omega}}{\Omega \leq B_2 \to \dots \to B_n \to \Omega}}{\Omega \to \Omega \leq B_1 \to B_2 \to \dots \to B_n \to \Omega}}{\Omega \leq B_1 \to B_2 \to \dots \to B_n \to \Omega}}{C \leq B_1 \to B_2 \to \dots \to B_n \to \Omega}$$

For the $(\cap R)$ rule, we have:

$$\dfrac{\dfrac{}{C \leq C \cap C} \quad \dfrac{C \leq B_1 \to \dots \to B_n \to A \qquad C \leq B_1 \to \dots \to B_n \to B}{C \cap C \leq (B_1 \to \dots \to B_n \to A) \cap (B_1 \to \dots \to B_n \to B)}}{C \leq (B_1 \to \dots \to B_n \to A) \cap (B_1 \to \dots \to B_n \to B)}$$

then, using transitivity, it is enough to prove $(B_1 \to \dots \to B_n \to A) \cap (B_1 \to \dots \to B_n \to B) \leq B_1 \to \dots \to B_n \to (A \cap B)$. We do it by induction on n. For $n = 1$, we simply have:

$$\dfrac{}{(B_n \to A) \cap (B_n \to B) \leq B_n \to (A \cap B)}$$

Then, using the induction hypothesis, we can build:

$$\dfrac{\dfrac{}{B_1 \leq B_1} \qquad \overset{IH}{(B_2 \to \dots \to B_n \to A) \cap (B_2 \to \dots \to B_n \to B) \leq B_2 \to \dots \to B_n \to (A \cap B)}}{B_1 \to ((B_2 \to \dots \to B_n \to A) \cap (B_2 \to \dots \to B_n \to B)) \leq B_1 \to \dots \to B_n \to (A \cap B)}$$

and we conclude by transitivity since we have:

$$\dfrac{}{(B_1 \to \dots \to B_n \to A) \cap (B_1 \to \dots \to B_n \to B) \leq B_1 \to ((B_2 \to \dots \to B_n \to A) \cap (B_2 \to \dots \to B_n \to B))}$$

The $(\cap L_1)$ and $(\cap L_2)$ rules are mapped to:

$$\dfrac{\dfrac{}{A \cap B \leq A} \quad A \leq B_1 \to \dots \to B_n \to C}{A \cap B \leq B_1 \to \dots \to B_n \to C} \qquad \dfrac{\dfrac{}{A \cap B \leq B} \quad B \leq B_1 \to \dots \to B_n \to C}{A \cap B \leq B_1 \to \dots \to B_n \to C}$$

The interpretations of the premise and of the conclusion of the $(\to R)$ rule are the same. The $(\to L)$ rule is mapped to:

$$\dfrac{C \leq A \qquad B \leq B_1 \to \dots \to B_n \to D}{A \to B \leq C \to B_1 \to \dots \to B_n \to D}$$

◀

A.2 Lemma 6

Proof. By induction on the derivation of $(\bigcap_{i\in I} A_i \to B_i) \trianglelefteq A, \Gamma \vdash_{\prec} B$, we look at the last rule:

- If the last rule is (ΩR), we have $B = \Omega$, we choose $J = \emptyset$ and we have:

$$\frac{}{A \trianglelefteq\ \vdash_{\prec} \Omega}\ \Omega R \qquad\qquad \frac{}{\Omega \trianglelefteq \Gamma \vdash_{\prec} \Omega}\ \Omega R$$

- If the last rule is $(\cap R)$, we have $B = C \cap D$ with $(\bigcap_{i\in I} A_i \to B_i) \trianglelefteq A, \Gamma \vdash_{\prec} C$ and $(\bigcap_{i\in I} A_i \to B_i) \trianglelefteq A, \Gamma \vdash_{\prec} D$, by induction hypothesis, we have two sets $J_1 \subseteq I$ and $J_2 \subseteq I$ allowing us to build:

$$\frac{A \trianglelefteq\ \vdash_{\prec} \bigcap_{j\in J_1} A_j \qquad A \trianglelefteq\ \vdash_{\prec} \bigcap_{j\in J_2} A_j}{A \trianglelefteq\ \vdash_{\prec} \bigcap_{j\in J_1\cup J_2} A_j}\ \cap R$$

$$\frac{\dfrac{\bigcap_{j\in J_1} B_j \trianglelefteq \Gamma \vdash_{\prec} C}{\bigcap_{j\in J_1\cup J_2} B_j \trianglelefteq \Gamma \vdash_{\prec} C}\ \cap L_1 \qquad \dfrac{\bigcap_{j\in J_2} B_j \trianglelefteq \Gamma \vdash_{\prec} D}{\bigcap_{j\in J_1\cup J_2} B_j \trianglelefteq \Gamma \vdash_{\prec} D}\ \cap L_2}{\bigcap_{j\in J_1\cup J_2} B_j \trianglelefteq \Gamma \vdash_{\prec} C \cap D}\ \cap R$$

  We are hiding here behind double lines the property $\bigcap_{j\in J_1\cup J_2} A_j \sim \left(\bigcap_{j\in J_1} A_j\right) \cap \left(\bigcap_{j\in J_2} A_j\right)$, which can be proved by induction on J_2 and holds easily because of the associativity, commutativity and idempotency of $\cap$.

- If the last rule is $(\cap L_1)$, we have $I = I_1 \cup I_2$ with $(\bigcap_{i\in I_1} A_i \to B_i) \trianglelefteq A, \Gamma \vdash_{\prec} B$, by induction hypothesis, we have $J \subseteq I_1 \subseteq I$ such that $A \trianglelefteq\ \vdash_{\prec} \bigcap_{j\in J} A_j$ and $\bigcap_{j\in J} B_j \trianglelefteq \Gamma \vdash_{\prec} B$.
- If the last rule is $(\to R)$, we have $B = C \to D$ with $(\bigcap_{i\in I} A_i \to B_i) \trianglelefteq A, \Gamma, C \vdash_{\prec} D$, by induction hypothesis, we have $J \subseteq I$ allowing us to build:

$$A \trianglelefteq\ \vdash_{\prec} \bigcap_{j\in J} A_j \qquad\qquad \frac{\bigcap_{j\in J} B_j \trianglelefteq \Gamma, C \vdash_{\prec} D}{\bigcap_{j\in J} B_j \trianglelefteq \Gamma \vdash_{\prec} C \to D}\ \to R$$

- If the last rule is $(\to L)$, I is a singleton $\{i\}$, with $A \trianglelefteq\ \vdash_{\prec} A_i$ and $B_i \trianglelefteq \Gamma \vdash_{\prec} B$, we have the result with $J = \{i\}$. ◀

A.3 Theorem 16

Proof. We prove the more general statement: if $A \trianglelefteq B_1, \ldots, B_n \vdash^{\delta}_{\prec} B$ has an $\mathsf{IS}^{\delta}_{\prec}$-derivation π then we have $A \leq B_1 \to \cdots \to B_n \to B$ in $\mathsf{BCD}^{\delta}_{\prec}$, by induction on the prefix weight of π. We consider each possible last rule.

The $(\prec^*)$, (ΩR), $(\cap R)$, $(\cap L_1)$, $(\cap L_2)$, $(\to R)$, and $(\to L)$ rules are treated as in Appendix A.1.

If π ends with a checkpoint, then $A = X, n = 0$, and $B = Y$ with $X \prec^* Y$. We have thus $A \leq B$ in $\mathsf{BCD}^{\delta}_{\prec}$.

The $(\mathcal{A}R)$ rule is mapped to:

$$\frac{A \leq B_1 \to \cdots \to B_n \to \delta X \qquad \dfrac{\dfrac{}{B_1 \leq B_1} \qquad \dfrac{\dfrac{\dfrac{}{B_n \leq B_n} \qquad \dfrac{}{\delta X \leq X}}{B_n \to \delta X \leq B_n \to X}}{\vdots \atop B_2 \to \cdots \to B_n \to \delta X \leq B_2 \to \cdots \to B_n \to X}}{B_1 \to \cdots \to B_n \to \delta X \leq B_1 \to \cdots \to B_n \to X}}{A \leq B_1 \to \cdots \to B_n \to X}$$

Finally, the $(\mathcal{A}L)$ rule is mapped to:

$$\dfrac{\dfrac{}{X \leq \delta X} \qquad \delta X \leq B_1 \to \cdots \to B_n \to B}{X \leq B_1 \to \cdots \to B_n \to B}$$

◀

B Safety of Instances

We give the appropriate missing $\mathsf{IS}_{\prec}$-derivations for the safety of the systems presented in Section 5. They all rely on (at least) two atoms φ and ψ satisfying $\varphi \prec \psi$.

CDZ:

$$\dfrac{\dfrac{\varphi \prec \psi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \psi}\prec^* \qquad \varphi \prec \psi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \psi}\prec^*}{\psi \to \varphi \trianglelefteq \varphi \vdash_{\prec} \psi}\to L}{\psi \to \varphi \trianglelefteq\ \vdash_{\prec} \varphi \to \psi}\to R$$

DHM:

$$\dfrac{\dfrac{\dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \Omega}\Omega R \qquad \varphi \prec \psi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \psi}\prec^*}{\Omega \to \varphi \trianglelefteq \varphi \vdash_{\prec} \psi}\to L}{\Omega \to \varphi \trianglelefteq\ \vdash_{\prec} \varphi \to \psi}\to R$$

Other instances:

$$\dfrac{\dfrac{\dfrac{}{\Omega \trianglelefteq\ \vdash_{\prec} \Omega}\Omega R \qquad \varphi \prec \psi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \psi}\prec^*}{\Omega \to \varphi \trianglelefteq \Omega \vdash_{\prec} \psi}\to L}{\Omega \to \varphi \trianglelefteq\ \vdash_{\prec} \Omega \to \psi}\to R$$

$$\dfrac{\dfrac{\kappa = \kappa \ \dfrac{}{\kappa \trianglelefteq\ \vdash_{\prec} \kappa}\prec^* \qquad \varphi \prec \psi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \psi}\prec^*}{\kappa \to \varphi \trianglelefteq \kappa \vdash_{\prec} \psi}\to L}{\kappa \to \varphi \trianglelefteq\ \vdash_{\prec} \kappa \to \psi}\to R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi = \varphi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \varphi}\prec^* \quad \dfrac{\varphi = \varphi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \varphi}\prec^* \quad \varphi = \varphi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \varphi}\prec^*}{\varphi \to \varphi \trianglelefteq \varphi \vdash_{\prec} \varphi}\to L}{\varphi \to \varphi \to \varphi \trianglelefteq \varphi, \varphi \vdash_{\prec} \varphi}\to L}{(\varphi \to \varphi \to \varphi) \cap (\varphi \to \varphi \to \psi) \trianglelefteq \varphi, \varphi \vdash_{\prec} \varphi}\cap L_1 \quad \dfrac{\dfrac{\varphi = \varphi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \varphi}\prec^* \quad \dfrac{\varphi = \varphi \ \dfrac{}{\varphi \trianglelefteq\ \vdash_{\prec} \varphi}\prec^* \quad \psi = \psi \ \dfrac{}{\psi \trianglelefteq\ \vdash_{\prec} \psi}\prec^*}{\varphi \to \psi \trianglelefteq \varphi \vdash_{\prec} \psi}\to L}{\varphi \to \varphi \to \psi \trianglelefteq \varphi, \varphi \vdash_{\prec} \psi}\to L}{(\varphi \to \varphi \to \varphi) \cap (\varphi \to \varphi \to \psi) \trianglelefteq \varphi, \varphi \vdash_{\prec} \psi}\cap L_2}{\dfrac{(\varphi \to \varphi \to \varphi) \cap (\varphi \to \varphi \to \psi) \trianglelefteq \varphi, \varphi \vdash_{\prec} \varphi \cap \psi}{\dfrac{(\varphi \to \varphi \to \varphi) \cap (\varphi \to \varphi \to \psi) \trianglelefteq \varphi \vdash_{\prec} \varphi \to (\varphi \cap \psi)}{(\varphi \to \varphi \to \varphi) \cap (\varphi \to \varphi \to \psi) \trianglelefteq\ \vdash_{\prec} \varphi \to \varphi \to (\varphi \cap \psi)}\to R}\to R}\cap R$$