Embedding Classical Linear Logic into Intuitionistic Linear Logic

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1 Classical Linear Logic

Given a set of atoms whose elements are denoted X, Y, ..., formulas of classical linear logic are given by:

$$A, B \quad ::= \quad X \mid X^{\perp} \mid A \parr B \mid A \otimes B \mid \perp \mid 1 \mid ?A \mid !A \mid A \,\&\, B \mid A \oplus B \mid \top \mid 0$$

The orthogonal construction $\_ \mapsto \_^{\perp}$ is extended from atoms to all formulas by:

$$\begin{array}{rcl@{\qquad}rcl}
(X^{\perp})^{\perp} & = & X & & & \\
(A \parr B)^{\perp} & = & A^{\perp} \otimes B^{\perp} & (A \otimes B)^{\perp} & = & A^{\perp} \parr B^{\perp} \\
\perp^{\perp} & = & 1 & 1^{\perp} & = & \perp \\
(?A)^{\perp} & = & !A^{\perp} & (!A)^{\perp} & = & ?A^{\perp} \\
(A \,\&\, B)^{\perp} & = & A^{\perp} \oplus B^{\perp} & (A \oplus B)^{\perp} & = & A^{\perp} \,\&\, B^{\perp} \\
\top^{\perp} & = & 0 & 0^{\perp} & = & \top
\end{array}$$

This operation satisfies $A^{\perp\perp} = A$.

2 Intuitionistic Linear Logic

Intuitionistic atoms are those from classical linear logic plus an additional one R. Formulas of intuitionistic linear logic are then obtained by:

$$A, B \quad ::= \quad R \mid X \mid A \otimes B \mid 1 \mid A \multimap B \mid !A \mid A \,\&\, B \mid \top$$

3 The Embedding

We consider the following embedding of classical formulas into intuitionistic ones:

$$\begin{array}{rcl@{\qquad}rcl}
(X^{\perp})^{\star} & = & X & X^{\star} & = & X \multimap R \\
(A \parr B)^{\star} & = & A^{\star} \otimes B^{\star} & (A \otimes B)^{\star} & = & ((A^{\star} \multimap R) \otimes (B^{\star} \multimap R)) \multimap R \\
\perp^{\star} & = & 1 & 1^{\star} & = & 1 \multimap R \\
(?A)^{\star} & = & !A^{\star} & (!A)^{\star} & = & !(A^{\star} \multimap R) \multimap R \\
(A \oplus B)^{\star} & = & A^{\star} \,\&\, B^{\star} & (A \,\&\, B)^{\star} & = & ((A^{\star} \multimap R) \,\&\, (B^{\star} \multimap R)) \multimap R \\
0^{\star} & = & \top & \top^{\star} & = & \top \multimap R
\end{array}$$

Theorem.
$\vdash A_1, \ldots, A_n$ *in classical linear logic* $\iff$ $A_1^{\star}, \ldots, A_n^{\star} \vdash R$ *in intuitionistic linear logic.*

Proof. *(Sketch)* This first direction is obtained by induction on a cut-free proof of $\vdash A_1, \dots, A_n$ in classical linear logic using only atomic axiom rules.

The second direction comes from the fact $(A^\star[^\perp/_R])^\perp \multimapboth A$ in classical linear logic (using the property that up to the coding $A \multimap B = A^\perp \parr B$ an intuitionistic formula without R can be considered as a classical formula). If $A_1^\star, \dots, A_n^\star \vdash R$ is derivable in intuitionistic linear logic then $\vdash (A_1^\star)^\perp[^\perp/_R], \dots, (A_n^\star)^\perp[^\perp/_R]$ is derivable in classical linear logic. $\square$

A more general development of a similar idea is presented in: `http://www.cs.cmu.edu/~fp/papers/CMU-CS-03-131R.pdf`.

A Classical Linear Logic Rules

$$\frac{}{\vdash X, X^{\perp}}\ ax$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot \qquad \frac{}{\vdash 1}\ 1$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ !$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_2 \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \frac{}{\vdash \Gamma, \top}\ \top$$

B Intuitionistic Linear Logic Rules

$$\frac{}{X \vdash X}\ ax \qquad \frac{}{R \vdash R}\ ax_R$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes_r \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes_l \qquad \frac{}{\vdash 1}\ 1_r \qquad \frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}\ 1_l$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap_r \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ \multimap_l$$

$$\frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C}\ !d \qquad \frac{\Gamma \vdash C}{\Gamma, !A \vdash C}\ !w \qquad \frac{\Gamma, !A, !A \vdash C}{\Gamma, !A \vdash C}\ !c \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ !$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ \&_r \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \,\&\, B \vdash C}\ \&_{1l} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C}\ \&_{2l} \qquad \frac{}{\Gamma \vdash \top}\ \top_r$$

C Proof of the Theorem

C.1 From classical to intuitionistic

$$\frac{\dfrac{}{X \vdash X}\ ax \qquad \dfrac{}{R \vdash R}\ ax_R}{X^{\star}, (X^{\perp})^{\star} \vdash R}\ \multimap_l$$

$$\frac{\Gamma^{\star}, A^{\star}, B^{\star} \vdash R}{\Gamma^{\star}, (A \parr B)^{\star} \vdash R}\ \otimes_l \qquad \frac{\Gamma^{\star} \vdash R}{\Gamma^{\star}, \bot^{\star} \vdash R}\ 1_l$$

$$\frac{\Gamma^{\star}, A^{\star} \vdash R}{\Gamma^{\star}, (?A)^{\star} \vdash R}\ !d \qquad \frac{\Gamma^{\star} \vdash R}{\Gamma^{\star}, (?A)^{\star} \vdash R}\ !w \qquad \frac{\Gamma^{\star}, (?A)^{\star}, (?A)^{\star} \vdash R}{\Gamma^{\star}, (?A)^{\star} \vdash R}\ !c$$

$$\frac{\Gamma^{\star}, A^{\star} \vdash R}{\Gamma^{\star}, (A \oplus B)^{\star} \vdash R}\ \&_{1l} \qquad \frac{\Gamma^{\star}, B^{\star} \vdash R}{\Gamma^{\star}, (A \oplus B)^{\star} \vdash R}\ \&_{2l}$$

$$\dfrac{\dfrac{\dfrac{\Gamma^\star, A^\star \vdash R}{\Gamma^\star \vdash A^\star \multimap R}\multimap_r \qquad \dfrac{\Delta^\star, B^\star \vdash R}{\Delta^\star \vdash B^\star \multimap R}\multimap_r}{\Gamma^\star, \Delta^\star \vdash (A^\star \multimap R) \otimes (B^\star \multimap R)}\otimes_r \qquad \dfrac{}{R \vdash R}ax_R}{\Gamma^\star, \Delta^\star, (A \otimes B)^\star \vdash R}\multimap_l$$

$$\dfrac{\dfrac{}{\vdash 1}1_r \qquad \dfrac{}{R \vdash R}ax_R}{1^\star \vdash R}\multimap_l$$

$$\dfrac{\dfrac{\dfrac{(?\Gamma)^\star, A^\star \vdash R}{(?\Gamma)^\star \vdash A^\star \multimap R}\multimap_r}{(?\Gamma)^\star \vdash !(A^\star \multimap R)}! \qquad \dfrac{}{R \vdash R}ax_R}{(?\Gamma)^\star, (!A)^\star \vdash R}\multimap_l$$

$$\dfrac{\dfrac{\dfrac{\Gamma^\star, A^\star \vdash R}{\Gamma^\star \vdash A^\star \multimap R}\multimap_r \qquad \dfrac{\Gamma^\star, B^\star \vdash R}{\Gamma^\star \vdash B^\star \multimap R}\multimap_r}{\Gamma^\star \vdash (A^\star \multimap R) \,\&\, (B^\star \multimap R)}\&_r \qquad \dfrac{}{R \vdash R}ax_R}{\Gamma^\star, (A \,\&\, B)^\star \vdash R}\multimap_l$$

$$\dfrac{\dfrac{}{\Gamma^\star \vdash \top}\top_r \qquad \dfrac{}{R \vdash R}ax_R}{\Gamma^\star, \top^\star \vdash R}\multimap_l$$

C.2 From intuitionistic to classical

Lemma.
If A is a classical formula, $(A^\star[^\perp/_R])^\perp \multimapboth A$ in classical linear logic.

Proof. A simple induction on A using $(A \multimap R)[^\perp/_R] \multimapboth A[^\perp/_R]^\perp$ in classical linear logic. □

From a proof of $A_1^\star, \ldots, A_n^\star \vdash R$ in intuitionistic linear logic, one can obtain a proof of $\vdash (A_1^\star)^\perp, \ldots, (A_n^\star)^\perp, R$ in classical linear logic (we just consider R as a fresh atom in classical linear logic), then one deduce that $\vdash (A_1^\star)^\perp[^\perp/_R], \ldots, (A_n^\star)^\perp[^\perp/_R], \perp$ is derivable in classical linear logic and finally that $\vdash (A_1^\star)^\perp[^\perp/_R], \ldots, (A_n^\star)^\perp[^\perp/_R]$ is also provable. By applying the lemma, we conclude that $\vdash A_1, \ldots, A_n$ is derivable in classical linear logic.