Mechanizing Cut-Elimination in Coq via Relational Phase Semantics

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Introduction

- Linear logic introduced by Girard
  - both classical and intuitionistic
  - separate multiplicatives ($\otimes, \multimap, \epsilon$)
  - from additives ($\&, \oplus, \bot, \top$)
- ILL via its sequent calculus
  - multiplicatives split the context
  - additives share the context
- formulas cannot be freely duplicated or discarded
  - no weakening (C) or contraction rule (W)
  - exponentials $!\,A$ re-introduce controlled C&W
  - generally undecidable
- Mechanized cut-elimination via phase semantics
  - A relational phase semantics (no monoid)
  - via Okada's lemma, both in `Prop` and `Type`

ILL sequent calculus (multiplicatives)

$$\frac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C} \langle \otimes_L \rangle \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \langle \otimes_R \rangle$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{A \multimap B, \Gamma, \Delta \vdash C} \langle \multimap_L \rangle \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B} \langle \multimap_R \rangle$$

$$\frac{\Gamma \vdash A}{\epsilon, \Gamma \vdash A} \langle \epsilon_L \rangle \qquad \frac{}{\vdash \epsilon} \langle \epsilon_R \rangle$$

ILL sequent calculus (additives)

$$\frac{A, \Gamma \vdash C}{A \,\&\, B, \Gamma \vdash C}\ \langle \&_L^1 \rangle \qquad \frac{B, \Gamma \vdash C}{A \,\&\, B, \Gamma \vdash C}\ \langle \&_L^2 \rangle$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ \langle \&_R \rangle \qquad \frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \oplus B, \Gamma \vdash C}\ \langle \oplus_L \rangle$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ \langle \oplus_R^1 \rangle \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ \langle \oplus_R^2 \rangle$$

$$\frac{}{\Gamma, \bot \vdash A}\ \langle \bot_L \rangle \qquad \frac{}{\Gamma \vdash \top}\ \langle \top_R \rangle$$

ILL (exponentials and structural)

$$\frac{A, \Gamma, \vdash B}{!\,A, \Gamma \vdash B}\ \langle !_L \rangle \qquad \frac{!\,\Gamma \vdash A}{!\,\Gamma \vdash !\,A}\ \langle !_R \rangle$$

$$\frac{}{A \vdash A}\ \langle \mathrm{id} \rangle \qquad \frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \langle \mathrm{cut} \rangle$$

$$\frac{\Gamma, \vdash B}{!\,A, \Gamma \vdash B}\ \langle W \rangle \qquad \frac{!\,A, !\,A, \Gamma \vdash B}{!\,A, \Gamma \vdash B}\ \langle C \rangle$$

$$\frac{\Gamma \vdash A}{\Delta \vdash A}\ \langle \Gamma \sim_p \Delta \rangle$$

Relational Phase semantics (overview)

- It is an algebraic semantics
  - Comparable to Lindenbaum construction
  - Interpret formula by "themselves" (completeness)
  - **does not require** $\langle cut \rangle$ (cut-admissibility)
- Usual phase semantics based on
  - commutative monoidal structure (contexts)
  - a stable closure
- Relational phase semantics
  - a composition relation (no axiom)
  - closure axioms absord the monoidal structure

Relational Phase Semantics (details)

- Closure $\mathrm{cl} : (M \to \mathtt{Prop}) \to (M \to \mathtt{Prop})$
  - with predicates $\mathcal{X}, \mathcal{Y} : M \to \mathtt{Prop}$

$$\mathcal{X} \subseteq \mathrm{cl}\,\mathcal{X} \quad \mathcal{X} \subseteq \mathcal{Y} \to \mathrm{cl}\,\mathcal{X} \subseteq \mathrm{cl}\,\mathcal{Y} \quad \mathrm{cl}(\mathrm{cl}\,\mathcal{X}) \subseteq \mathrm{cl}\,\mathcal{X}$$

- Composition $\bullet : M \to M \to M \to \mathtt{Prop}$, $\mathrm{e} : M$
  - extended to predicates $M \to \mathtt{Prop}$ by

$$\begin{array}{rcl} \mathcal{X} \bullet \mathcal{Y} & := & \bigcup\{x \bullet y \mid x \in \mathcal{X}, y \in \mathcal{Y}\} \\ \mathcal{X} \multimap \mathcal{Y} & := & \{z \mid z \bullet \mathcal{X} \subseteq \mathcal{Y}\} \end{array}$$

  - $x \in \mathrm{cl}(\mathrm{e} \bullet x)$ (neutral1)
  - $\mathrm{e} \bullet x \subseteq \mathrm{cl}\{x\}$ (neutral2)
  - $x \bullet y \subseteq \mathrm{cl}(y \bullet x)$ (commutativity)
  - $x \bullet (y \bullet z) \subseteq \mathrm{cl}((x \bullet y) \bullet z)$ (associativity)
- Stability: $(\mathrm{cl}\,\mathcal{X}) \bullet \mathcal{Y} \subseteq \mathrm{cl}(\mathcal{X} \bullet \mathcal{Y})$

Rel. Phase Sem. (exponential, soundness)

- Let $\mathcal{J} := \{x \mid x \in \mathrm{cl}\{\mathrm{e}\} \wedge x \in \mathrm{cl}(x \bullet x)\}$
- Choose $\mathcal{K} \subseteq \mathcal{J}$ such that $\mathrm{e} \in \mathrm{cl}\,\mathcal{K}$ and $\mathcal{K} \bullet \mathcal{K} \subseteq \mathcal{K}$
- Semantics for variables: $[\![\cdot]\!] : \mathsf{Var} \to M \to \mathtt{Prop}$
  - which is closed: $\mathrm{cl}[\![V]\!] \subseteq [\![V]\!]$
  - extended to formulas

$$[\![A \otimes B]\!] := \mathrm{cl}([\![A]\!] \bullet [\![B]\!]) \quad [\![A \multimap B]\!] := [\![A]\!] \multimap\!\!\bullet\, [\![B]\!]$$

$$[\![A \,\&\, B]\!] := [\![A]\!] \cap [\![B]\!] \quad [\![A \oplus B]\!] := \mathrm{cl}([\![A]\!] \cup [\![B]\!])$$

$$[\![\bot]\!] := \mathrm{cl}\,\emptyset \quad [\![\top]\!] := M \quad [\![\epsilon]\!] := \mathrm{cl}\{\mathrm{e}\}$$

$$[\![!\,A]\!] := \mathrm{cl}(\mathcal{K} \cap [\![A]\!]) \quad [\![\Gamma_1, \ldots, \Gamma_n]\!] := [\![\Gamma_1 \otimes \cdots \otimes \Gamma_n]\!]$$

- Soundness: if $\Gamma \vdash A$ has a proof then $[\![\Gamma]\!] \subseteq [\![A]\!]$

Relational Phase Sem. (cut-admissibility)

- A syntactic model $M := \texttt{list Form}$
- for $\Gamma, \Delta, \Theta \in M$

$$\Theta \in \Gamma \bullet \Delta \iff \lfloor \Gamma, \Delta \rfloor \sim_p \Theta$$

- $\mathcal{K} := \{!\,\Gamma \mid \Gamma \in M\}$ ($\emptyset \in \mathcal{K}$ and $\mathcal{K} \bullet \mathcal{K} \subseteq \mathcal{K}$)
- *contextual closure* $\mathrm{cl} : (M \to \texttt{Prop}) \to (M \to \texttt{Prop})$

$$\Delta \in \mathrm{cl}\,\mathcal{X} \iff \boxed{\forall \Gamma\, A,\ \mathcal{X}, \Gamma \models A \to \Delta, \Gamma \models A}$$

- where $\models : \texttt{list Form} \to \texttt{Form} \to \texttt{Prop}$
  - $\models$ is a deduction relation
  - such as provability or *cut-free provability*
  - permutations: $\Gamma \sim_p \Delta \to \Gamma \models A \to \Delta \models A$

Rules as algebraic equations

- Define $\downarrow A := \{\Gamma \mid \Gamma \models A\}$, then $\mathrm{cl}(\downarrow A) \subseteq \downarrow A$
- $\downarrow A \bullet \downarrow B \subseteq \downarrow(A \otimes B)$ iff

$$\frac{\Gamma \models A \qquad \Delta \models B}{\Gamma, \Delta \models A \otimes B} \text{ for any } \Gamma, \Delta$$

- $\lfloor A \otimes B \rfloor \in \mathrm{cl}\{\lfloor A, B \rfloor\}$ iff

$$\frac{A, B, \Gamma \models C}{A \otimes B, \Gamma \models C} \text{ for any } \Gamma, C$$

- $\mathcal{K} \subseteq \mathcal{J}$ iff $\models$ closed under W and C.

Okada's lemma

- For $\models$ defined as $\vdash_{\mathrm{cf}}$ closed under cut-free ILL

$$\boxed{\forall A,\ \lfloor A \rfloor \in [\![A]\!] \subseteq {\downarrow} A} \quad \text{and} \quad \forall \Gamma,\ \Gamma \in [\![\Gamma]\!]$$

- By induction on A, then by induction on Γ
- By soundness, from a (cut using) proof of $\Gamma \vdash A$
  - we deduce $\Gamma \in [\![\Gamma]\!] \subseteq [\![A]\!] \subseteq {\downarrow} A$
  - hence $\Gamma \vdash_{\mathrm{cf}} A$
  - hence $\Gamma \vdash A$ is cut-free provable
- Hence a semantic proof of cut-admissibility

Extensions, other logics, cut-elimination

- Extensions to other logics:
  - Phase semantics, contextual closure very generic
  - of course fragments of ILL, but also CLL
  - ILL with modality, Linear time ILL
  - Bunched Implications (BI)
  - Relevance logic, prop. Intuitionistic Logic
  - Display calculi (context = consecutions)?
- Computational content
  - `Prop` $\rightsquigarrow$ `Type` gives cut-elimination algo.
  - can be extracted (you do not want to read it...)
- Coq development
  - `@GH/DmxLarchey/Coq-Phase-Semantics`
  - Around 1300 loc for specs and 1000 loc for proofs
  - 2/5 of which are libraries (lists, permutations ...)