Mathematical Structures in Computer (), 1–
doi:

CAMBRIDGE
UNIVERSITY PRESS

Multiplicative Linear Logic with Self-Dual Unit

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Abstract

In multiplicative linear logic, the admissibility of the mix rules is related with the provability of the sequents $\bot \vdash 1$ and $1 \vdash \bot$. We show, both syntactically and semantically, how this is related with the stronger property $\bot \simeq 1$. Starting from this remark, we introduce and study MLLM: a variant of multiplicative linear logic where $\bot = 1$. We survey sequent calculus, proof nets and categorical semantics of MLLM.

Keywords: Multiplicative Linear Logic; Mix Rule; Proof Nets; Star-Autonomous Categories

Multiplicative linear logic without units (MLL^-) built with the connectives $\otimes$ and $\parr$ is usually considered as the core part of the theory of linear logic. It is for example the setting in which the theory of proof nets (Girard (1987); Danos and Regnier (1989)) is the most (and best) developed. Two independent extensions are very common in the literature: units and (*mix*) rules.

First, it is very natural to extend the binary (associative and commutative) connectives of a logic (here $\otimes$ and $\parr$) with associated units (here 1 and $\bot$). From a semantic point of view it is usually harmless, and it makes things a lot simpler for defining categorical models for example. The categorical structure of proofs of multiplicative linear logic *with units* (MLL) comes from star-autonomous categories, the unit-free case being much more involved (Houston (2008)). On the proof-net side, being able to distinguish proofs considered distinct by the categorical interpretation requires to introduce some additional device such as jumps (Girard (1996); Blute et al. (1996); Lamarche and Straßburger (2006); Hughes (2012)). Then proof nets have to be considered up to some equivalence relation hiding part of the information introduced by jumps (or other device). No simple alternative could exist since the underlying notion of equality of proofs is PSPACE-complete (Heijltjes and Houston (2016)).

The second extension of MLL^- consists in adding the so called (*mix*) rules (see for example Fleury and Retoré (1994); Bellin and Scott (1994); Bellin (1997); Cockett and Seely (1997*a*); Nguyen (2020) for additional discussions). The (mix_2) rule $\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\, mix_2$ corresponds to the provability of $A \otimes B \vdash A \parr B$ and then defines a logic between MLL and the logic of compact closed categories (where $\parr = \otimes$). In the language of proof nets, MLL proofs satisfy acyclicity and connectedness in the Danos-Regnier criterion (Danos and Regnier (1989)), adding (mix_2) allows to forget connectedness, and no condition remains in the compact closed case. The sibling rule (mix_0) $\dfrac{}{\vdash}\, mix_0$ adds the provability of the empty sequent $\vdash$ (and makes the empty proof net

acceptable). From both (mix_2) and (mix_0), one can deduce the general n-ary variants. For this reason we often use the generic wording "(mix) rules" for all of them (and thus for (mix_2) together with (mix_0)).

In the case units are considered, we have a correspondence in provability between (mix_2) and $\bot \vdash 1$, and between (mix_0) and $1 \vdash \bot$. So that considering all (mix) rules has exactly the same provability power as assuming the logical equivalence $\bot \dashv\vdash 1$. This is how things are usually described in the syntactic sequent-calculus world. In the denotational world, this is often translated as $\bot \simeq 1$. Note there is a (potentially big) gap between $\bot \dashv\vdash 1$ and $\bot \simeq 1$: the existence of proofs/morphisms in each direction *vs* the fact that they moreover compose into the identity in both directions. The key contribution of the present work is to fill this gap. Moving between provability and isomorphism requires to consider a notion of equality of proofs. In the spirit of the Curry-Howard-Lambek correspondence (Lambek and Scott (1988)), we consider proofs up to expansion of axioms (η-equivalence) and up to cut elimination (β-equivalence). Our main remark is that simple equations (already validated by the proof-net syntax) suffice then to turn the equiprovability $\bot \dashv\vdash 1$ into an isomorphism. We call the induced congruence on proofs the *ρ-equivalence*, it is generated by:

$$\dfrac{\vdash \Gamma \qquad \dfrac{}{\vdash}\, mix_0}{\vdash \Gamma}\, mix_2 \quad =_\rho \quad \vdash \Gamma \qquad\qquad \dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Gamma}{\vdash \Gamma}\, mix_2 \quad =_\rho \quad \vdash \Gamma$$

which stipulate that (mix_0) acts as a unit for (mix_2). The existence of the isomorphism $\bot \simeq 1$ is studied on the syntactic side by proving the uniqueness (up to $\beta\eta\rho$-equivalence) of the proof of $\vdash \bot, 1$ (Lemma 9 and Theorem 10). On the semantic side, we prove that from the existence of morphisms $m : \bot \otimes \bot \longrightarrow \bot$ (for interpreting (mix_2)) and $n : 1 \longrightarrow \bot$ (for interpreting (mix_0)), one can build an isomorphism $\bot \simeq 1$ in any star-autonomous category as soon as n acts as a unit for m (Lemma 40 and Proposition 42).

Built around these two results, the paper revisits the theory of multiplicative linear logic: sequent calculus, proof nets, categorical semantics, etc. This is mostly a survey of known results from the literature which we choose to formulate in the setting of a variant of MLL with (mix) in which we have a self dual unit, denoted $\mathtt{I}$, satisfying $\mathtt{I} = \mathtt{I}^\bot = \bot = 1$. It strengthens $\bot \simeq 1$ into $\bot = 1$ and gives a common unit to $\otimes$ and $\parr$.

In this framework, proof nets do not need jumps and provide canonical representatives of proofs up to $\beta\eta\rho$-equivalence (without introducing any additional quotient as opposed to the case of MLL with units). In the same time, units make (categorical) semantics easier to describe. We get the best of the two worlds. Note this also happens in the case of the logic of compact closed categories, but we here have a less degenerate setting in which $\parr \neq \otimes$. This structure (with two distinct dual monoidal operations) is directly related with the Danos-Regnier acyclicity condition on proof nets which is a central property of the theory (it avoids the creation of loops in MLL proof nets, it can be used for proving normalization in presence of exponentials, etc.) while connectedness (or weaker counting constraints with units (Fleury and Retoré (1994))) happened to have a much weaker impact on the (computational) properties of proof nets.

In the presentation of the MLLM system, we choose to introduce a self-dual unit, but we also make specific choices for the management of the exchange structural rule. Indeed the treatment of the exchange rule in the sequent calculus of linear logic and in the theory of proof nets is often pretty loose. We try to be more meticulous here: it helps being precise on the definition of congruences on proofs (and it would become necessary for moving towards more accurate formalizations of linear logic). Our design choice is to incorporate (a minimal amount of) exchange rules in the other rules of the system so that exchange becomes admissible (thus avoiding the addition of one more rule to the system). This is done by relying on a shuffle operation on contexts for binary multiplicative rules.

Content and Related Works. Section 1.1 introduces multiplicative linear logic with units and (*mix*) rules. Sequent-calculus rules are formatted in a way which makes the exchange rule admissible (while preserving the computational content of proofs). Basic standard properties are reviewed and, relying on the ρ-equivalence, we introduce the syntactic isomorphism $\bot \cong 1$ (Theorem 10).

Section 1.2 briefly presents some links with intuitionistic linear logic in presence of the (*mix*) rules (following Laurent (2018)).

Section 1.3 defines the variant MLLM with self-dual unit $\mathtt{I} = \mathtt{I}^{\bot}$. Basic properties are proved in a way very similar to Section 1.1.

In Section 2, we study proof nets for MLLM. The framework comes from proof nets for MLL^{-} with (*mix*), extended with I which has mostly no impact. Results are then adapted from those in the corresponding literature (see Girard (1987); Danos and Regnier (1989); Danos (1990); Fleury and Retoré (1994); Bellin (1997) and more). To sum up, sequent-calculus proofs can be translated into proof nets by means of a surjective translation $(\_)^{\bullet}$ (Proposition 22). Through this translation, one gets different characterizations of the equivalence of proofs in Section 3.1:

$$\begin{aligned}\pi_1 =_{\beta\rho} \pi_2 &\iff {\pi_1}^{\bullet} =_{\beta} {\pi_2}^{\bullet}\\ &\iff \mathrm{NF}({\pi_1}^{\bullet}) = \mathrm{NF}({\pi_2}^{\bullet}) \iff (\mathrm{NF}(\pi_1))^{\bullet} = (\mathrm{NF}(\pi_2))^{\bullet}\\ &\iff \mathrm{NF}(\pi_1) =_{\sigma\rho} \mathrm{NF}(\pi_2)\end{aligned}$$

(with $\mathrm{NF}(\_)$ for "normal form").

The short Section 3.2 takes advantage of the common unit for both multiplicative binary connectives to introduce a canonical simplification in formulas. This allows in particular to interpret MLLM in any denotational model of MLL with strict and equal multiplicative units (*i.e.* $[\![\bot]\!] = [\![1]\!]$, $[\![A \otimes 1]\!] = [\![A]\!]$ and $[\![A \parr \bot]\!] = [\![A]\!]$). Moreover results (such as injectivity or full completeness) from the unit-free case can then be lifted to MLLM (Pagani (2007) for example).

Section 4 focuses on categorical semantics and its relation with syntax. This is a widely studied topic for MLL (Seely (1989); Blute et al. (1996); Blute and Scott (2004); Lamarche and Straßburger (2006); Melliès (2009); Hughes (2012)), for MLL^{-} (Houston (2008); Heijltjes and Straßburger (2016)), and for MLL with (*mix*) (Cockett and Seely (1997*a*)). Interpreting the (*mix*) rules thanks to a magma structure on $\bot$, we give a necessary and sufficient condition on $\bot$ to have $\bot \simeq 1$ in a star-autonomous category. This is split into a first result in the monoidal setting: a unitary magma M such that $1 \simeq M \multimap M$ is isomorphic to 1 (Lemma 40). Then, moving to the star-autonomous setting, we can use $1 \simeq \bot \multimap \bot$ to get $\bot \simeq 1$ (Proposition 42).

Finally, following the results from the literature cited above, we give the ingredients for the use of proof nets as a presentation of the free star-autonomous category with 1 as dualizing object.

1. Multiplicative Linear Logic with Mix

1.1 The Sequent Calculus MLL_{mix}

We consider (classical) multiplicative linear logic (MLL) with units extended with (*mix*) rules. We call this system MLL_{mix}.

Given a denumerable set of atoms $\mathscr{X}$ (whose elements are denoted X Y, etc.), **formulas of** MLL_{mix} are given by:

$$A ::= X \mid \bar{X} \mid A \otimes A \mid A \parr A \mid 1 \mid \bot$$

The **dual** of formulas is the involution defined by:

$$\begin{array}{lll} X^{\bot} = \bar{X} & (A \otimes B)^{\bot} = A^{\bot} \parr B^{\bot} & 1^{\bot} = \bot \\ \bar{X}^{\bot} = X & (A \parr B)^{\bot} = A^{\bot} \otimes B^{\bot} & \bot^{\bot} = 1 \end{array}$$

Sequents are written $\vdash \Gamma$ where Γ is a *list* of formulas. The order of formulas is important but it can usually be modified through *exchange* rules. We choose here a presentation of rules where exchange happens to be an admissible rule. This is done by using a *shuffle* operation (Blute and Scott (1998)) when merging sequents. Given two lists Γ and Δ, their **shuffle** $\Gamma \sqcup\!\sqcup \Delta$ (see also Appendix A) is the set of all lists which are permutations of the concatenation of Γ and Δ such that the order of elements in Γ and the order of elements in Δ are *not modified* ($[a, b, \alpha, c, \beta]$ is a shuffle of $[a, b, c]$ and $[\alpha, \beta]$ but $[a, \alpha, c, \beta, b]$ is not).

The **rules** of MLL_{mix} are:

$$\frac{}{\vdash \bar{X}, X}\, ax_l \qquad \frac{}{\vdash X, \bar{X}}\, ax_r \qquad \frac{}{\vdash}\, mix_0 \qquad \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup\!\sqcup \Delta}\, mix_2$$

$$\frac{\vdash \Gamma, A, \Gamma' \qquad \vdash \Delta, B, \Delta'}{\vdash \Gamma \sqcup\!\sqcup \Delta, A \otimes B, \Gamma' \sqcup\!\sqcup \Delta'}\, \otimes \qquad \frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, A \parr B, \Gamma'}\, \parr \qquad \frac{}{\vdash 1}\, 1 \qquad \frac{\vdash \Gamma, \Gamma'}{\vdash \Gamma, \bot, \Gamma'}\, \bot$$

$$\frac{\vdash \Gamma, \underline{A}, \Gamma' \qquad \vdash \Delta, \underline{A^\bot}, \Delta'}{\vdash \Gamma \sqcup\!\sqcup \Delta, \Gamma' \sqcup\!\sqcup \Delta'}\, cut$$

where $\Gamma \sqcup\!\sqcup \Delta$ means that the have *an element of* the shuffle of Γ and Δ.

To help reading the proofs, we mark A (in blue) the main formulas in the conclusion sequent of each rule, and we mark $\underline{A}$ (underlined) the cut formulas in the premises of (*cut*) rules.

Remark 1. Even if we do not write it explicitly, each rule containing the $\sqcup\!\sqcup$ symbol must come with the information of how exactly the two lists are mixed. That is, which one of the two lists, each element in the shuffle comes from. This is required to get proof relevance, as otherwise we would loose the computational content of proofs by making identical, for example, the two proofs of the multiplicative Booleans:

$$\dfrac{\dfrac{\dfrac{}{\vdash \bar{X}, X}\, ax_l \qquad \dfrac{}{\vdash \bar{X}, X}\, ax_l}{\vdash \bar{X}, \bar{X}, X \otimes X}\, \otimes}{\vdash \bar{X} \parr \bar{X}, X \otimes X}\, \parr$$

The shuffling information is required in the ($\otimes$) rule to distinguish between the following two proofs:

$$\dfrac{\dfrac{\dfrac{}{\vdash \bar{X}, X}\, ax_l \qquad \dfrac{}{\vdash \bar{X}, X}\, ax_l}{\vdash \bar{X}, \bar{X}, X \otimes X}\, \otimes}{\vdash \bar{X} \parr \bar{X}, X \otimes X}\, \parr \qquad \dfrac{\dfrac{\dfrac{}{\vdash \bar{X}, X}\, ax_l \qquad \dfrac{}{\vdash \bar{X}, X}\, ax_l}{\vdash \bar{X}, \bar{X}, X \otimes X}\, \otimes}{\vdash \bar{X} \parr \bar{X}, X \otimes X}\, \parr$$

where colors allow to follow occurrences. In the first proof, we shuffle $\bar{X}$ and $\bar{X}$ into $\bar{X}, \bar{X}$, while we shuffle them into $\bar{X}, \bar{X}$ in the second proof.

The rules are presented in a rather constrained way to make the system easier to analyse. However various standard additional rules are derivable or admissible.

Lemma 2 (Axiom Expansion)**.** For any formula A, there exists a proof $\boldsymbol{id}_A$ of $\vdash A^\bot, A$ (*i.e.* the rule $\dfrac{}{\vdash A^\bot, A}\, ax$ is derivable).

Proof. The rewrite rules of Appendices B.1 and B.2 define a strongly confluent and strongly normalizing rewriting system. Starting from $\dfrac{}{\vdash A^\bot, A}\, ax$, the (unique) obtained normal form is the proof id_A of $\vdash A^\bot, A$ in MLL_{mix}. □

Lemma 3 (General Mix Rules)**.** For any natural number n, the following n-ary mix rule is derivable:

$$\frac{\vdash \Gamma_1 \qquad \cdots \qquad \vdash \Gamma_n}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_n}\, mix_n$$

Proof. By induction on n with (mix_0) for $n = 0$ and (mix_2) for the induction step. □

Lemma 4 (Exchange)**.** The exchange rule $\dfrac{\vdash \Gamma}{\vdash \Gamma\varsigma}\, ex$ is admissible in MLL_{mix} (where ς is a permutation).

Proof. By induction on the proof π of $\vdash \Gamma$. We consider each possible last rule of π:

- (ax_l) If ς is the identity, there is nothing to modify in π. If ς permutes $\bar{X}$ and X, we replace π with $\dfrac{}{\vdash X, \bar{X}}\, ax_r$.
- (ax_r) If ς is the identity, there is nothing to modify in π. If ς permutes $\bar{X}$ and X, we replace π with $\dfrac{}{\vdash \bar{X}, X}\, ax_l$.
- (1) ς is the identity on the singleton list, there is nothing to modify in π.
- (mix_0) ς is the identity on the empty list, there is nothing to modify in π.
- (mix_2) By Property A(8), we can apply the induction hypothesis to the premises of the (mix_2) rule and apply a (mix_2) rule to the obtained proofs.
- $(\parr)$ We have $\Sigma = (\Gamma' \cdot [A \parr B] \cdot \Delta')\varsigma$ which means $\Sigma = \Sigma_1 \cdot [A \parr B] \cdot \Sigma_2$ with $\Sigma_1 \cdot \Sigma_2 \sim \Gamma' \cdot \Delta'$ and then $\Sigma_1 \cdot [A] \cdot [B] \cdot \Sigma_2 \sim \Gamma' \cdot [A] \cdot [B] \cdot \Delta'$. We apply the induction hypothesis to the premise of the $(\parr)$ rule, and then we apply a $(\parr)$ rule to the obtained proof.
- $(\bot)$ Very similar to $(\parr)$.
- $(\otimes)$ We look for a conclusion $\Sigma = \Xi\varsigma$ with $\Xi \in (\Gamma' \sqcup \Gamma'' \cdot [A \otimes B] \cdot \Delta' \sqcup \Delta'')$ which means $\Sigma = \Sigma_1 \cdot [A \otimes B] \cdot \Sigma_2$ with $\Sigma_1 \cdot \Sigma_2 \sim \Xi_0 \in (\Gamma' \sqcup \Gamma'') \cdot (\Delta' \sqcup \Delta'') \subseteq (\Gamma' \sqcup \Gamma'') \sqcup (\Delta' \sqcup \Delta'') = (\Gamma' \sqcup \Delta') \sqcup (\Gamma'' \sqcup \Delta'')$ (Properties A(4), A(3) and A(2)), thus $\Xi_0 \in \Xi'_0 \sqcup \Xi''_0$ (with $\Xi'_0 \in \Gamma' \sqcup \Delta'$ and $\Xi''_0 \in \Gamma'' \sqcup \Delta''$). By Property A(8), we get Σ' and Σ'' such that $\Sigma_1 \cdot \Sigma_2 \in \Sigma' \sqcup \Sigma''$, $\Sigma' \sim \Xi'_0 \in \Gamma' \sqcup \Delta'$ (thus $\Sigma' \sim \Gamma' \cdot \Delta'$ by Property A(7)) and $\Sigma'' \sim \Xi''_0 \in \Gamma'' \sqcup \Delta''$ (thus $\Sigma'' \sim \Gamma'' \cdot \Delta''$ by Property A(7)). Now by Property A(5), we get $\Sigma'_1, \Sigma'_2, \Sigma''_1$ and Σ''_2 with $\Sigma' = \Sigma'_1 \cdot \Sigma'_2$, $\Sigma'' = \Sigma''_1 \cdot \Sigma''_2$, $\Sigma_1 \in \Sigma'_1 \sqcup \Sigma''_1$ and $\Sigma_2 \in \Sigma'_2 \sqcup \Sigma''_2$. We then conclude by applying the induction hypothesis to the premises of the $(\otimes)$ rule and then applying a $(\otimes)$ rule to the obtained proofs:

$$\frac{\dfrac{\vdash \Gamma', A, \Delta'}{\vdash \Sigma'_1, A, \Sigma'_2}\, (IH) \qquad \dfrac{\vdash \Gamma'', B, \Delta''}{\vdash \Sigma''_1, B, \Sigma''_2}\, (IH)}{\vdash \Sigma_1, A \otimes B, \Sigma_2}\, \otimes$$

- (cut) Very similar to $(\otimes)$. □

Note that, except for transforming some (ax_r) rules into (ax_l) and conversely, the structure of the proof is unchanged through this exchange transformation.

Proposition 5 (Cut Admissibility)**.** The cut rule is admissible in MLL_{mix} without (cut).

Proof. The rewrite rules of Appendices C.1 and C.2 (also called **reduction steps**) define a strongly normalizing rewriting system. This can be seen by considering, as termination measure, the lexicographic order on pairs of natural numbers with:

- first component, the size (*i.e.* number of rules) of the proof;
- second component, the sum over all cuts c of the size of the sub-proof with c as last rule.

The first component is strictly decreasing in key steps. The first component is not modified in commutation steps, and there is one cut which becomes conclusion of a strictly smaller proof (the sizes of the other sub-proofs ending with cuts are not modified), thus the second component strictly decreases.

A normal form of a proof is a cut-free proof of the same sequent: if there is a cut in a proof, then there is at least one cut with no cut above it, and there is at least one reduction step involving this cut which can be applied. □

From the proof of Proposition 5, one gets that the congruence $=_\beta$ generated by $\to_\beta$ (the reduction relation described in Appendices C.1 and C.2) contains at least one cut-free proof in each equivalence class. This is a crucial notion of equality of proofs. Note, because we restrict axiom rules to atoms, proofs are all "η-expanded", so that the λ-calculus analogue (through the Curry-Howard-Lambek correspondence) of this equational theory is the $\beta\eta$-equivalence of the λ-calculus.

From category theory (see Section 4 and in particular Section 4.4), we can import the idea of isomorphic objects into the world of logical systems (see for example Di Cosmo (1995); Balat and Di Cosmo (1999); Di Guardia and Laurent (2025)). Given a congruence $\equiv$ on proofs, two formulas A and B are **isomorphic** (denoted $A \cong B$) if there exist two proofs π_1 with conclusion $\vdash A^\perp, B$ and π_2 with conclusion $\vdash B^\perp, A$ such that:

$$\dfrac{\dfrac{\pi_1}{\vdash A^\perp, \underline{B}} \qquad \dfrac{\pi_2}{\vdash \underline{B^\perp}, A}}{\vdash A^\perp, A}\, cut \quad \equiv \quad \dfrac{}{\vdash A^\perp, A}\, id_A$$

and

$$\dfrac{\dfrac{\pi_2}{\vdash B^\perp, \underline{A}} \qquad \dfrac{\pi_1}{\vdash \underline{A^\perp}, B}}{\vdash B^\perp, B}\, cut \quad \equiv \quad \dfrac{}{\vdash B^\perp, B}\, id_B$$

We are now going to study the impact of the (*mix*) rules on the units 1 and $\perp$. It is well known (see for example Bellin (1997); Cockett and Seely (1997*a*)) that (mix_0) entails the provability of $\vdash \perp, \perp$ and (mix_2) entails the provability of $\vdash 1, 1$:

$$\dfrac{\dfrac{\dfrac{}{\vdash}\, mix_0}{\vdash \underline{\perp}}\, \perp}{\vdash \perp, \underline{\perp}}\, \perp \qquad\qquad \dfrac{\dfrac{}{\vdash \underline{1}}\, 1 \qquad \dfrac{}{\vdash \underline{1}}\, 1}{\vdash 1, 1}\, mix_2$$

and, using (*cut*) rules, the converse also holds: if $\vdash \perp, \perp$ is provable then (mix_0) is derivable, and if $\vdash 1, 1$ is provable then (mix_2) is derivable:

$$\dfrac{\dfrac{}{\vdash \underline{1}}\, 1 \qquad \dfrac{\dfrac{}{\vdash \underline{1}}\, 1 \qquad \vdash \underline{\perp}, \perp}{\vdash \underline{\perp}}\, cut}{\vdash}\, cut \qquad\qquad \dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, \underline{\perp}}\, \perp \qquad \dfrac{\dfrac{\vdash \Delta}{\vdash \Delta, \underline{\perp}}\, \perp \qquad \vdash \underline{1}, 1}{\vdash \Delta, \underline{1}}\, cut}{\vdash \Gamma \sqcup \Delta}\, cut$$

We are going to prove that, adding to $=_\beta$ a very simple quotient on proofs, provides an isomorphism $1 \cong \perp$ (which is a much stronger property than the logical equivalence $\vdash \perp, \perp$ and $\vdash 1, 1$ described above). In presence of the mix rules, the following equalities on proofs are natural to consider as they remove useless (mix_0) rules, and integrate the idea of (mix_0) acting as a unit for

(mix_2) (they correspond to Property A(1) of $\sqcup$):

$$\dfrac{\vdash \Gamma \qquad \dfrac{}{\vdash}\, mix_0}{\vdash \Gamma}\, mix_2 \qquad =_{\rho_1} \qquad \vdash \Gamma$$

$$\dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Gamma}{\vdash \Gamma}\, mix_2 \qquad =_{\rho_2} \qquad \vdash \Gamma$$

The extension of $=_\beta$ incorporating these two (resp. first, resp. second) equations is denoted $=_{\beta\rho}$ (resp. $=_{\beta\rho_1}$, resp. $=_{\beta\rho_2}$). The congruence generated by both $=_{\rho_1}$ and $=_{\rho_2}$ is called $=_\rho$.

Lemma 6 (Properties of (mix_2))**.** The following equations about (mix_2) can be derived:

$$\dfrac{\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad \vdash \Sigma}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2 \qquad =_\beta \qquad \dfrac{\vdash \Gamma \qquad \dfrac{\vdash \Delta \qquad \vdash \Sigma}{\vdash \Delta \sqcup \Sigma}\, mix_2}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2$$

$$\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad =_{\beta\rho_i} \qquad \dfrac{\vdash \Delta \qquad \vdash \Gamma}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad\qquad i \in \{1,2\}$$

$$\dfrac{\vdash \Gamma \qquad \dfrac{}{\vdash}\, mix_0}{\vdash \Gamma}\, mix_2 \qquad =_{\beta\rho_2} \qquad \vdash \Gamma$$

$$\dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Gamma}{\vdash \Gamma}\, mix_2 \qquad =_{\beta\rho_1} \qquad \vdash \Gamma$$

Proof. The following proof:

$$\dfrac{\vdash \Gamma \qquad \dfrac{\vdash \Delta}{\vdash \Delta, \bot}\, \bot}{\vdash \Gamma \sqcup \Delta, \underline{\bot}}\, mix_2 \qquad \dfrac{\dfrac{}{\vdash 1}\, 1 \qquad \vdash \Sigma}{\vdash \Sigma, \underline{1}}\, mix_2}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, cut$$

reduces by $\to_\beta$ to both:

$$\dfrac{\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad \vdash \Sigma}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2 \qquad \text{and} \qquad \dfrac{\vdash \Gamma \qquad \dfrac{\vdash \Delta \qquad \vdash \Sigma}{\vdash \Delta \sqcup \Sigma}\, mix_2}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2$$

thus they are equal up to $=_\beta$.

Assuming $i = 2$ (the case $i = 1$ is simular), the following proof:

$$\dfrac{\dfrac{}{\vdash 1}\, 1 \qquad \vdash \Gamma}{\vdash \underline{1}, \Gamma}\, mix_2 \qquad \dfrac{\dfrac{\dfrac{}{\vdash}\, mix_0}{\vdash \bot}\, \bot \qquad \vdash \Delta}{\vdash \underline{\bot}, \Delta}\, mix_2}{\vdash \Gamma \sqcup \Delta}\, cut$$

reduces by $\to_\beta$ to both:

$$\dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Gamma}{\vdash \Gamma}\, mix_2 \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad \text{and} \qquad \dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Delta}{\vdash \Delta}\, mix_2 \qquad \vdash \Gamma}{\vdash \Gamma \sqcup \Delta}\, mix_2$$

thus, by applying $=_{\rho_2}$ to each of them, we get

$$\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad =_{\beta\rho_2} \qquad \dfrac{\vdash \Delta \qquad \vdash \Gamma}{\vdash \Gamma \sqcup \Delta}\, mix_2$$

We conclude with:

$$\dfrac{\vdash \Gamma \qquad \dfrac{}{\vdash}\, mix_0}{\vdash \Gamma}\, mix_2 \quad =_{\beta\rho_2} \quad \dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Gamma}{\vdash \Gamma}\, mix_2 \quad =_{\rho_2} \quad \vdash \Gamma$$

and the last case is similar. □

In particular, $=_{\beta\rho}$, $=_{\beta\rho_1}$ and $=_{\beta\rho_2}$ coincide. In MLL_{mix}, we consider the congruence $=_{\beta\rho}$ as our notion of equality of proofs. Isomorphisms between formulas are then considered in this setting and denoted $A \cong B$.

We now explore the whole set of proofs of some simple sequents when proofs are considered up to $=_{\beta\rho}$.

Lemma 7 (Uniqueness for $\vdash$)**.** Any cut-free proof of $\vdash$ is equal to $\dfrac{}{\vdash}\, mix_0$, up to $=_\rho$. So that, up to $=_{\beta\rho}$, this is the only proof of $\vdash$.

Proof. We prove the first point by induction on the size of the proof. The only possible last rules of a cut-free proof of $\vdash$ are (mix_0) and (mix_2). In the first case, we are done. In the second case, by induction hypothesis applied to the two premises, we get a proof which, up to $=_\rho$, is:

$$\dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \dfrac{}{\vdash}\, mix_0}{\vdash}\, mix_2 \qquad =_\rho \qquad \dfrac{}{\vdash}\, mix_0$$

Up to $=_\beta$, any proof is equal to a cut-free proof (Proposition 5) thus, up to $=_{\beta\rho}$, any proof of $\vdash$ is equal to $\dfrac{}{\vdash}\, mix_0$. □

Lemma 8 (Non-Trivial (mix_2) Rules)**.** Any cut-free proof is equal, up to $=_\rho$, to a proof in which no premise of a (mix_2) rule is an empty sequent.

Proof. By induction on the size of the proof: if it contains such a (mix_2) rule, then the corresponding premise must be equal to $\dfrac{}{\vdash}\, mix_0$ up to $=_\rho$ by Lemma 7, and we can remove these (mix_2) and (mix_0) rules thanks to $=_\rho$ (and we get a smaller proof). □

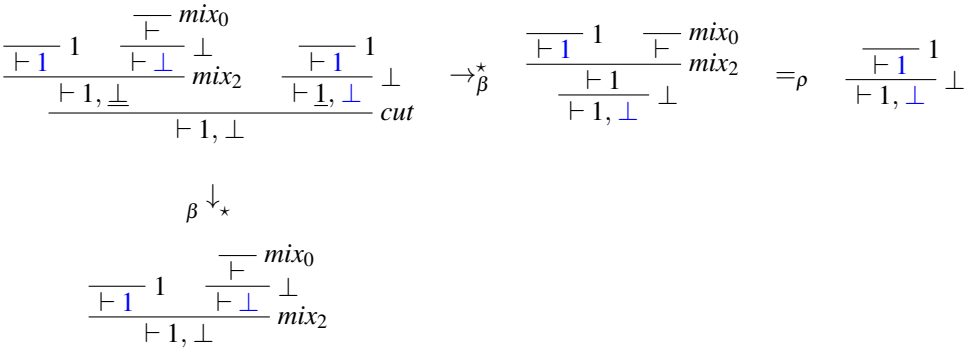
Note that a (mix_2) rule with no empty premise must have a conclusion with at least two formulas.

Lemma 9 (Uniqueness for $\vdash 1, \bot$)**.** Up to $=_{\beta\rho}$, there is exactly one proof of $\vdash 1, \bot$ which is $id_\bot$ (and the same with id_1 for $\vdash \bot, 1$).

Proof. We can focus on cut-free proofs (thanks to $=_\beta$ and Proposition 5) in which no premise of a (mix_2) rule is empty (thanks to Lemma 8). Then we find only the following three possibilities:

$$\dfrac{\dfrac{}{\vdash 1}\, 1}{\vdash 1, \bot}\, \bot \qquad \dfrac{\dfrac{}{\vdash 1}\, 1 \qquad \dfrac{\dfrac{}{\vdash}\, mix_0}{\vdash \bot}\, \bot}{\vdash 1, \bot}\, mix_2 \qquad \dfrac{\dfrac{\dfrac{}{\vdash}\, mix_0}{\vdash \bot}\, \bot \qquad \dfrac{}{\vdash 1}\, 1}{\vdash 1, \bot}\, mix_2$$

The last two are equal up to $=_{\beta\rho}$ by Lemma 6. The first two are equal up to $=_{\beta\rho}$:



□

Theorem 10 (Isomorphism between $\bot$ and 1)**.** In MLL_{mix} up to $=_{\beta\rho}$, we have $\bot \cong 1$.

Proof. We have already seen that $\vdash 1, 1$ and $\vdash \bot, \bot$ are provable. Then, thanks to Lemma 9, their composition in each direction must be the identity proof up to $=_{\beta\rho}$. □

Remember that $\bot \cong 1$ entails, in MLL, that both (mix_0) and (mix_2) are derivable. Starting from Section 1.3, we are going to study the extension of MLL with $\bot \cong 1$. In fact to make things slightly simpler we consider the case $\bot = 1$.

1.2 Double Negation Translation

In this section, we follow Laurent (2018) in comparing provability in classical MLL and intuitionistic IMLL extended with (*mix*) rules. Remember that intuitionistic multiplicative linear logic (IMLL) is a particularly natural variant of MLL to consider when dealing with categorical semantics (see Section 4).

It is not immediate to define a (mix_2) rule in intuitionistic linear logic since something like $\dfrac{\vdash A \qquad \vdash B}{\vdash A, B}$ is meaningless in an intuitionistic setting (*i.e.* with exactly one formula on the right-hand side of sequents). However, by fixing a particular intuitionistic linear formula $\mathtt{R}$, one can introduce the system $\mathsf{IMLL}_{mix(\mathtt{R})}$ which includes some mix behaviour.

Given a denumerable set of atoms $\mathscr{X}$, **intuitionistic** (multiplicative linear) **formulas** are defined by:

$$F ::= X \mid F \otimes F \mid 1 \mid F \multimap F$$

Given a formula $\mathtt{R}$ in this grammar, the system $\mathsf{IMLL}_{mix(\mathtt{R})}$ is defined with sequents of the shape $\Gamma \vdash F$ (with Γ a list of intuitionistic formulas and F an intuitionistic formula) by means of the

following **rules** (extending IMLL):

$$\frac{}{X \vdash X}\, ax_a \qquad \frac{\Gamma, \Sigma \vdash F \qquad \Delta, F, \Xi \vdash G}{\Gamma \sqcup \Delta, \Sigma \sqcup \Xi \vdash G}\, cut$$

$$\frac{\Gamma \vdash F \qquad \Delta \vdash G}{\Gamma \sqcup \Delta \vdash F \otimes G} \otimes R \qquad \frac{\Gamma, F, G, \Delta \vdash H}{\Gamma, F \otimes G, \Delta \vdash H} \otimes L \qquad \frac{}{\vdash 1}\, 1R \qquad \frac{\Gamma, \Delta \vdash F}{\Gamma, 1, \Delta \vdash F}\, 1L$$

$$\frac{\Gamma, F \vdash G}{\Gamma \vdash F \multimap G} \multimap R \qquad \frac{\Gamma, \Sigma \vdash F \qquad \Delta, G, \Xi \vdash H}{\Gamma \sqcup \Delta, F \multimap G, \Sigma \sqcup \Xi \vdash H} \multimap L$$

$$\frac{}{\vdash \mathtt{R}}\, \mathtt{R} \qquad \frac{\Gamma \vdash \mathtt{R} \qquad \Delta \vdash \mathtt{R}}{\Gamma \sqcup \Delta \vdash \mathtt{R}}\, mix_{\mathtt{R}}$$

Lemma 11 (Axiom Expansion)**.** For any formula F, the rule $\frac{}{F \vdash F}\, ax$ is derivable in $\mathsf{IMLL}_{mix(\mathtt{R})}$ (we note id_F the associated proof).

Proof. It is derivable in IMLL thus in $\mathsf{IMLL}_{mix(\mathtt{R})}$. □

In $\mathsf{IMLL}_{mix(\mathtt{R})}$, we have proofs of $1 \vdash \mathtt{R}$ and $\mathtt{R} \otimes \mathtt{R} \vdash \mathtt{R}$:

$$\frac{\frac{}{\vdash \mathtt{R}}\, \mathtt{R}}{1 \vdash \mathtt{R}}\, 1L \qquad \frac{\frac{id_{\mathtt{R}} \qquad id_{\mathtt{R}}}{\mathtt{R} \vdash \mathtt{R} \qquad \mathtt{R} \vdash \mathtt{R}}\, mix_{\mathtt{R}}}{\mathtt{R}, \mathtt{R} \vdash \mathtt{R}}}{\mathtt{R} \otimes \mathtt{R} \vdash \mathtt{R}} \otimes L$$

The translation of MLL into IMLL defined in Laurent (2018) can be turned into a translation of MLL_{mix} into $\mathsf{IMLL}_{mix(\mathtt{R})}$. On formulas, we define:

$$X^\dagger = X \multimap \mathtt{R} \qquad \bar{X}^\dagger = X$$
$$1^\dagger = 1 \multimap \mathtt{R} \qquad \bot^\dagger = 1$$
$$(A \otimes B)^\dagger = ((A^\dagger \multimap \mathtt{R}) \otimes (B^\dagger \multimap \mathtt{R})) \multimap \mathtt{R} \qquad (A \parr B)^\dagger = A^\dagger \otimes B^\dagger$$

Lemma 12 (Translation of Dual (Laurent, 2018, Lemma 2.3))**.** Given a formula A of MLL_{mix}, $A^\dagger \multimap \mathtt{R}$, $(A^\bot)^\dagger \multimap \mathtt{R} \vdash \mathtt{R}$ is derivable in IMLL.

An MLL_{mix} proof of $\vdash \Gamma$ is then translated into an $\mathsf{IMLL}_{mix(\mathtt{R})}$ proof of $\Gamma^\dagger \vdash \mathtt{R}$. For MLL, each rule is translated into a sequence of rules (see Laurent (2018)), and Lemma 12 is used for the (*cut*) case. We give here the translation of the new rules for MLL_{mix}:

$$\frac{}{\vdash}\, mix_0 \quad \mapsto \quad \frac{}{\vdash \mathtt{R}}\, \mathtt{R}$$

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \quad \mapsto \quad \frac{\Gamma^\dagger \vdash \mathtt{R} \qquad \Delta^\dagger \vdash \mathtt{R}}{\Gamma^\dagger \sqcup \Delta^\dagger \vdash \mathtt{R}}\, mix_{\mathtt{R}}$$

In the particular case $\mathtt{R} = 1$, (Laurent, 2018, Proposition 2.14) tells us that $\vdash \Gamma$ is provable in MLL_{mix} if and only if $\Gamma^\dagger \vdash 1$ is provable in $\mathsf{IMLL}_{mix(1)}$ if and only if $\vdash \Gamma$ is provable in MLL extended with the equivalence between 1 and $\bot$.

1.3 The Sequent Calculus MLLM

Following the idea that the one-sided presentation of linear logic can be obtained from the two-sided sequent calculus by turning the isomorphism $A^{\bot\bot} \cong A$ into an equality $A^{\bot\bot} = A$, we will

now give a presentation of MLL_{mix} in which we have not only $\bot \cong 1$ but directly $\bot = 1$, *i.e.* $1^\bot = 1$ (and thus $\bot^\bot = \bot$). We denote by $\mathtt{I}$ this unique unit, called *bone*, and satisfying $\mathtt{I}^\bot = \mathtt{I}$.

Given a denumerable set of atoms $\mathscr{X}$ (whose elements are denoted X, Y, etc.), **formulas of** MLLM are given by:

$$A ::= X \mid \bar{X} \mid A \otimes A \mid A \parr A \mid \mathtt{I}$$

The **dual** of formulas is the involution defined by:

$$X^\bot = \bar{X} \qquad (A \otimes B)^\bot = A^\bot \parr B^\bot \qquad \mathtt{I}^\bot = \mathtt{I}$$
$$\bar{X}^\bot = X \qquad (A \parr B)^\bot = A^\bot \otimes B^\bot$$

There are different possibilities for selecting the rules. In particular one can choose to remove the (1) rule or the ($\bot$) rule from MLL_{mix}. One could also choose to keep both and to remove the (mix_0) and (mix_2) rule (but they are necessary for cut elimination to hold). Because of better induced properties, we make the choice of removing the ($\bot$) rule while keeping the others. Indeed we want cut-elimination and moreover ($\bot$) and (mix_2) are the only rules in MLL_{mix} which can be applied with an empty sequent as premise, thus below a (mix_0) rule. Avoiding the ($\bot$) rule, and reasoning up to $=_\rho$, provides strong constraints on the use of the (mix_0) rule, see Proposition 17.

The **rules** of MLLM are:

$$\frac{}{\vdash \bar{X}, X}\, ax_l \qquad \frac{}{\vdash X, \bar{X}}\, ax_r \qquad \frac{}{\vdash}\, mix_0 \qquad \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2$$

$$\frac{\vdash \Gamma, A, \Gamma' \qquad \vdash \Delta, B, \Delta'}{\vdash \Gamma \sqcup \Delta, A \otimes B, \Gamma' \sqcup \Delta'} \otimes \qquad \frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, A \parr B, \Gamma'} \parr \qquad \frac{}{\vdash \mathtt{I}}\, \mathtt{I}$$

$$\frac{\vdash \Gamma, \underline{A}, \Gamma' \qquad \vdash \Delta, \underline{A^\bot}, \Delta'}{\vdash \Gamma \sqcup \Delta, \Gamma' \sqcup \Delta'}\, cut$$

The unit $\mathtt{I}$ replaces both 1 and $\bot$ as they become identified. Note that $\mathtt{I} \mapsto 1$ defines a canonical embedding of cut-free proofs of MLLM into cut-free proofs of MLL_{mix}. Conversely, mapping both 1 and $\bot$ to $\mathtt{I}$, and by translating the ($\bot$) rule by:

$$\frac{\vdash \Gamma, \Delta}{\vdash \Gamma, \bot, \Delta}\, \bot \qquad \mapsto \qquad \frac{\vdash \Gamma, \Delta \qquad \dfrac{}{\vdash \mathtt{I}}\, \mathtt{I}}{\vdash \Gamma, \mathtt{I}, \Delta}\, mix_2$$

one can map MLL_{mix} proofs (including cuts since the dual is mapped to the dual) into MLLM.

In MLLM, we get similar properties as in MLL_{mix}. Let us recall some of them.

Lemma 13 (Axiom Expansion)**.** For any formula A, there exists a proof id_A of $\vdash A^\bot, A$ (*i.e.* the rule $\dfrac{}{\vdash A^\bot, A}\, ax$ is derivable).

Proof. As in Lemma 2 using the rewrite rules of Appendices B.1 and B.3. □

Lemma 14 (General Mix Rules)**.** For any natural number n, the following n-ary mix rule is derivable:

$$\frac{\vdash \Gamma_1 \qquad \cdots \qquad \vdash \Gamma_n}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_n}\, mix_n$$

Lemma 15 (Exchange)**.** The exchange rule is admissible in MLLM.

Proof. Same proof as Lemma 4. □

The constraints on the rules of MLLM allow us to strengthen the situation regarding (mix_0) and empty sequents.

Lemma 16 (Asocial (mix_0))**.** If the conclusion of π is not the empty sequent then any (mix_0) rule can be removed from π using $=_\rho$.

Proof. In MLLM, a (mix_0) rule can only be the premise of a (mix_2) rule and $=_\rho$ equations can be applied to erase them. □

Proposition 17 (Cut Admissibility)**.** If Γ is a non-empty list of formulas then $\vdash \Gamma$ is provable in MLLM if and only if it is provable without using the (*cut*) rule and with non-empty sequents only.

Proof. Using the rules from Appendices C.1 and C.3, one can turn any proof into a cut-free proof in the same way as in Proposition 5.

In a cut-free proof, we can apply Lemma 16 and get a proof with no empty sequent if the conclusion is not empty (in MLLM, an empty sequent can only be the conclusion of a (*cut*) rule or of a (mix_0) rule). □

This means that proofs can restricted to be a (mix_0) rule alone (which gives a cut-free proof of the empty sequent) or not to use it at all. In fact the proof of Proposition 17 tells us that at least one of these proofs can be found in each equivalence class for $=_{\beta\rho}$ (where $=_{\beta\rho}$ is defined as in MLL_{mix} from the relation presented in Appendices C.1 and C.3 by adding the two $=_\rho$ equations).

In Section 2, we are going to define the alternative syntax of proof nets which provides a quotient of sequent-calculus proofs up to rule permutations. Rule permutations are the equations defined in Appendix D. The induced congruence is denoted $=_\sigma$ (in relation with the work in the λ-calculus (Regnier (1994))). We focus here on the permutations of rules in cut-free proofs as it is sufficient for the results we want to derive. It is possible to take cuts into account, but this leads to additional technical difficulties: is the (*cut*) rule commutative (*i.e.* do we identify two proofs where the premises of a (*cut*) rule are permuted)? what is the desired quotient on proofs with respect to cuts and exchanges (*i.e.* for example which proofs below which all reduce to (ax_l) should be identified up to rule permutations)?

$$\dfrac{\dfrac{}{\vdash \bar{X}, \underline{X}}\,ax_l \qquad \dfrac{}{\vdash \underline{\bar{X}}, X}\,ax_l}{\vdash \bar{X}, X}\,cut \qquad\qquad \dfrac{\dfrac{}{\vdash \underline{\bar{X}}, X}\,ax_l \qquad \dfrac{}{\vdash \bar{X}, \underline{X}}\,ax_l}{\vdash \bar{X}, X}\,cut$$

$$\dfrac{\dfrac{}{\vdash \bar{X}, \underline{X}}\,ax_l \qquad \dfrac{}{\vdash X, \underline{\bar{X}}}\,ax_r}{\vdash \bar{X}, X}\,cut \qquad\qquad \dfrac{\dfrac{}{\vdash \underline{X}, \bar{X}}\,ax_r \qquad \dfrac{}{\vdash \underline{\bar{X}}, X}\,ax_l}{\vdash \bar{X}, X}\,cut$$

Lemma 18 (From $=_{\sigma\rho}$ to $=_{\beta\rho}$)**.** The congruence $=_{\sigma\rho}$ is included in $=_{\beta\rho}$.

Proof. Here are two typical examples (see Lemma 6). The proof:

$$\dfrac{\dfrac{\vdash \Gamma \qquad \dfrac{\dfrac{}{\vdash \mathtt{I}}\,\mathtt{I} \qquad \vdash \Delta}{\vdash \mathtt{I}, \Delta}\,mix_2}{\vdash \underline{\mathtt{I}}, \Gamma \sqcup \Delta}\,mix_2 \qquad \dfrac{\dfrac{}{\vdash \mathtt{I}}\,\mathtt{I} \qquad \vdash \Sigma}{\vdash \underline{\mathtt{I}}, \Sigma}\,mix_2}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\,cut$$

reduces by $\to_\beta$ to both:

$$\dfrac{\dfrac{\vdash \Gamma \qquad \dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Delta}{\vdash \Delta}\, mix_2}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad \vdash \Sigma}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2 \quad \text{and} \quad \dfrac{\vdash \Gamma \qquad \dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Delta}{\vdash \Delta}\, mix_2 \qquad \vdash \Sigma}{\vdash \Delta \sqcup \Sigma}\, mix_2}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2$$

and we conclude by $=_\rho$ that we have:

$$\dfrac{\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad \vdash \Sigma}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2 \quad =_{\beta\rho} \quad \dfrac{\vdash \Gamma \qquad \dfrac{\vdash \Delta \qquad \vdash \Sigma}{\vdash \Delta \sqcup \Sigma}\, mix_2}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2$$

Similarly, the proof:

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{I}}\, \mathtt{I} \qquad \vdash \Gamma}{\vdash \underline{\mathtt{I}}, \Gamma}\, mix_2 \qquad \dfrac{\dfrac{}{\vdash \mathtt{I}}\, \mathtt{I} \qquad \vdash \Delta}{\vdash \underline{\mathtt{I}}, \Delta}\, mix_2}{\vdash \Gamma \sqcup \Delta}\, cut$$

reduces by $\to_\beta$ to both:

$$\dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Gamma}{\vdash \Gamma}\, mix_2 \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \quad \text{and} \quad \dfrac{\dfrac{}{\vdash}\, mix_0 \qquad \vdash \Delta}{\vdash \Delta}\, mix_2 \qquad \vdash \Gamma}{\vdash \Gamma \sqcup \Delta}\, mix_2$$

thus, by applying $=_\rho$ to both of them, we get

$$\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \quad =_{\beta\rho} \quad \dfrac{\vdash \Delta \qquad \vdash \Gamma}{\vdash \Gamma \sqcup \Delta}\, mix_2$$

See Di Guardia and Laurent (2025) for other cases. □

The translation from MLL_{mix} into $\mathsf{IMLL}_{mix(\mathtt{R})}$ described in Section 1.2 can be adapted to MLLM with $\mathtt{I}^\dagger = 1$ $(= (\mathtt{I}^\perp)^\dagger)$. Lemma 12 is extended with:

$$\dfrac{\dfrac{}{\vdash 1}\, 1R \qquad \dfrac{id_{\mathtt{R}}}{\mathtt{R} \vdash \mathtt{R}}}{1 \multimap \mathtt{R} \vdash \mathtt{R}}\, \multimap L \qquad \dfrac{\dfrac{}{\vdash 1}\, 1R \qquad \dfrac{id_{\mathtt{R}}}{\mathtt{R} \vdash \mathtt{R}}}{1 \multimap \mathtt{R} \vdash \mathtt{R}}\, \multimap L}{1 \multimap \mathtt{R}, 1 \multimap \mathtt{R} \vdash \mathtt{R}}\, mix_{\mathtt{R}}$$

The translation of proofs comes with:

$$\dfrac{}{\vdash \mathtt{I}}\, \mathtt{I} \quad \mapsto \quad \dfrac{\dfrac{}{\vdash \mathtt{R}}\, \mathtt{R}}{1 \vdash \mathtt{R}}\, 1L$$

2. Proof Nets

We introduce a notion of proof net for MLLM which corresponds exactly to those used for MLL with (*mix*) and 1 as unique unit (except for cut elimination with relies on self duality $\mathtt{I}^\perp = \mathtt{I}$). Avoiding the use of $\perp$ makes things a lot simpler as no notion of jump for example is required.

Definition 19 (Proof Net)**.** A **proof net** is a target-partial directed graph with edges labeled with formulas and with vertices labeled with *ax*, $\mathtt{I}$, $\otimes$, $\parr$ or *cut*. By *target-partial*, we mean that edges may have no target (they all have a source). Local constraints are applied to each kind of vertex:

- *ax*-vertices have no input edge and two output edges labeled with an atom and its dual;

- `I`-vertices have no input edge and one output edge labeled `I`;
- $\otimes$-vertices have two ordered input edges and one output edge with label $A \otimes B$ (where A is the label of the first input edge and B is the label of the second input edge);
- $\parr$-vertices have two ordered input edges and one output edge with label $A \parr B$ (where A is the label of the first input edge and B is the label of the second input edge);
- *cut*-vertices have two input edges with labels A and $A^\perp$ and no output edge.

A total order is given on the set of all edges with no target.

These graphs are also subject to the **Danos-Regnier acyclicity condition** (Danos and Regnier (1989); Danos (1990)): every (undirected) cycle contains the two input edges of some $\parr$-vertex.

In a proof net, the input edges of a vertex are called its **premises** and the output edges of a vertex are called its **conclusions**. The edges with no target are called the **conclusions** of the proof net. Given a proof net $\mathscr{R}$ with ordered conclusions $e_1, \ldots, e_n$ with labels $A_1, \ldots, A_n$, the sequent $\vdash A_1, \ldots, A_n$ is called the **conclusion sequent** of $\mathscr{R}$. A $\otimes$- or $\parr$-vertex whose conclusion is a conclusion of the proof net is called **terminal**.

Remark 20. In a proof net:

- any edge with label X or $\bar{X}$ is conclusion of an *ax*-vertex;
- any edge with label `I` is conclusion of a `I`-vertex;
- any edge with label $A \otimes B$ is conclusion of a $\otimes$-vertex (and its first premise is labeled A and the second one is labeled B);
- any edge with label $A \parr B$ is conclusion of a $\parr$-vertex (and its first premise is labeled A and the second one is labeled B).

As a consequence, given a sequence of formulas Γ containing at most one occurrence of each atom X and of each dual $\bar{X}$, there is at most one cut-free proof net with conclusion sequent $\vdash \Gamma$.

Indeed the absence of *cut*-vertex guarantees that any edge can be reached from a conclusion of the proof net through an anti-directed path. Then starting from a conclusion edge, its source vertex is uniquely determined by the label of the edge, and we can iterate this process until we reach *ax*-vertices and `I`-vertices. The only missing information is how *ax*-vertices are pairing edges of dual types X and $\bar{X}$, but since there is at most one of each of them, there is at most one possibility. Note the graph produced this way, if it exists (*i.e.* if each X comes with some $\bar{X}$), does not necessarily validate the Danos-Regnier acyclicity condition.

2.1 Desequentialization and Sequentialization

Given a proof π of $\vdash \Gamma$, one can define, by induction on π, a proof net $\pi^\bullet$ with conclusion sequent $\vdash \Gamma$ in such a way that there is a bijection between the non-(*mix*) rules of π and the vertices of $\pi^\bullet$. $\pi^\bullet$ is sometimes called the **desequentialization** of π. For that, we look at the last rule of π:

(ax_l) If the conclusion of π is $\vdash \bar{X}, X$, $\pi^\bullet$ contains one *ax*-vertex v and two edges e_1 (labeled $\bar{X}$) and e_2 (labeled X). These edges have v as source and no target, they are ordered by $e_1 < e_2$.

(ax_r) If the conclusion of π is $\vdash X, \bar{X}$, $\pi^\bullet$ contains one *ax*-vertex v and two edges e_1 (labeled $\bar{X}$) and e_2 (labeled X). These edges have v as source and no target, they are ordered by $e_2 < e_1$.

(mix_0) $\pi^\bullet$ is the empty proof net.

(mix_2) If π is obtained from π_1 with conclusion $\vdash \Gamma$ and π_2 with conclusion $\vdash \Delta$ by applying a (mix_2) rule, we obtain $\pi^\bullet$ from the disjoint union of ${\pi_1}^\bullet$ and ${\pi_2}^\bullet$, by re-ordering the conclusions according to the shuffle associated with the rule.

($\otimes$) If π is obtained from π_1 with conclusion $\vdash \Gamma, A, \Gamma'$ and π_2 with conclusion $\vdash \Delta, B, \Delta'$ by applying a ($\otimes$) rule introducing $A \otimes B$, we obtain $\pi^\bullet$ from the disjoint union of ${\pi_1}^\bullet$ and ${\pi_2}^\bullet$ by adding a new $\otimes$-vertex v and a new edge e labeled $A \otimes B$ with source v and no target. The two conclusions edges e_1 of ${\pi_1}^\bullet$ with label A and e_2 of ${\pi_2}^\bullet$ with label B now have target v and are appropriately ordered as premises of v: $e_1 < e_2$. The total order on the conclusions of $\pi^\bullet$ is obtained by starting from the new edge e, and by ordering the edges smaller than e_1 in ${\pi_1}^\bullet$ and those smaller than e_2 in ${\pi_2}^\bullet$ below e according to the shuffle associated with the rule (and the same with edges bigger than e_1 and those bigger than e_2, all put above e).

($\parr$) If π is obtained from π_1 with conclusion $\vdash \Gamma, A, B, \Delta$ by applying a ($\parr$) rule introducing $A \parr B$, we obtain $\pi^\bullet$ from ${\pi_1}^\bullet$ by adding a new $\parr$-vertex v and a new edge e labeled $A \parr B$ with source v and no target. The two conclusions edges e_1 and e_2 of ${\pi_1}^\bullet$ with labels A and B now have target v and are appropriately ordered as premises of v: $e_1 < e_2$. The total order on the conclusions of $\pi^\bullet$ is obtained by replacing the two (consecutive) edges labeled A and B by the new edge e.

(I) $\pi^\bullet$ contains one I-vertex v and one edge labeled I with v as source and no target.

(*cut*) If π is obtained from π_1 with conclusion $\vdash \Gamma, A, \Gamma'$ and π_2 with conclusion $\vdash \Delta, A^\perp, \Delta'$ by applying a (*cut*) rule, we obtain $\pi^\bullet$ from the disjoint union of ${\pi_1}^\bullet$ and ${\pi_2}^\bullet$ by adding a new *cut*-vertex v as target of the conclusion edge e_1 labeled A of ${\pi_1}^\bullet$ and of the conclusion edge e_2 labeled $A^\perp$ of ${\pi_2}^\bullet$. The total order on the conclusions of $\pi^\bullet$ is obtained by ordering the edges smaller than e_1 in ${\pi_1}^\bullet$ and those smaller than e_2 in ${\pi_2}^\bullet$ below the edges bigger than e_1 and those bigger than e_2 (on each side, edges are ordered according to the shuffle).

We want now, conversely, to extract a sequent calculus proof form any proof net. That is to prove that $(\_)^\bullet$ is surjective.

Lemma 21 (Splitting Tensor/Cut)**.** In a proof net with no terminal $\parr$-vertex which contains at least one $\otimes$-vertex or *cut*-vertex, there is a terminal $\otimes$-vertex whose premises are not connected by a (undirected) path out of this vertex, or a *cut*-vertex whose premises are not connected by a (undirected) path out of this vertex.

Such a *cut*-vertex or terminal $\otimes$-vertex with no cycle containing it is called **splitting**.

Proof. Note that if there is a $\otimes$-vertex and no *cut*-vertex, there is a terminal $\otimes$-vertex. Indeed the maximal directed path starting from the output edge of the $\otimes$-vertex must reach a conclusion of the proof net (since there is no *cut*-vertex, thus no vertex without output edge). The last vertex it crosses cannot be a $\parr$-vertex thus is a $\otimes$-vertex. Then see for example Bellin (1997); Di Guardia et al. (2025). □

Proposition 22 (Sequentialization)**.** Any proof net is the image of a sequent calculus proof from MLLM.

Proof. We prove the result by induction on the number of vertices of a proof net $\mathscr{R}$. A proof π such that $\pi^\bullet = \mathscr{R}$ is called a **sequentialization** of $\mathscr{R}$.

If $\mathscr{R}$ is empty, it is the image of $\dfrac{}{\vdash}\, mix_0$.

If $\mathscr{R}$ contains more than one connected component, we split it into two non-empty disconnected sub-graphs $\mathscr{R}_1$ and $\mathscr{R}_2$. If we project the order on the conclusions labeled Σ of $\mathscr{R}$ to the conclusions labeled Γ of $\mathscr{R}_1$ and to the conclusions labeled Δ of $\mathscr{R}_2$, we get $\Sigma \in \Gamma \sqcup\!\sqcup \Delta$. Let π_1 and π_2 be sequentializations of $\mathscr{R}_1$ and $\mathscr{R}_2$, $\mathscr{R}$ is the image of the proof obtained by applying a (mix_2) rule to π_1 and π_2 with the shuffle leading to $\vdash \Sigma$.

If $\mathscr{R}$ contains a terminal $\parr$-vertex v with conclusion e labeled $A \parr B$, it has conclusion sequent $\vdash \Gamma, A \parr B, \Delta$ and we consider the graph $\mathscr{R}_1$ obtained by erasing v and e. We order the two

premises e_1 labeled A and e_2 labeled B of v with $e_1 < e_2$ and with all conclusions smaller (resp. bigger) than e, *i.e.* with labels in Γ (resp. Δ), which become smaller than e_1 (resp. bigger than e_2). $\mathscr{R}_1$ is a proof net and we can consider a sequentialization π_1 of $\mathscr{R}_1$ with conclusion $\vdash \Gamma, A, B, \Delta$. $\mathscr{R}$ is the image of the proof obtained by adding to π_1 a ($\parr$) rule introducing $A \parr B$.

If $\mathscr{R}$ contains no terminal $\parr$-vertex, but contains at least one $\otimes$-vertex or *cut*-vertex, we can apply Lemma 21. If we have a terminal splitting $\otimes$-vertex v with conclusion e labeled $A \otimes B$ and premises e_1 and e_2, removing v and e splits $\mathscr{R}$ into two (not uniquely defined in general) disconnected sub-graphs $\mathscr{R}_1$ and $\mathscr{R}_2$. We order the conclusions of $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) by starting from the order on the conclusions of $\mathscr{R}$, by replacing e with e_1 (resp. e_2) and by projecting the order to the conclusions edges of $\mathscr{R}_1$ (resp. $\mathscr{R}_2$). $\mathscr{R}_1$ and $\mathscr{R}_2$ are proof nets and, by induction hypothesis, we get two sequentializations π_1 with conclusion $\vdash \Gamma, A, \Gamma'$ and π_2 with conclusion $\vdash \Delta, B, \Delta'$ of $\mathscr{R}_1$ and $\mathscr{R}_2$. $\pi^\bullet$ is the translation of the following proof:

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2 \\ \vdash \Gamma, A, \Gamma' & \vdash \Delta, B, \Delta'\end{array}}{\vdash \Gamma \sqcup \Delta, A \otimes B, \Gamma' \sqcup \Delta'} \otimes$$

If applying Lemma 21 gives us a splitting *cut*-vertex v with premises e_1 (labeled A) and e_2 (labeled $A^\perp$), we remove v which splits $\mathscr{R}$ into two (not uniquely defined in general) disconnected sub-graphs $\mathscr{R}_1$ and $\mathscr{R}_2$. We order the conclusions of $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) by projecting the order from $\mathscr{R}$ to the conclusions edges of $\mathscr{R}_1$ (resp. $\mathscr{R}_2$). We then add e_1 (resp. e_2) as a least element. By induction hypothesis, we get two sequentializations π_1 with conclusion $\vdash A, \Gamma$ and π_2 with conclusion $\vdash A^\perp, \Delta$ of $\mathscr{R}_1$ and $\mathscr{R}_2$. $\pi^\bullet$ is the translation of the following proof:

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2 \\ \vdash \underline{A}, \Gamma & \vdash \underline{A^\perp}, \Delta\end{array}}{\vdash \Gamma \sqcup \Delta} \mathit{cut}$$

If none of the cases above applies, $\mathscr{R}$ contains only `I`-vertices and *ax*-vertices: there is no terminal $\parr$-vertex and, by Lemma 21, no $\otimes$-vertex and no *cut*-vertex, thus no $\parr$-vertex at all (as all other kinds of possible vertices do not have any input edge, a $\parr$-vertex would lead, by following the maximal directed path from its conclusion, to another $\parr$-vertex source of a conclusion edge). Moreover if we have exactly one connected component, there is only one such `I`- or *ax*-vertex since no vertex has an input edge. In the first case (one `I`-vertex), it is the translation of $\dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}$. In the second case (one *ax*-vertex), the conclusions e_1 and e_2 of v are labeled, respectively, $\bar{X}$ and X (for some X). If $e_1 < e_2$, $\mathscr{R}$ is the translation of $\dfrac{}{\vdash \bar{X}, X}\,ax_l$, while if $e_2 < e_1$, it is the translation of $\dfrac{}{\vdash X, \bar{X}}\,ax_r$. □

We now know what is the image of $(\_)^\bullet$, but the process of computing a sequentialization of a given proof net is highly non-deterministic: choices in the successive selections of vertices and in the decompositions into connected components. We want to characterize the kernel of $(\_)^\bullet$: what is the quotient induced on the sequent calculus through the translation into proof nets? We prove it is exactly $=_{\sigma\rho}$ for cut-free proofs (Bellin (1997)).

Lemma 23 (Correctness of $=_\sigma$)**.** If $\pi_1 =_{\sigma\rho} \pi_2$ then ${\pi_1}^\bullet = {\pi_2}^\bullet$.

Proof. Simple verification for the two equations defining $=_\rho$ and for each equality from Appendix D. □

Lemma 24 (Connected Components)**.** If the connected components of $\pi^\bullet$ are the proof nets $\mathscr{R}_1$, $\ldots$, $\mathscr{R}_k$ then π is equal up to $=_{\sigma\rho}$ to:

$$\dfrac{\begin{array}{ccc}\pi_1 & & \pi_k\\ \vdash \Gamma_1 & \cdots & \vdash \Gamma_k\end{array}}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k}\, mix_k$$

with $\pi_1^\bullet = \mathscr{R}_1, \ldots, \pi_k^\bullet = \mathscr{R}_k$.

Proof. Note that, thanks to the equations on (mix_0) and (mix_2) in $=_{\sigma\rho}$, the exact way of defining (mix_k) (see Lemma 14) does not matter: we can reason up to associativity, commutativity (of (mix_2)) and unitality (of (mix_0)).

We reason by induction on π. If the last rule of π is a (mix) rule, the result is easy by induction hypothesis. Otherwise, we apply the induction hypothesis on the premises of the last rule of π and this last rule must commute up with the obtained (mix) rules which leads to disconnected components. The only non trivial case (see Appendix D) is when the last rule is a $(\parr)$ rule. By induction hypothesis, we get:

$$\dfrac{\dfrac{\begin{array}{ccc}\pi_1 & & \pi_k\\ \vdash \Gamma_1, \Xi_1, \Gamma_1' & \cdots & \vdash \Gamma_k, \Xi_k, \Gamma_k'\end{array}}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k, A, B, \Gamma_1' \sqcup \cdots \sqcup \Gamma_k'}\, mix_k}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k, A \parr B, \Gamma_1' \sqcup \cdots \sqcup \Gamma_k'}\, \parr$$

where the list $\Xi_1 \cdot \cdots \cdot \Xi_k$ is exactly $[A, B]$ (thus at most two of them are not empty).

If A and B belong to the same Ξ_i (then the other Ξ_j are empty), we can commute the rules:

$$\dfrac{\begin{array}{ccccccc}\pi_1 & & & & \pi_i & & \pi_k\\ \vdash \Gamma_1, \Gamma_1' & \cdots & \vdash \Gamma_i, A, B, \Gamma_i' & & \vdash \Gamma_i, A \parr B, \Gamma_i' & \cdots & \vdash \Gamma_k, \Gamma_k'\end{array}}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k, A \parr B, \Gamma_1' \sqcup \cdots \sqcup \Gamma_k'}\, mix_k$$

The connected components of the images of the proofs with and without the $(\parr)$ rule are the same (up to adding the $\parr$-vertex to the ith connected component).

If A and B belong to two distinct Ξ_i and Ξ_j, up to $=_{\sigma\rho}$, we can assume $i = 1$ and $j = 2$ and rearrange the rules in (mix_k) to get:

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdash \Gamma_1, A, \Gamma_1' & \vdash \Gamma_2, B, \Gamma_2'\end{array}}{\vdash \Gamma_1 \sqcup \Gamma_2, A, B, \Gamma_1' \sqcup \Gamma_2'}\, mix_2 \qquad \begin{array}{ccc}\pi_3 & & \pi_k\\ \vdash \Gamma_3, \Gamma_3' & \cdots & \vdash \Gamma_k, \Gamma_k'\end{array}}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k, A, B, \Gamma_1' \sqcup \cdots \sqcup \Gamma_k'}\, mix_{k-1}}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k, A \parr B, \Gamma_1' \sqcup \cdots \sqcup \Gamma_k'}\, \parr$$

We now commute the $(\parr)$ rule up:

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdash \Gamma_1, A, \Gamma_1' & \vdash \Gamma_2, B, \Gamma_2'\end{array}}{\vdash \Gamma_1 \sqcup \Gamma_2, A, B, \Gamma_1' \sqcup \Gamma_2'}\, mix_2}{\vdash \Gamma_1 \sqcup \Gamma_2, A \parr B, \Gamma_1' \sqcup \Gamma_2'}\, \parr \qquad \begin{array}{ccc}\pi_3 & & \pi_k\\ \vdash \Gamma_3, \Gamma_3' & \cdots & \vdash \Gamma_k, \Gamma_k'\end{array}}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k, A \parr B, \Gamma_1' \sqcup \cdots \sqcup \Gamma_k'}\, mix_{k-1}$$

By induction hypothesis, $\pi_1^\bullet, \ldots, \pi_k^\bullet$ correspond to the k connected components of $\pi^\bullet$ after the removal of the $\parr$-vertex v associated with the last rule of π. In $\pi^\bullet$, $\pi_1^\bullet$ and $\pi_2^\bullet$ are connected through the $\parr$-vertex v and we are done with $k-1$ connected components. □

Lemma 25 (Terminal Splitting $\otimes$)**.** If $\pi^\bullet$ is obtained by connecting two disjoint proof nets $\mathscr{R}_1$ and $\mathscr{R}_2$ by means of a terminal $\otimes$-vertex then π is equal up to $=_\sigma$ to:

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdash \Gamma, A, \Gamma' & \vdash \Delta, B, \Delta'\end{array}}{\vdash \Gamma \sqcup \Delta, A \otimes B, \Gamma' \sqcup \Delta'}\otimes$$

with ${\pi_1}^\bullet = \mathscr{R}_1$ and ${\pi_2}^\bullet = \mathscr{R}_2$.

Proof. Let v be the $\otimes$-vertex under consideration. We prove the result by induction on π. We look at the last rule of π. If it is the $(\otimes)$ rule mapped to v, we are done. Otherwise, it can be a $(\otimes)$, (cut), $(\parr)$ or (mix_2) rule:

- $(\otimes)$ (not mapped to v) One of the premises contains the $(\otimes)$ rule mapped to v. We apply the induction hypothesis to this premise. The two $(\otimes)$ rules commute through $=_\sigma$ and we get the requested proof.
- (cut) Very similar to the $(\otimes)$ case.
- $(\parr)$ We apply the induction hypothesis to the premise of the rule. Since the vertex associated with the $(\parr)$ rule and its two premises are all contained in $\mathscr{R}_1$ or all contained in $\mathscr{R}_2$ (otherwise v cannot be splitting in $\pi^\bullet$), the formulas associated with the two premises both belong to π_1 or to π_2 which means that we can commute the $(\parr)$ rule up with the $(\otimes)$ rule up to $=_\sigma$. This is the only tricky case.
- (mix_2) One of the premises contains the $(\otimes)$ rule mapped to v. We apply the induction hypothesis to this premise. The (mix_2) rule commutes up with the $(\otimes)$ rule up to $=_\sigma$ (there are two ways, and we do not care which one is chosen). □

Lemma 26 (Terminal $\parr$)**.** If $\pi^\bullet$ is obtained by adding a terminal $\parr$-vertex to a proof net $\mathscr{R}_1$ then π is equal up to $=_\sigma$ to:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, A, B, \Gamma'\end{array}}{\vdash \Gamma, A \parr B, \Gamma'}\parr$$

with ${\pi_1}^\bullet = \mathscr{R}_1$.

Proof. The $(\parr)$ rule mapped to a terminal vertex commutes down with any rule below it (Appendix D). By moving it at the bottom of π, we get the desired proof. □

Theorem 27 (Completeness of $=_\sigma$)**.** All cut-free proofs with the same image proof net by $(\_)^\bullet$ are equal up to $=_{\sigma\rho}$.

Proof. By induction on the number of vertices of the image (cut-free) proof net $\mathscr{R}$.

If $\mathscr{R}$ is empty, a proof π with image $\mathscr{R}$ has conclusion sequent $\vdash$ and contains no (cut) rule thus is equal to $\dfrac{}{\vdash} mix_0$ up to $=_\rho$ (Lemma 7 adapted to MLLM).

If $\mathscr{R}$ has at least two connected components, and π and π' are two proofs with image $\mathscr{R}$, by Lemma 24, we have:

$$\pi =_{\sigma\rho} \dfrac{\begin{array}{ccc}\pi_1 & & \pi_k\\ \vdash \Gamma_1 & \cdots & \vdash \Gamma_k\end{array}}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k} mix_k \qquad \text{and} \qquad \pi' =_{\sigma\rho} \dfrac{\begin{array}{ccc}\pi'_1 & & \pi'_k\\ \vdash \Gamma_1 & \cdots & \vdash \Gamma_k\end{array}}{\vdash \Gamma_1 \sqcup \cdots \sqcup \Gamma_k} mix_k$$

with $\pi_1^\bullet = \mathscr{R}_1 = \pi_1'^\bullet, \ldots, \pi_k^\bullet = \mathscr{R}_k = \pi_k'^\bullet$. By induction hypothesis applied to $\mathscr{R}_i$, we get $\pi_i =_{\sigma\rho} \pi_i'$ and we conclude.

If $\mathscr{R}$ has exactly one connected component and has a terminal splitting $\otimes$-vertex connecting the proof nets $\mathscr{R}_1$ and $\mathscr{R}_2$, then by Lemma 25:

$$\pi = _\sigma \dfrac{\begin{matrix}\pi_1 & \pi_2\\ \vdash \Gamma, A, \Gamma' & \vdash \Delta, B, \Delta'\end{matrix}}{\vdash \Gamma \sqcup \Delta, A \otimes B, \Gamma' \sqcup \Delta'}\otimes \qquad \text{and} \qquad \pi' = _\sigma \dfrac{\begin{matrix}\pi_1' & \pi_2'\\ \vdash \Gamma, A, \Gamma' & \vdash \Delta, B, \Delta'\end{matrix}}{\vdash \Gamma \sqcup \Delta, A \otimes B, \Gamma' \sqcup \Delta'}\otimes$$

with $\pi_1^\bullet = \mathscr{R}_1 = \pi_1'^\bullet$ and $\pi_2^\bullet = \mathscr{R}_2 = \pi_2'^\bullet$. By induction hypothesis applied to $\mathscr{R}_i$, we get $\pi_i =_{\sigma\rho} \pi_i'$ and we conclude.

If $\mathscr{R}$ has exactly one connected component and has a terminal $\parr$-vertex (with $\mathscr{R}_1$ the proof net obtained by removing this $\parr$-vertex), then by Lemma 26:

$$\pi = _\sigma \dfrac{\begin{matrix}\pi_1\\ \vdash \Gamma, A, B, \Gamma'\end{matrix}}{\vdash \Gamma, A \parr B, \Gamma'}\parr \qquad \text{and} \qquad \pi' = _\sigma \dfrac{\begin{matrix}\pi_1'\\ \vdash \Gamma, A, B, \Gamma'\end{matrix}}{\vdash \Gamma, A \parr B, \Gamma'}\parr$$

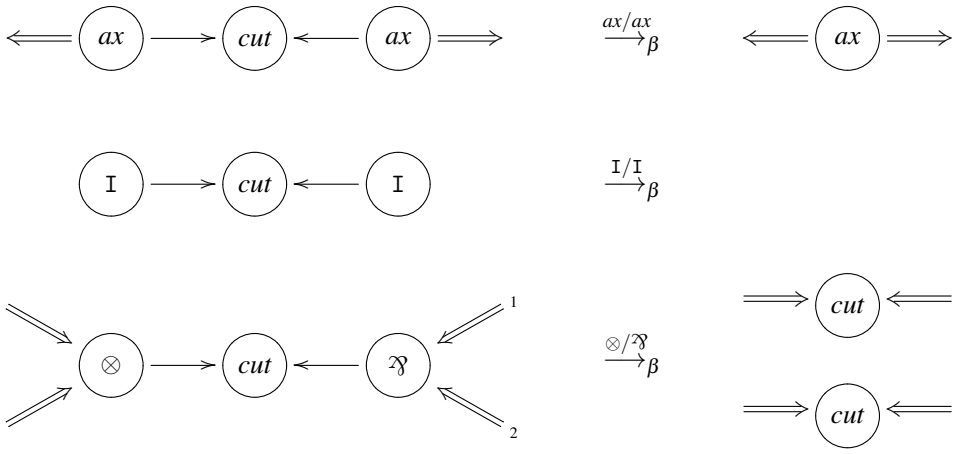
with $\pi_1^\bullet = \mathscr{R}_1 = \pi_1'^\bullet$. By induction hypothesis applied to $\mathscr{R}_1$, we get $\pi_1 =_{\sigma\rho} \pi_1'$ and we conclude.

If $\mathscr{R}$ has exactly one connected component with no terminal splitting $\otimes$- or terminal $\parr$-vertex, then it is reduced to one *ax*-vertex or to one $\mathtt{I}$-vertex (see proof of Proposition 22). The only proofs with such an associated proof net are those built from an (*ax*) rule (resp. ($\mathtt{I}$) rule) by adding (mix_0) and (mix_2) rules which can be erased up to $=_\rho$ (Lemma 16) leading to $\dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}$, $\dfrac{}{\vdash \bar{X}, X}\,ax_l$ or $\dfrac{}{\vdash X, \bar{X}}\,ax_r$. In the last two cases, only one can be mapped to $\mathscr{R}$: the one with appropriate conclusion sequent. □

From Lemma 23 and Theorem 27, we obtain a linear-time decision procedure for testing equality of cut-free proofs up to $=_{\sigma\rho}$, by simply comparing the proof nets (starting from the (ordered) conclusions). This is very close to the situation for MLL without units, but very different from the case with $\bot$ (Heijltjes and Houston (2016)).

2.2 Cut Elimination

A cut elimination procedure is defined on proof nets by means of the following graph rewriting steps:



Let us comment on these rewriting steps. In the $\mathtt{I}/\mathtt{I}$ step, the three vertices and two edges are replaced by an empty sub-graph (*i.e.* erased). Indices 1 and 2 in the $\otimes/\parr$ step denote the ordering

of premises of the $\otimes$- and $\parr$-vertices. Double-arrow edges are preserved and keep their label unchanged. The conclusions of the proof net cannot be modified (only "internal" edges are erased) thus the order on conclusions is preserved.

The new vertices have appropriate numbers of input and output edges, with appropriate labels. This comes from Remark 20 and the fact that since the two input edges of *cut*-vertices have dual labels, they must be unordered pairs X and $\bar{X}$, or $\mathtt{I}$ and $\mathtt{I}$, or $A \otimes B$ and $A^\perp \parr B^\perp$ (with first (resp. second) premise of the associated $\otimes$-vertex labeled A (resp. B) and the first (resp. second) premise of the associated $\parr$-vertex labeled $A^\perp$ (resp. $B^\perp$)). This also shows that any *cut*-vertex in a proof net must belong to exactly one of the above reduction patterns (thanks to the Danos-Regnier condition, the premises of a *cut*-vertex cannot be conclusions of the same *ax*-vertex).

Finally if the reduct of a proof net contains an undirected cycle not containing the two premises of any $\parr$-vertex, then there is also such a cycle in the starting proof net. It is immediate for the ax/ax (which is just shortening some paths) and $\mathtt{I}/\mathtt{I}$ steps (which does not touch any cycle). For a $\otimes/\parr$ step, a cycle in the reduct going through one or none of the *cut*-vertices is easily mapped to a similar cycle in the starting proof net. If a cycle uses both *cut*-vertices, we can map it to a cycle connecting the two premises of the $\otimes$-vertex in the starting proof net.

By existence of a reduction step for each *cut*-vertex, a normal form is cut-free. By uniqueness of the reduction step in which a given *cut*-vertex can be involved, we can prove the rewriting system is strongly confluent. Indeed, using the Danos-Regnier condition, the only possible overlapping redexes are:

and this proof net reduces in both ways to the same one. Moreover the system is strongly normalizing since any step makes the number of vertices of the proof net strictly decrease.

This means any proof net $\mathscr{R}$ can be turned into a unique cut-free one $\mathbf{NF}(\mathscr{R})$ by applying the rewriting steps until reaching a normal form.

Lemma 28 (Simulation)**.** Cut elimination of proof nets simulates cut elimination in the sequent calculus:

$$\pi_1 \to_\beta \pi_2 \quad \Longrightarrow \quad {\pi_1}^\bullet \to_\beta^\star {\pi_2}^\bullet$$

Proof. Simple verification for each step from Appendices C.1 and C.3. More precisely, key steps are mapped to one reduction step in proof nets, and the source and target of commutation steps are mapped to the same proof net. This means in particular that the length of the reduction on the proof-net side is smaller or equal to the length on the sequent-calculus side. □

For any formula A, we denote by $\boldsymbol{id}_A^\bullet$ the proof net associated with the proof id_A (Lemma 13). Given two proof nets $\mathscr{R}_1$ with conclusion sequent $\vdash A, B$ and $\mathscr{R}_2$ with conclusion sequent $\vdash B^\perp, C$, $\boldsymbol{cut}_B^\bullet(\mathscr{R}_1, \mathscr{R}_2)$ is the proof net with conclusion sequent $\vdash A, C$ obtained by considering the disjoint union of $\mathscr{R}_1$ and $\mathscr{R}_2$ and by adding a *cut*-vertex with premises the conclusion of $\mathscr{R}_1$ labeled B and the conclusion of $\mathscr{R}_2$ labeled $B^\perp$.

Lemma 29 (Identity Cut)**.** Given a proof net $\mathscr{R}$ with conclusion sequent $\vdash A^\perp, B$:

$$cut_A^\bullet(id_A^\bullet, \mathscr{R}) \to_\beta^\star \mathscr{R} \qquad \text{and} \qquad cut_B^\bullet(\mathscr{R}, id_B^\bullet) \to_\beta^\star \mathscr{R}$$

Proof. Simple induction on A (resp. B), using the structure of $id_A^\bullet$ (resp. $id_B^\bullet$). □

3. Properties of MLLM from Proof Nets

We now take advantage of the proof net syntax and of the links between proof nets and the sequent calculus to prove properties about MLLM. Let us sum up the key ingredients we already have:

- id_A is mapped to $id_A^\bullet$ (by definition of $id_A^\bullet$)
- cut-free proofs are mapped to cut-free proof nets (by definition of $(\_)^\bullet$)
- if $\mathscr{R}_1$ and $\mathscr{R}_2$ are cut-free and $\mathscr{R}_1 =_\beta \mathscr{R}_2$ then $\mathscr{R}_1 = \mathscr{R}_2$ (confluence of $\to_\beta$ for proof nets)
- if $\pi_1 =_\beta \pi_2$ then $\pi_1{}^\bullet =_\beta \pi_2{}^\bullet$ (Lemma 28)
- if $\pi_1 =_{\sigma\rho} \pi_2$ then $\pi_1 =_{\beta\rho} \pi_2$ (Lemma 18)
- if π_1 and π_2 are cut-free proofs, $\pi_1 =_{\sigma\rho} \pi_2$ if and only if $\pi_1{}^\bullet = \pi_2{}^\bullet$ (Lemma 23 and Theorem 27)

3.1 Properties of Cut Elimination

We develop a few additional properties.

Lemma 30 (Cut-Free Proofs Equality)**.** If π_1 and π_2 are cut-free proofs, $\pi_1 =_{\sigma\rho} \pi_2$ if and only if $\pi_1 =_{\beta\rho} \pi_2$ if and only if $\pi_1{}^\bullet = \pi_2{}^\bullet$.

Proof. If $\pi_1 =_{\sigma\rho} \pi_2$ then $\pi_1 =_{\beta\rho} \pi_2$ by Lemma 18. If $\pi_1 =_{\beta\rho} \pi_2$ then, by Lemma 28, $\pi_1{}^\bullet =_\beta \pi_2{}^\bullet$, and these proof nets are cut-free thus $\pi_1{}^\bullet = \pi_2{}^\bullet$. If $\pi_1{}^\bullet = \pi_2{}^\bullet$ then, by Theorem 27, $\pi_1 =_{\sigma\rho} \pi_2$. □

As already seen in Lemma 18, cut elimination is not confluent in the sequent calculus. Here is another (the smallest) example from MLL:

$$\dfrac{\dfrac{\dfrac{}{\vdash \bar{X}, X}\,ax_l \quad \dfrac{}{\vdash Y, \bar{Y}}\,ax_r}{\vdash \bar{X}, X \otimes Y, \underline{\bar{Y}}}\otimes \quad \dfrac{\dfrac{}{\vdash Y, \bar{Y}}\,ax_r \quad \dfrac{}{\vdash \bar{Z}, Z}\,ax_l}{\vdash \underline{Y}, \bar{Y} \otimes \bar{Z}, Z}\otimes}{\vdash \bar{X}, X \otimes Y, \bar{Y} \otimes \bar{Z}, Z}\,cut$$

reduces to both

$$\dfrac{\dfrac{}{\vdash \bar{X}, X}\,ax_l \quad \dfrac{\dfrac{}{\vdash Y, \bar{Y}}\,ax_r \quad \dfrac{}{\vdash \bar{Z}, Z}\,ax_l}{\vdash Y, \bar{Y} \otimes \bar{Z}, Z}\otimes}{\vdash \bar{X}, X \otimes Y, \bar{Y} \otimes \bar{Z}, Z}\otimes \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \bar{X}, X}\,ax_l \quad \dfrac{}{\vdash Y, \bar{Y}}\,ax_r}{\vdash \bar{X}, X \otimes Y, \bar{Y}}\otimes \quad \dfrac{}{\vdash \bar{Z}, Z}\,ax_l}{\vdash \bar{X}, X \otimes Y, \bar{Y} \otimes \bar{Z}, Z}\otimes$$

This default of confluence can be evaluated: normal forms differ only up to permutations of rules.

Lemma 31 (Confluence up to $=_{\sigma\rho}$)**.** If π_1 and π_2 are two normal forms for $\to_\beta$ of a given proof π of MLLM, then $\pi_1 =_{\sigma\rho} \pi_2$.

Proof. π_1 and π_2 are cut-free and $\pi_1 =_\beta \pi_2$ thus, by Lemma 30, $\pi_1 =_{\sigma\rho} \pi_2$. □

Lemma 32 (Two-Way Simulation)**.** $\pi_1 =_{\beta\rho} \pi_2$ if and only if $\pi_1{}^\bullet =_\beta \pi_2{}^\bullet$.

Proof. The left-to-right direction comes from Lemma 28.

For the reverse direction, let π_1' be a normal form of π_1 and π_2' be a normal form of π_2 for $\to_\beta$. We have $\pi_1'{}^\bullet =_\beta \pi_1{}^\bullet =_\beta \pi_2{}^\bullet =_\beta \pi_2'{}^\bullet$ (by assumption and Lemma 28). But $\pi_1'{}^\bullet$ and $\pi_2'{}^\bullet$ are cut-free thus $\pi_1'{}^\bullet = \pi_2'{}^\bullet$. Then by Lemma 30, $\pi_1 =_{\beta\rho} \pi_1' =_{\beta\rho} \pi_2' =_{\beta\rho} \pi_2$. □

3.2 Unit Elimination

Thanks to $\mathtt{I}^\perp = \mathtt{I}$, $\mathtt{I}$ is the unit of both $\otimes$ and $\parr$ in MLLM. We consider the following rewriting system on formulas:

$$A \otimes \mathtt{I} \mapsto A \qquad \mathtt{I} \otimes A \mapsto A$$
$$A \parr \mathtt{I} \mapsto A \qquad \mathtt{I} \parr A \mapsto A$$

which can be easily proved to be strongly normalizing (the size of formulas strictly decreases) and strongly confluent (the critical pairs are trivial: $\mathtt{I} \otimes \mathtt{I} \mapsto \mathtt{I}$ and $\mathtt{I} \parr \mathtt{I} \mapsto \mathtt{I}$). Given a formula A, we denote by A^u its unique normal form with respect to the rewriting system above.

Lemma 33 (Unit-Free Normal Form)**.** For any A, either A^u is unit-free (*i.e.* contains no $\mathtt{I}$) or $A^u = \mathtt{I}$.

Proof. By induction on A. If A is X, $\bar{X}$ or $\mathtt{I}$, $A^u = A$ and the result is immediate. If $A = B \otimes C$ (resp. $A = B \parr C$), we apply the induction hypothesis to B and C. If B^u and C^u are unit-free, then $A^u = B^u \otimes C^u$ (resp. $A^u = B^u \parr C^u$) is unit-free as well. If $B^u = \mathtt{I}$ then $A^u = (B^u \otimes C^u)^u = (\mathtt{I} \otimes C^u)^u = C^u$ (resp. $A^u = (B^u \parr C^u)^u = (\mathtt{I} \parr C^u)^u = C^u$) is unit-free or equal to $\mathtt{I}$. If $C^u = \mathtt{I}$, the result comes in the same way. □

Lemma 34 (Unit-Elimination Isomorphism)**.** For any A, $A \cong A^u$.

Proof. By using the following two proof schemes, it is easy to define the isomorphisms corresponding to the four equations defining $(\_)^u$:

$$\dfrac{\dfrac{id_A}{\vdash A^\perp, A} \qquad \dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}}{\vdash A^\perp, A \otimes \mathtt{I}}\,\otimes \qquad\qquad \dfrac{\dfrac{\dfrac{id_A}{\vdash A^\perp, A} \qquad \dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}}{\vdash A^\perp, A, \mathtt{I}}\,mix_2}{\vdash A^\perp, A \parr \mathtt{I}}\,\parr$$

□

Using Lemma 34, it is possible to transform any proof π of $\vdash \Gamma$ into a proof π^u of $\vdash \Gamma^u$ by introducing the appropriate cuts on formulas of Γ with the proofs of the associated isomorphisms. Let Γ_0 be the list of formulas A of Γ such that $A^u \neq \mathtt{I}$, so that $\Gamma^u \in {\Gamma_0}^u \sqcup [\mathtt{I}, \ldots, \mathtt{I}]$ (with ${\Gamma_0}^u$ unit-free). We have:

$$\pi^u \quad =_{\beta\rho} \quad \dfrac{\dfrac{\dfrac{\pi_0}{\vdash {\Gamma_0}^u} \qquad \dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}}{\vdash {\Gamma_0}^u \sqcup \mathtt{I}}\,mix_2 \qquad \dfrac{\vdots \qquad\qquad \dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}}{\vdash {\Gamma_0}^u \sqcup \mathtt{I} \sqcup \cdots \sqcup \mathtt{I}}\,mix_2}$$

with π_0 containing no $\mathtt{I}$.

We can prove this through proof nets: $\mathscr{R} = \mathrm{NF}((\pi^u)^\bullet)$ is a proof net with conclusion edges labeled with the formulas ${\Gamma_0}^u, \mathtt{I}, \cdots, \mathtt{I}$. Each conclusion labeled $\mathtt{I}$ is the conclusion of a $\mathtt{I}$-vertex which constitutes a connected component of $\mathscr{R}$. Let $\mathscr{R}_0$ be the sub-graph of $\mathscr{R}$ obtained by removing all these disconnected $\mathtt{I}$-vertices. Let π_0 be a sequentialization of $\mathscr{R}_0$, the proof

$$\dfrac{\dfrac{\pi_0}{\vdash \Gamma_0{}^u} \quad \dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}}{\vdash \Gamma_0{}^u \amalg \mathtt{I}}\,mix_2 \qquad \dfrac{\dfrac{\vdots}{\vdash \Gamma_0{}^u \amalg \mathtt{I} \amalg \cdots \amalg \mathtt{I}} \quad \dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}}{}\,mix_2$$

translates by $(\_)^\bullet$ into $\mathscr{R}$ thus π^u is equal to it up to $=_{\beta\rho}$ (Lemma 32). Now $\mathscr{R}_0$ is cut-free and has no sub-formula $\mathtt{I}$ in its conclusion sequent $\vdash \Gamma_0{}^u$, thus it does not contain any $\mathtt{I}$-vertex. This means π_0 contains no $\mathtt{I}$.

Before moving to the general setting of categorical models, we can mention that both the relational model and the coherent model of linear logic are models of MLLM since $[\![\bot]\!] = [\![1]\!]$ (which moreover is a strict unit for $\otimes$ and $\parr$ in these models) so we can choose $[\![\mathtt{I}]\!] = [\![\bot]\!] = [\![1]\!]$. Injectivity and full completeness for the coherent semantics as in Pagani (2007) can be easily extended from unit-free MLL with (*mix*) to MLLM thanks to unit elimination: a proof of $\vdash \Gamma$ is interpreted as a clique of $[\![\Gamma_0{}^u]\!]$ (with Γ_0 defined as above and unit-free).

It is also possible to replay the relation between coherence and acyclicity (Retoré (1997)) in the context of MLLM, since $\mathtt{I}$ has no impact.

4. Categorical Semantics

The standard notion of categorical model of intuitionistic multiplicative linear logic is given by *symmetric monoidal closed categories* (SMCC) (see for example Blute and Scott (2004)) with constructors $\otimes$, 1 and $\multimap$ (we use the same notations as for the corresponding connectives of IMLL). For classical multiplicative linear logic, one then moves to *star-autonomous categories* (Seely (1989)): SMCC with a distinguished object $\bot$ subject to a canonical isomorphism $(A \multimap \bot) \multimap \bot \simeq A$. Towards modelling the (*mix*) rules, we study conditions on such categories leading to an isomorphism $\bot \simeq 1$.

4.1 Internal Magmas and Monoids

We recall how some algebraic structures can be represented as objects in (symmetric) monoidal categories.

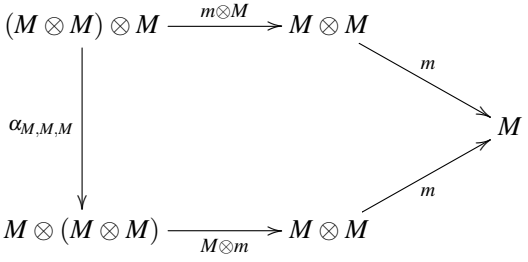
Definition 35 (Internal Magma and Monoid)**.** Given an object M in a monoidal category,

- an **internal magma** is a pair $(M, M \otimes M \xrightarrow{m} M)$;
- an **internal unitary** (resp. left-unitary, resp. right-unitary) **magma** is a triple $(M, M \otimes M \xrightarrow{m} M, 1 \xrightarrow{n} M)$ such that the two (resp. first, resp. second) diagrams below:

$$\begin{array}{ccc} 1 \otimes M & \xrightarrow{n \otimes M} & M \otimes M \\ & \searrow^{\lambda_M} & \downarrow m \\ & & M \end{array} \qquad \begin{array}{ccc} M \otimes 1 & \xrightarrow{M \otimes n} & M \otimes M \\ & \searrow^{\rho_M} & \downarrow m \\ & & M \end{array}$$

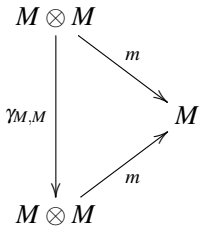
commute;

- an **internal monoid** is an internal unitary magma such that:



commutes.

When the monoidal category is *symmetric*, an internal magma (or monoid) is **commutative** if:



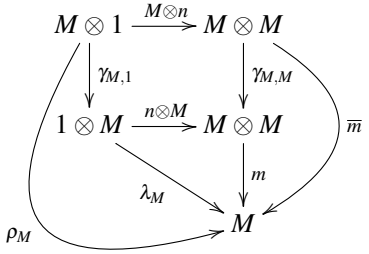
commutes.

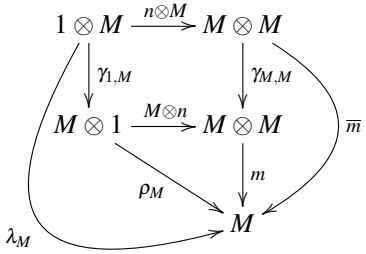
Example 36. A terminal object $\top$ in a (symmetric) monoidal category is an internal (commutative) monoid: m and n are the unique morphisms into $\top$, and the diagrams immediately commute since they all have $\top$ as target.

In any (symmetric) monoidal category, the tensor unit 1 is an internal (commutative) monoid with: $m := \lambda_1 = \rho_1 : 1 \otimes 1 \longrightarrow 1$ and $n := id_1 : 1 \longrightarrow 1$. The diagrams commute thanks to Mac Lane's theorem and the definition of symmetry in a monoidal category.

In a *symmetric* monoidal category, the tensor product of two internal (commutative) monoids is an internal (commutative) monoid (see for example (Melliès, 2009, Section 6.2)).

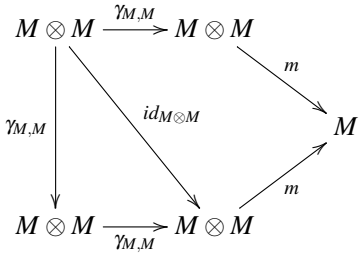
In a *symmetric* monoidal category, given an internal unitary (resp. left-unitary, resp. right-unitary) magma $(M, M \otimes M \xrightarrow{m} M, 1 \xrightarrow{n} M)$, we can define its **symmetric** $(M, M \otimes M \xrightarrow{\overline{m}=\gamma_{M,M};m} M, 1 \xrightarrow{n} M)$ which is an internal unitary (resp. right-unitary, resp. left-unitary) magma. Indeed we have:





by properties of symmetric monoidal categories.

$(M, M\otimes M \xrightarrow{m} M, 1 \xrightarrow{n} M)$ is commutative when it is equal to its symmetric, and then $(M, M\otimes M \xrightarrow{\overline{m}} M, 1 \xrightarrow{n} M)$ is commutative as well (*i.e.* $\overline{\overline{m}} = \overline{m}$):



4.2 The Intuitionistic / Monoidal Case

Let us consider a symmetric monoidal closed category $\mathscr{C}$. Given a valuation ν from the set of atoms $\mathscr{X}$ to objects of $\mathscr{C}$, we can interpret each $\mathsf{IMLL}_{mix(\mathtt{R})}$ formula A as an object $[\![A]\!]$ of $\mathscr{C}$ and then, assuming $R = [\![\mathtt{R}]\!]$ is an *internal unitary magma*, we can also interpret each proof of $\Gamma \vdash A$ in $\mathsf{IMLL}_{mix(\mathtt{R})}$ as a morphism from $[\![\bigotimes \Gamma]\!] := \bigotimes_{C\in\Gamma}[\![C]\!]$ to $[\![A]\!]$ in $\mathscr{C}$. One should be more precise on what $\bigotimes_{C\in\Gamma}[\![C]\!]$ means but, up to Mac Lane's coherence results for symmetric monoidal categories (Mac Lane, 1998, Theorem XI.1), any two choices of ordering (and associating) elements of Γ before gluing them with $\otimes$ would lead to isomorphic objects with a unique canonical isomorphism relating them. In the same spirit, given two lists Γ and Δ and an element Σ of $\Gamma \sqcup\!\sqcup \Delta$, by identifying exactly which element of Γ and Δ is mapped to each element of Σ, one defines a unique canonical isomorphism from $\bigotimes \Sigma$ to $\bigotimes \Gamma \otimes \bigotimes \Delta$.

The rules of $\mathsf{IMLL}_{mix(\mathtt{R})}$ coming from IMLL can be interpreted in $\mathscr{C}$ in the usual way for an SMCC (see for example Blute and Scott (2004) and (Melliès, 2009, Chapter 2)). We describe how to interpret the two additional rules by using the unitary magma structure of R:

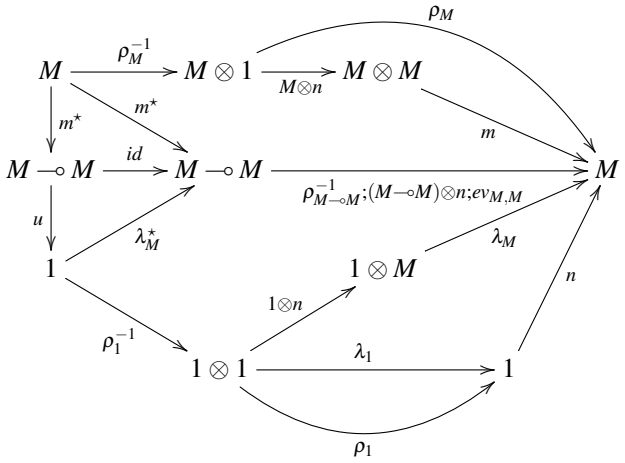
$$\frac{}{\vdash \mathtt{R}}\,\mathtt{R} \quad\mapsto\quad 1 \xrightarrow{n} R$$

$$\frac{\pi_1 \qquad \pi_2}{\Gamma\vdash \mathtt{R} \qquad \Gamma\vdash \mathtt{R}}\,mix_2 \quad\mapsto\quad \bigotimes(\Gamma \sqcup\!\sqcup \Delta) \xrightarrow{\sim} \bigotimes\Gamma\otimes\bigotimes\Delta \xrightarrow{[\![\pi_1]\!]\otimes[\![\pi_2]\!]} R\otimes R \xrightarrow{m} R$$

In this general setting (an SMCC with a selected internal unitary magma R), there can be a lot of morphisms into R. We want to restrict to the case where R is "small enough". In a monoidal category $\mathscr{C}$, elements of the hom-set $\mathscr{C}(1,1)$ are known as **scalars**. We say that an object S of $\mathscr{C}$ is **endo-scalar** if $\lambda^\star_S : 1 \longrightarrow S \multimap S$ has an inverse u. The object 1 is always endo-scalar. If S is an endo-scalar object, there is a bijection between $\mathscr{C}(1,1)$ (scalars) and $\mathscr{C}(S,S)$: $\mathscr{C}(1,1) \simeq \mathscr{C}(1, S\multimap S) \simeq \mathscr{C}(S,S)$.

Lemma 37. If $(M, M\otimes M \xrightarrow{m} M, 1 \xrightarrow{n} M)$ is an internal right-unitary magma such that $\lambda^\star_M : 1 \longrightarrow M \multimap M$ has a left inverse u (*i.e.* $u\,;\lambda^\star_M = id_{M\multimap M}$), then $m^\star\,;u\,;n = id_M$.

Proof.

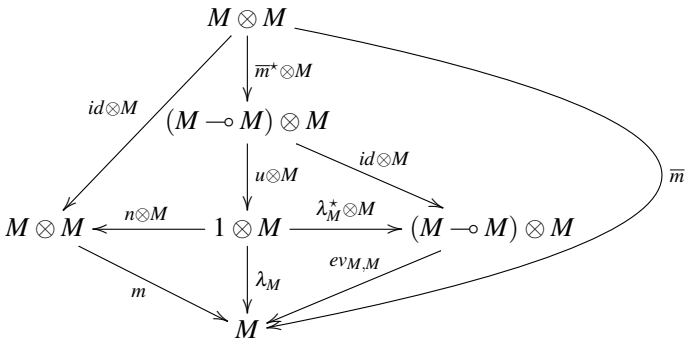


The two pentagons are Lemma 53 (Appendix E). The square is naturality of λ. The top right triangle is the definition of an internal right-unitary magma. The left bottom triangle is the assumption $u\,;\lambda^\star_M = id_{M\multimap M}$. By properties of inverses, we get the identity on M on the top side of the diagram and $m^\star\,;u\,;n$ on the bottom side. □

Lemma 38. If $(M, M\otimes M \xrightarrow{m} M, 1 \xrightarrow{n} M)$ is an internal left-unitary (resp. right-unitary) magma such that $\lambda^\star_M : 1 \longrightarrow M \multimap M$ has a left inverse u, then $(M, M\otimes M \xrightarrow{m} M, 1 \xrightarrow{n} M)$ is an internal commutative unitary magma.

Proof. Without loss of generality, we consider only the case of an internal left-unitary magma since it covers the right-unitary case by moving from (M, m, n) to $(M, \overline{m}, n)$.

We prove $\overline{m} = m$:



The left top triangle is Lemma 37 applied to the internal right-unitary magma $(M, \overline{m}, n)$.

The left bottom triangle comes from the definition of an internal left-unitary magma.

The central triangle is the fact that u is left inverse to $\lambda^\star_M$.

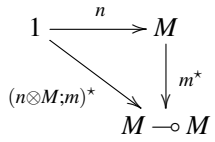
The other commutations come from properties of symmetric monoidal closed categories.

Now (M, m, n) is left-unitary thus $(M, \overline{m}, n)$ is right-unitary and we conclude that (M, m, n) is unitary since $\overline{m} = m$. □

We can now prove a categorical analogue of Theorem 10.

Proposition 39 (Endo-Scalar Unitary Magma)**.** If an internal left-unitary (resp. right-unitary) magma (M, m, n) is endo-scalar then n is an isomorphism.

Proof. First, by Lemma 38, (M, m, n) is an internal unitary magma.
Then we prove that $m^\star ; u : M \longrightarrow 1$ is an inverse of n.
We know, by Lemma 37, that $m^\star ; u ; n = id_M$.
Then, the following diagram commutes by property of curryfication:



and we thus have $n ; m^\star ; u = (n \otimes M ; m)^\star ; u = \lambda_M^\star ; u = id_1$ since M is an internal unitary magma and M is endo-scalar. □

4.3 The Classical / Star-Autonomous Case

We now look at the star-autonomous case, towards categorical models of MLLM.

Lemma 40 ($\perp$ is Endo-Scalar)**.** In a star-autonomous category, $\perp$ is an endo-scalar object.

Proof. In any star-autonomous category:

- $\rho_A^\star : A \longrightarrow 1 \multimap A$ is an isomorphism, and then $\rho_A^\star \multimap \perp : (1 \multimap A) \multimap \perp \longrightarrow A \multimap \perp$ is an isomorphism since $\_ \multimap \perp$ is a (contravariant) functor (by properties of symmetric monoidal closed categories);
- $(\gamma_{1,1\multimap\perp} ; ev_{1,\perp})^\star : 1 \longrightarrow (1 \multimap \perp) \multimap \perp$ is an isomorphism (by definition of star-autonomous categories).

The composition $(\gamma_{1,1\multimap\perp} ; ev_{1,\perp})^\star ; \rho_\perp^\star \multimap \perp$ is then also an isomorphism. By properties of symmetric monoidal closed categories, $(\gamma_{1,1\multimap\perp} ; ev_{1,\perp})^\star ; \rho_\perp^\star \multimap \perp = (1 \otimes \rho_\perp^\star ; \gamma_{1,1\multimap\perp} ; ev_{1,\perp})^\star : 1 \longrightarrow \perp \multimap \perp$ and we conclude, by Lemma 52 (Appendix E), that $\lambda_\perp^\star$ is an isomorphism. □

Definition 41 (Star-Autonomous Categories with Self-Dual Unit)**.** A **star-autonomous category with self-dual unit** is a star-autonomous category such that $\perp \simeq 1$.

This is the natural notion of categorical model of MLLM (see Cockett and Seely (1997*a*) for the very similar negation-free setting). Now our Proposition 39 provides a sufficient condition for a star-autonomous category to have a self-dual unit.

Proposition 42 (Unitary Magma on $\perp$)**.** A star-autonomous category in which $\perp$ comes with a structure of internal left-unitary (resp. right-unitary) magma is a star-autonomous category with self-dual unit.

Proof. By Lemma 40 and Proposition 39. □

The converse of Proposition 42 also holds since 1 is an internal commutative monoid in any symmetric monoidal category (Example 36). By composing the two results, we see that if $\perp$ is an internal left-unitary (resp. right-unitary) magma then it must be in fact an internal commutative monoid.

4.4 Proofs as Morphisms: Syntactic Categories

Following the Curry-Howard-Lambek correspondence (Lambek and Scott (1988)), it is common practice to consider proofs as morphisms to build a category from a proof system. Regarding (fragments of) classical linear logic, the idea is to consider a category with:

- linear logic formulas as objects;
- linear logic proofs of $\vdash A^\perp, B$ as morphisms from A to B;
- the axiom proof $\dfrac{}{\vdash A^\perp, A}\, id_A$ as the identity morphism from A to A;
- the cut of two proofs $\dfrac{\vdash A^\perp, B \qquad \vdash B^\perp, C}{\vdash A^\perp, C}\, cut$ as their composition.

However for this to define a category, one needs to quotient proofs otherwise, for example, the identity is not a unit of composition:

$$\dfrac{\dfrac{}{\vdash A^\perp, B}\,\pi \qquad \dfrac{}{\vdash B^\perp, B}\, id_B}{\vdash A^\perp, B}\, cut \quad \neq \quad \dfrac{}{\vdash A^\perp, B}\,\pi$$

The standard such equational theory on proofs is $=_{\beta\eta}$ which equates proofs up to cut-elimination steps (see Appendix C) and axiom expansions (see Appendix B). Since we restrict here axiom rules to atoms, we can ignore the axiom-expansion equations (see also Di Guardia and Laurent (2025)).

The induced notion of isomorphic objects $\simeq$ coming from category theory corresponds here to the syntactic notion of isomorphism $\cong$ introduced in Section 1.1, which is also referred to as *type isomorphisms* (Di Cosmo (1995)) (by analogy with the λ-calculus through the Curry-Howard-Lambek correspondence).

In the present work, we focus on one-sided presentations of the logics which come with negation as an involution up to *equality*: $A^{\perp\perp} = A$. For this reason we want to replace some isomorphisms with equalities in the definitions of categorical models of linear logic. Considering star-autonomous categories in which $(A \multimap \perp) \multimap \perp = A$ is natural in the context of proof nets (see for example Heijltjes and Straßburger (2016)). This additional constraint is studied and discussed in Cockett et al. (2006).

However this does not perfectly fit our goal of using linear logic formulas as objects of such a category. If $A \multimap B$ is defined as $A^\perp \parr B$ then $(A \multimap \perp) \multimap \perp = (A \otimes 1) \parr \perp \neq A$. For this reason let us first mention an alternative definition of star-autonomous categories: a symmetric monoidal category $\mathscr{C}$ equipped with an equivalence $(\_)^\perp : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{C}$ such that $(A \otimes (\_)^\perp)^\perp$ is a right adjoint (thus denoted $A \multimap \_$) to $\_ \otimes A$ for each object A (thus providing a SMCC structure). We call **star-autonomous categories with strict involution** such a category in which $A^{\perp\perp} = A$ and the canonical morphism $\rho_A^\star ; (\lambda_{A^\perp}^{-1})^\perp : A \longrightarrow A^{\perp\perp}$ is the identity (see also Lamarche and Straßburger (2006)). When considering units such that $1^\perp = \perp \simeq 1$, we can then naturally restrict to the strict case where $1^\perp = 1$.

Definition 43 (Star-Autonomous Categories with Strict Self-Dual Unit)**.** A **star-autonomous category with strict self-dual unit** (SACSSDU) is a monoidal category with an equivalence $(\_)^\perp : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{C}$ such that, for any A:

- $(A \otimes (\_)^\perp)^\perp$ is a right adjoint to $\_ \otimes A$
- $A^{\perp\perp} = A$ and $\rho_A^\star ; (\lambda_{A^\perp}^{-1})^\perp = id_A : A \longrightarrow A$
- $1^\perp = 1$

i.e. a star-autonomous categories with strict involution such that $1^\perp = 1$ (and the selected isomorphism $1^\perp \simeq 1$ is id_1).

We can rely on our two notions of syntactic proofs (proofs in the sequent calculus and proof nets) to define such syntactic categories.

Definition 44 (Category of Proof Nets)**.** Given a set of atoms $\mathscr{X}$, we define the category $\mathscr{P}_\mathscr{X}$ with:

- objects are MLLM formulas built from $\mathscr{X}$;
- morphisms from A to B are cut-free proof nets with conclusion sequent $\vdash A^\perp, B$;
- the identity morphism on A is $id^\bullet_A$;
- composition of $\mathscr{R}_1$ and $\mathscr{R}_2$ with conclusion sequents $\vdash A^\perp, B$ and $\vdash B^\perp, C$ is $\mathrm{NF}(cut^\bullet_B(\mathscr{R}_1, \mathscr{R}_2))$.

Proposition 45 (Structure of $\mathscr{P}_\mathscr{X}$)**.** $\mathscr{P}_\mathscr{X}$ is a star-autonomous category with strict self-dual unit.

Proof. It is easily seen on objects that negation $(\_)^\perp$ is a strict involution. We use $\mathtt{I}$ as unit for $\otimes$ and we have $\mathtt{I}^\perp = \mathtt{I}$. It remains to prove that $(\mathscr{P}_\mathscr{X}, \_ \otimes \_, \mathtt{I}, (\_ \otimes (\_)^\perp)^\perp = (\_)^\perp \parr \_)$ is an SMCC and to compute $\rho^\star_A \,;\, (\lambda^{-1}_{A^\perp})^\perp$.

The morphisms λ_X, ρ_X, $\alpha_{X,Y,Z}$, $\gamma_{X,Y}$ (X, Y and Z atoms) are obtained as the unique cut-free proof nets (see Remark 20) with conclusion sequents: $\vdash \mathtt{I} \parr \bar{X}, X$ and $\vdash \bar{X} \parr \mathtt{I}, X$ and $\vdash (\bar{X} \parr \bar{Y}) \parr \bar{Z}, X \otimes (Y \otimes Z)$ and $\vdash \bar{X} \parr \bar{Y}, Y \otimes X$. The general cases λ_A, ρ_A, $\alpha_{A,B,C}$, $\gamma_{A,B}$ are obtained from the atomic cases by replacing ax-vertices by $id^\bullet$ proof nets. Simple computations show that all the coherence conditions for symmetric monoidal categories are satisfied.

The curryfication operation is obtained by transforming a proof net with conclusion sequent $\vdash A \parr B, C$ into a proof net with conclusion sequent $\vdash A, B \parr C$ (this is done by removing the appropriate $\parr$-vertex in the starting proof net and by adding a new one with premises the second and third conclusions of the obtained proof net). Again one can easily check that the necessary equalities hold.

Let us now compute $\rho^\star_A \,;\, (\lambda^{-1}_{A^\perp})^\perp$. $\rho^\star_A$ with conclusion sequent $\vdash A^\perp, \mathtt{I} \parr A$ is $id^\bullet_A$ together with a $\mathtt{I}$-vertex and a $\parr$-vertex. $(\lambda^{-1}_{A^\perp})^\perp$ with conclusion sequent $\vdash \mathtt{I} \otimes A^\perp, A$ is $id^\bullet_A$ together with a $\mathtt{I}$-vertex and a $\otimes$-vertex. Introducing a cut and reducing it leads to $\mathrm{NF}(cut^\bullet_A(id^\bullet_A, id^\bullet_A)) = id^\bullet_A$ (Lemma 29). □

We now move to the sequent calculus.

Definition 46 (Category of Sequent Calculus Proofs)**.** Given a set of atoms $\mathscr{X}$, we define the category $\mathscr{S}_\mathscr{X}$ with:

- objects are MLLM formulas built from $\mathscr{X}$;
- morphisms from A to B are *equivalence classes for* $=_{\sigma\rho}$ of cut-free proofs of the sequent calculus MLLM with conclusion $\vdash A^\perp, B$;
- the identity morphism on A is (the equivalence class of) the proof id_A with conclusion $\vdash A^\perp, A$;
- given two proofs π_1 and π_2 with conclusions $\vdash A^\perp, B$ and $\vdash B^\perp, C$, composition is induced on equivalence classes by (the equivalence class of) the normal form of:

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdash A^\perp, \underline{B} & \vdash \underline{B^\perp}, C\end{array}}{\vdash A^\perp, C}\ cut$$

This definition requires to check that composition is compatible with $=_{\sigma\rho}$ (or with $=_{\beta\rho}$ thanks to Lemma 30), that identity proofs id_A are units for composition, etc. This structure (and more) is deduced from the categorical structure of $\mathscr{P}_{\mathscr{X}}$.

Proposition 47 (Isomorphism between $\mathscr{S}_{\mathscr{X}}$ and $\mathscr{P}_{\mathscr{X}}$)**.** *$\mathscr{S}_{\mathscr{X}}$ and $\mathscr{P}_{\mathscr{X}}$ are isomorphic categories.*

Proof. $\mathscr{S}_{\mathscr{X}}$ and $\mathscr{P}_{\mathscr{X}}$ are isomorphic categories through $(\_)^\bullet$. Objects are the same. Equivalence classes of cut-free proofs of $\vdash A^\perp, B$ up to $=_{\sigma\rho}$ exactly correspond to a unique proof net with conclusion sequent $\vdash A^\perp, B$ (Proposition 22, Lemma 23, and Theorem 27). The (equivalence class) of the identity proof id_A is mapped to the identity proof net $id_A^\bullet$. Composition is mapped to composition (Lemma 32). □

Lemma 48 (Structure of $\mathscr{S}_{\mathscr{X}}$)**.** *$\mathscr{S}_{\mathscr{X}}$ is a star-autonomous category with strict self-dual unit.*

Proof. By Proposition 47, the structure of $\mathscr{P}_{\mathscr{X}}$ (Proposition 45) can be moved to $\mathscr{S}_{\mathscr{X}}$. □

Lemma 49 (Definability)**.** *Each morphism in $\mathscr{S}_{\mathscr{X}}$ can be defined using the constructors of star-autonomous categories: id, $\_;\_$, $\_\otimes\_$, $\mathtt{I}$, λ, ρ, α, γ, ev, $(\_)^\star$, $(\_)^{-1}$ and $(\_)^\perp$.*

Proof. It is convenient to describe a bit more the structure of $\mathscr{S}_{\mathscr{X}}$ first. Star-autonomous categories come with a structure of symmetric linearly distributive category (Cockett and Seely (1997*b*)). In particular, $\_\parr\_ := ((\_)^\perp \otimes (\_)^\perp)^\perp$ $(= (\_)^\perp \multimap \_)$ gives a bifunctor and, together with $1^\perp$, it defines a second symmetric monoidal structure (obtained from the structure of $\_\otimes\_$ through $(\_)^\perp$). This means, using Mac Lane's coherence theorem, that we can reason modulo ACU (associativity-commutativity-unitality) on both $\_\otimes\_$ and $\_\parr\_$ by not always mentioning the unique canonical isomorphisms allowing to move from one representation to another.

The two symmetric monoidal structures are linked by means of the linear distributivity natural transformation: $\delta_{A,B,C} = (\alpha_{C,A^\perp\multimap B,A^\perp}; C\otimes ev_{A^\perp,B})^\star : C\otimes(A^\perp\multimap B) \longrightarrow A^\perp \multimap (C\otimes B) = C\otimes(A\parr B) \longrightarrow A\parr(C\otimes B)$. Using the symmetry isomorphisms for $\_\otimes\_$ and $\_\parr\_$, we obtain similarly $(A\parr B)\otimes C \longrightarrow A\parr(B\otimes C)$, $C\otimes(B\parr A) \longrightarrow (C\otimes B)\parr A$ and $(B\parr A)\otimes C \longrightarrow (B\otimes C)\parr A$. For simplicity of notations, we use $\delta_{A,B,C}$ for all of them.

Let us now look at the morphisms we can compute from proofs. Given a proof π with conclusion $\vdash \Gamma$, we map it to a proof $\overline{\pi}$ with conclusion $\vdash \mathtt{I}, ?\Gamma$ (we use $\vdash \mathtt{I}, \mathtt{I}$ if Γ is empty) by building:

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{I}}\ \mathtt{I} \qquad \begin{array}{c}\pi\\ \vdash \Gamma\end{array}}{\vdash \mathtt{I}, \Gamma}\ mix_2}{\vdash \mathtt{I}, ?\Gamma}\ \parr$$

This maps any proof to a proof with exactly two formulas in conclusions, thus a morphism in $\mathscr{S}_{\mathscr{X}}$. We prove, by induction on a cut-free proof π with conclusion $\vdash \Gamma$, that $\overline{\pi}$ is equal up to $=_{\beta\rho}$ (thus up to $=_{\sigma\rho}$, Lemma 30) to a proof built from the listed primitives.

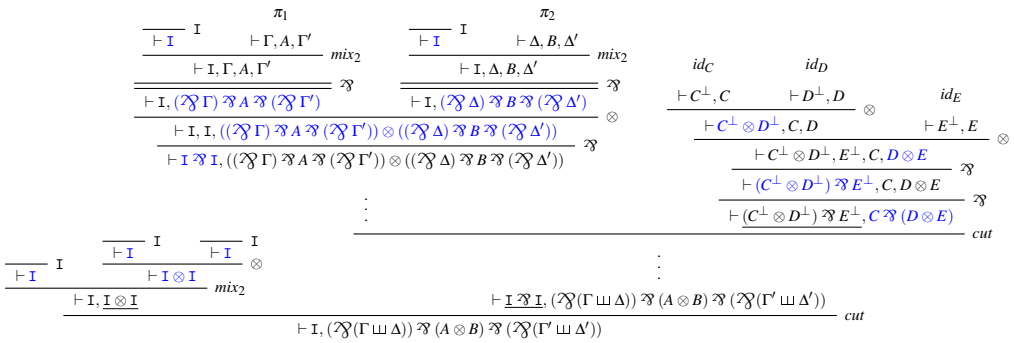
We look at the last rule of π:

(ax_l) We use $\lambda^\star_X$.
(ax_r) We use $\lambda^\star_X$.

(I) We use $id_{\mathtt{I}}$.

($\parr$) We apply the induction hypothesis and compose the resulting proof with associativity morphisms for $\_ \parr \_$.

($\otimes$) By induction hypothesis, we get (definable up to $=_{\beta\rho}$) proofs $\overline{\pi_1}$ and $\overline{\pi_2}$ with conclusions $\vdash \mathtt{I}, (\Gamma \parr \Gamma) \parr A \parr (\Gamma' \parr \Gamma')$ and $\vdash \mathtt{I}, (\Delta \parr \Delta) \parr B \parr (\Delta' \parr \Delta')$ from which we build the proof:



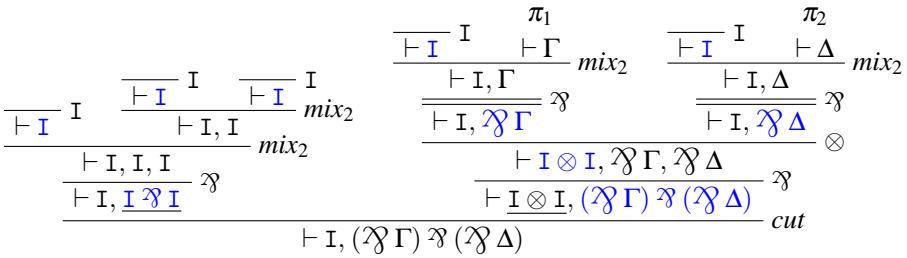
which corresponds to applying the $\_ \otimes \_$ bifunctor, post-composing multiple times with δ morphisms (with appropriate values of C, D, and E) and pre-composing with $\lambda_{\mathtt{I}}^{-1}$. It reduces by $\rightarrow_\beta$ to:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,mix_0 \qquad \overset{\pi_1}{\vdash \Gamma, A, \Gamma'}}{\vdash \Gamma, A, \Gamma'}\,mix_2 \qquad \dfrac{\dfrac{}{\vdash}\,mix_0 \qquad \overset{\pi_2}{\vdash \Delta, B, \Delta'}}{\vdash \Delta, B, \Delta'}\,mix_2}{\vdash \Gamma \sqcup \Delta, A \otimes B, \Gamma' \sqcup \Delta'}\,\otimes}{\vdash (\Gamma \sqcup \Delta) \parr (A \otimes B) \parr (\Gamma' \sqcup \Delta')}\,\parr \qquad \dfrac{}{\vdash \mathtt{I}}\,\mathtt{I}}{\vdash \mathtt{I}, (\Gamma \sqcup \Delta) \parr (A \otimes B) \parr (\Gamma' \sqcup \Delta')}\,mix_2$$

that is $\overline{\pi}$ up to $=_{\sigma\rho}$.

(mix_0) We use $id_{\mathtt{I}}$.

(mix_2) By induction hypothesis, we get (definable up to $=_{\beta\rho}$) proofs $\overline{\pi_1}$ and $\overline{\pi_2}$ with conclusions $\vdash \mathtt{I}, \Gamma$ and $\vdash \mathtt{I}, \Delta$ from which we build the proof:



which corresponds to applying the $\_ \parr \_$ bifunctor and pre-composing with $\lambda_{\mathtt{I}}^{\perp}$. It reduces by $\rightarrow_\beta$ to:

$$\dfrac{\dfrac{}{\vdash \mathtt{I}}\,\mathtt{I} \qquad \dfrac{\dfrac{\dfrac{}{\vdash}\,mix_0 \qquad \vdots\,\pi_1}{\vdash \Gamma}\,mix_2}{\vdash \Gamma}\,\parr \qquad \dfrac{\dfrac{\dfrac{}{\vdash}\,mix_0 \qquad \vdots\,\pi_2}{\vdash \Delta}\,mix_2}{\vdash \Delta}\,\parr}{\vdash \mathtt{I}, \Gamma \parr \Delta}\,mix_2}{\vdash \mathtt{I}, (\Gamma \parr \Gamma) \parr (\Delta \parr \Delta)}\,\parr$$

that is $\overline{\pi}$ up to $=_{\sigma\rho}$ (and modulo ACU for $\parr$). □

Assume given a SACSSDU $\mathscr{C}$ coming with an interpretation ν of elements of $\mathscr{X}$ as objects of $\mathscr{C}$, there is a unique way $[\![\_]\!]$ of mapping MLLM formulas into objects of $\mathscr{C}$ if we want to respect the interpretation of atoms and of each connective with respect to the SACSSDU constructions in $\mathscr{C}$ (in particular $[\![A^\perp]\!] = [\![A]\!]^\perp$):

$$[\![X]\!] = \nu(X) \qquad \bar{X} = \nu(X)^\perp \qquad [\![\mathtt{I}]\!] = 1$$
$$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!] \qquad [\![A \parr B]\!] = ([\![A]\!]^\perp \otimes [\![B]\!]^\perp)^\perp$$

Thanks to Lemma 49, there is at most one way to extend $[\![\_]\!]$ into an interpretation of proofs as morphisms of $\mathscr{C}$ in such a way that SACSSDU constructions are preserved. Using Lemma 48 and the invariance of the computed morphism under $=_{\sigma\rho}$, we get a SACSSDU-functor from $\mathscr{S}_\mathscr{X}$ to $\mathscr{C}$ compatible with ν on $\mathscr{X}$. This means that $\mathscr{P}_\mathscr{X}$, through Proposition 47, provides a canonical representation of the free SACSSDU over the set $\mathscr{X}$.

There should be no particular difficulty in extending this to a characterization of the free SACSSDU over a given category (as for example in Hughes (2012)).

5. Conclusion

We have presented and studied MLLM, a variant of multiplicative linear logic with (*mix*) rules and a self-dual unit. This logic lives in between standard multiplicative linear logic with units and the logic of compact closed categories ($\perp = 1$ but $\parr \neq \otimes$).

Unit-free multiplicative linear logic is often presented as the paradise in linear logic. Units however break the canonicity of the proof net representation (Blute et al. (1996); Heijltjes and Houston (2016)). We have tried to provide evidence that MLLM extends all good properties of unit-free MLL to a setting with units. We have developed proof nets and categorical semantics. One could continue with many topics from the linear logic toolbox: phase semantics (Fleury and Retoré (1994)), game semantics (Abramsky and Jagadeesan (1994)), geometry of interaction (Abramsky et al. (2002)), denotational models, etc.

Acknowledgements. Thanks to Thomas Ehrhard who suggested to me to look for a categorical version of Theorem 10. Thanks also to Rémi Di Guardia, Michele Pagani, Colin Riba and Tito Nguyen for various discussions around properties of multiplicative linear logic.

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Appendix A. Shuffle

We use the following notations for lists:

- $[\,]$ is the empty list;
- $[A_1, \ldots, A_n]$ is the list containing $A_1, \ldots, A_n$;
- $\Gamma \cdot \Delta$ is the concatenation of Γ and Δ (which can be lifted to sets of lists);
- $\Gamma \sim \Delta$ means that Γ is a permutation of Δ.

Definition 50 (Shuffle)**.** Given two lists Γ and Δ, their **shuffle** $\Gamma \sqcup \Delta$ is the following set of lists defined by induction on Γ and Δ:

- $[\,] \sqcup \Delta = \{\Delta\}$;
- $\Gamma \sqcup [\,] = \{\Gamma\}$;
- if $\Gamma = [A] \cdot \Gamma'$ and $\Delta = [B] \cdot \Delta'$ then $\Gamma \sqcup \Delta = \{[A] \cdot \Sigma \mid \Sigma \in \Gamma' \sqcup \Delta\} \cup \{[B] \cdot \Sigma \mid \Sigma \in \Gamma \sqcup \Delta'\}$.

Example 51.

$$\begin{aligned}
[a] \sqcup [\alpha] &= \{[a, \alpha], [\alpha, a]\} \\
[a, b] \sqcup [\alpha] &= \{[a, b, \alpha], [a, \alpha, b], [\alpha, a, b]\} \\
[a, b, c] \sqcup [\alpha, \beta] &= \{[a, b, c, \alpha, \beta], [a, b, \alpha, c, \beta], [a, b, \alpha, \beta, c], [a, \alpha, b, c, \beta], [a, \alpha, b, \beta, c], \\
&\qquad [a, \alpha, \beta, b, c], [\alpha, a, b, c, \beta], [\alpha, a, b, \beta, c], [\alpha, a, \beta, b, c], [\alpha, \beta, a, b, c]\}
\end{aligned}$$

Properties. We have the following results about $\sqcup$:

(1) $\Gamma \sqcup [\,] = [\,] \sqcup \Gamma = \{\Gamma\}$;
(2) $\Gamma \sqcup \Delta = \Delta \sqcup \Gamma$;
(3) $(\Gamma \sqcup \Delta) \sqcup \Sigma = \Gamma \sqcup (\Delta \sqcup \Sigma)$;
(4) $\Gamma \cdot \Delta \in \Gamma \sqcup \Delta$ and $\Delta \cdot \Gamma \in \Gamma \sqcup \Delta$;
(5) if $\Sigma_1 \cdot \Sigma_2 \in \Gamma \sqcup \Delta$, there exist Γ_1, Γ_2, Δ_1 and Δ_2 such that $\Gamma = \Gamma_1 \cdot \Gamma_2$, $\Delta = \Delta_1 \cdot \Delta_2$, $\Sigma_1 \in \Gamma_1 \sqcup \Delta_1$ and $\Sigma_2 \in \Gamma_2 \sqcup \Delta_2$;
(6) if $\Sigma_1 \cdot [C] \cdot \Sigma_2 \in \Gamma \sqcup \Delta$, there exist Γ_1, Γ_2, Δ_1 and Δ_2 such that $\Sigma_1 \in \Gamma_1 \sqcup \Delta_1$ and $\Sigma_2 \in \Gamma_2 \sqcup \Delta_2$ with:
   – either $\Gamma = \Gamma_1 \cdot [C] \cdot \Gamma_2$ and $\Delta = \Delta_1 \cdot \Delta_2$
   – or $\Gamma = \Gamma_1 \cdot \Gamma_2$ and $\Delta = \Delta_1 \cdot [C] \cdot \Delta_2$;
(7) if $\Sigma \in \Gamma \sqcup \Delta$ then $\Sigma \sim \Gamma \cdot \Delta$;
(8) if $\Sigma' \sim \Sigma \in \Gamma \sqcup \Delta$, there exist Γ' and Δ' such that $\Gamma' \sim \Gamma$, $\Delta' \sim \Delta$ and $\Sigma' \in \Gamma' \sqcup \Delta'$.

Appendix B. Axiom Expansion

B.1 Unit-Free Axiom Expansion

$$\frac{}{\vdash \bar{X}, X}\,ax \quad \to_\eta \quad \frac{}{\vdash \bar{X}, X}\,ax_l$$

$$\frac{}{\vdash X, \bar{X}}\,ax \quad \to_\eta \quad \frac{}{\vdash X, \bar{X}}\,ax_r$$

$$\frac{}{\vdash A^\perp \parr B^\perp, A \otimes B}\,ax \quad \to_\eta \quad \frac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,ax \qquad \dfrac{}{\vdash B^\perp, B}\,ax}{\vdash A^\perp, B^\perp, A \otimes B}\,\otimes}{\vdash A^\perp \parr B^\perp, A \otimes B}\,\parr$$

$$\frac{}{\vdash A^{\perp} \otimes B^{\perp}, A \parr B}\, ax \quad \to_{\eta} \quad \frac{\dfrac{}{\vdash A^{\perp}, A}\, ax \qquad \dfrac{}{\vdash B^{\perp}, B}\, ax}{\dfrac{\vdash A^{\perp} \otimes B^{\perp}, A, B}{\vdash A^{\perp} \otimes B^{\perp}, A \parr B}\, \parr}\, \otimes$$

B.2 Axiom Expansion for Units in MLL$_{mix}$

$$\frac{}{\vdash \perp, 1}\, ax \quad \to_{\eta} \quad \frac{\dfrac{}{\vdash 1}\, 1}{\vdash \perp, 1}\, \perp$$

$$\frac{}{\vdash 1, \perp}\, ax \quad \to_{\eta} \quad \frac{\dfrac{}{\vdash 1}\, 1}{\vdash 1, \perp}\, \perp$$

B.3 Axiom Expansion for Units in MLLM

$$\frac{}{\vdash \mathtt{I}, \mathtt{I}}\, ax \quad \to_{\eta} \quad \frac{\dfrac{}{\vdash \mathtt{I}}\, \mathtt{I} \qquad \dfrac{}{\vdash \mathtt{I}}\, \mathtt{I}}{\vdash \mathtt{I}, \mathtt{I}}\, mix_2$$

Appendix C. Cut Elimination

C.1 Unit-Free Cut Elimination

Key cases

$$\frac{\dfrac{}{\vdash \bar{X}, X}\, ax_l \qquad \dfrac{}{\vdash \bar{X}, \underline{X}}\, ax_l}{\vdash X, \bar{X}}\, cut \quad \to_{\beta} \quad \frac{}{\vdash X, \bar{X}}\, ax_r$$

$$\frac{\dfrac{}{\vdash \bar{X}, X}\, ax_l \qquad \dfrac{}{\vdash \bar{X}, \underline{X}}\, ax_l}{\vdash \bar{X}, X}\, cut \quad \to_{\beta} \quad \frac{}{\vdash \bar{X}, X}\, ax_l$$

$$\frac{\dfrac{}{\vdash \bar{X}, \underline{X}}\, ax_l \qquad \dfrac{}{\vdash \bar{X}, X}\, ax_l}{\vdash \bar{X}, X}\, cut \quad \to_{\beta} \quad \frac{}{\vdash \bar{X}, X}\, ax_l$$

$$\frac{\dfrac{}{\vdash \bar{X}, X}\, ax_l \qquad \dfrac{}{\vdash \bar{X}, \underline{X}}\, ax_l}{\vdash X, \bar{X}}\, cut \quad \to_{\beta} \quad \frac{}{\vdash X, \bar{X}}\, ax_r$$

$$\frac{\dfrac{}{\vdash \bar{X}, X}\, ax_l \qquad \dfrac{}{\vdash \underline{X}, \bar{X}}\, ax_r}{\vdash X, \bar{X}}\, cut \quad \to_{\beta} \quad \frac{}{\vdash X, \bar{X}}\, ax_r$$

$$\frac{\dfrac{}{\vdash \bar{X}, X}\, ax_l \qquad \dfrac{}{\vdash \underline{X}, \bar{X}}\, ax_r}{\vdash \bar{X}, X}\, cut \quad \to_{\beta} \quad \frac{}{\vdash \bar{X}, X}\, ax_l$$

$$\frac{\dfrac{}{\vdash \bar{X}, \underline{X}}\, ax_l \qquad \dfrac{}{\vdash X, \bar{X}}\, ax_r}{\vdash \bar{X}, X}\, cut \quad \to_{\beta} \quad \frac{}{\vdash \bar{X}, X}\, ax_l$$

$$\dfrac{\dfrac{}{\vdash \bar{X}, \underline{X}}\, ax_l \qquad \dfrac{}{\vdash X, \underline{\bar{X}}}\, ax_r}{\vdash X, \bar{X}}\, cut \quad \to_\beta \quad \dfrac{}{\vdash X, \bar{X}}\, ax_r$$

$$\dfrac{\dfrac{}{\vdash \underline{X}, \bar{X}}\, ax_r \qquad \dfrac{}{\vdash X, \underline{\bar{X}}}\, ax_r}{\vdash \bar{X}, X}\, cut \quad \to_\beta \quad \dfrac{}{\vdash \bar{X}, X}\, ax_l$$

$$\dfrac{\dfrac{}{\vdash \underline{X}, \bar{X}}\, ax_r \qquad \dfrac{}{\vdash X, \underline{\bar{X}}}\, ax_r}{\vdash X, \bar{X}}\, cut \quad \to_\beta \quad \dfrac{}{\vdash X, \bar{X}}\, ax_r$$

$$\dfrac{\dfrac{}{\vdash X, \underline{\bar{X}}}\, ax_r \qquad \dfrac{}{\vdash \underline{X}, \bar{X}}\, ax_r}{\vdash X, \bar{X}}\, cut \quad \to_\beta \quad \dfrac{}{\vdash X, \bar{X}}\, ax_r$$

$$\dfrac{\dfrac{}{\vdash X, \underline{\bar{X}}}\, ax_r \qquad \dfrac{}{\vdash \underline{X}, \bar{X}}\, ax_r}{\vdash \bar{X}, X}\, cut \quad \to_\beta \quad \dfrac{}{\vdash \bar{X}, X}\, ax_l$$

$$\dfrac{\dfrac{}{\vdash \underline{X}, \bar{X}}\, ax_r \qquad \dfrac{}{\vdash \underline{\bar{X}}, X}\, ax_l}{\vdash \bar{X}, X}\, cut \quad \to_\beta \quad \dfrac{}{\vdash \bar{X}, X}\, ax_l$$

$$\dfrac{\dfrac{}{\vdash \underline{X}, \bar{X}}\, ax_r \qquad \dfrac{}{\vdash \underline{\bar{X}}, X}\, ax_l}{\vdash X, \bar{X}}\, cut \quad \to_\beta \quad \dfrac{}{\vdash X, \bar{X}}\, ax_r$$

$$\dfrac{\dfrac{}{\vdash X, \underline{\bar{X}}}\, ax_r \qquad \dfrac{}{\vdash \underline{\bar{X}}, X}\, ax_l}{\vdash \bar{X}, X}\, cut \quad \to_\beta \quad \dfrac{}{\vdash \bar{X}, X}\, ax_l$$

$$\dfrac{\dfrac{}{\vdash X, \underline{\bar{X}}}\, ax_r \qquad \dfrac{}{\vdash \bar{X}, \underline{X}}\, ax_l}{\vdash \bar{X}, X}\, cut \quad \to_\beta \quad \dfrac{}{\vdash \bar{X}, X}\, ax_l$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2 \qquad \vdash \Delta_1, B, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, \underline{A \otimes B}, \Gamma_2 \sqcup \Delta_2}\, \otimes \qquad \dfrac{\vdash \Sigma_1, A^\perp, B^\perp, \Sigma_2}{\vdash \Sigma_1, \underline{A^\perp \parr B^\perp}, \Sigma_2}\, \parr}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\, cut \quad \to_\beta$$

$$\dfrac{\vdash \Gamma_1, \underline{A}, \Gamma_2 \qquad \dfrac{\vdash \Delta_1, \underline{B}, \Delta_2 \qquad \vdash \Sigma_1, A^\perp, \underline{B^\perp}, \Sigma_2}{\vdash \Delta_1 \sqcup \Sigma_1, \underline{A^\perp}, \Delta_2 \sqcup \Sigma_2}\, cut}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\, cut$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2}{\vdash \Gamma_1, \underline{A \parr B}, \Gamma_2}\, \parr \qquad \dfrac{\vdash \Delta_1, A^\perp, \Delta_2 \qquad \vdash \Sigma_1, B^\perp, \Sigma_2}{\vdash \Delta_1 \sqcup \Sigma_1, \underline{A^\perp \otimes B^\perp}, \Delta_2 \sqcup \Sigma_2}\, \otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\, cut \quad \to_\beta$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \underline{A}, B, \Gamma_2 \qquad \vdash \Delta_1, \underline{A^\perp}, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, \underline{B}, \Gamma_2 \sqcup \Delta_2}\, cut \qquad \vdash \Sigma_1, \underline{B^\perp}, \Sigma_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\, cut$$

Commutations

$$\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, C^{\perp}, \Delta_2, A, B, \Delta_3}{\vdash \Delta_1, \underline{C^{\perp}}, \Delta_2, A \parr B, \Delta_3}\parr}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, A \parr B, \Gamma_3 \sqcup \Delta_3}\,cut \quad \rightarrow_{\beta} \quad \dfrac{\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \vdash \Delta_1, \underline{C^{\perp}}, \Delta_2, A, B, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, A, B, \Gamma_3 \sqcup \Delta_3}\,cut}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, A \parr B, \Gamma_3 \sqcup \Delta_3}\parr$$

$$\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, A, B, \Delta_2, C^{\perp}, \Delta_3}{\vdash \Delta_1, A \parr B, \Delta_2, \underline{C^{\perp}}, \Delta_3}\parr}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\,cut \quad \rightarrow_{\beta} \quad \dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \vdash \Delta_1, A, B, \Delta_2, \underline{C^{\perp}}, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A, B, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\,cut}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\parr$$

$$\dfrac{\dfrac{\vdash \Gamma_1, C, \Gamma_2, A, B, \Gamma_3}{\vdash \Gamma_1, \underline{C}, \Gamma_2, A \parr B, \Gamma_3}\parr \qquad \vdash \Delta_1, \underline{C^{\perp}}, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, A \parr B, \Gamma_3 \sqcup \Delta_3}\,cut \quad \rightarrow_{\beta} \quad \dfrac{\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, A, B, \Gamma_3 \qquad \vdash \Delta_1, \underline{C^{\perp}}, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, A, B, \Gamma_3 \sqcup \Delta_3}\,cut}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, A \parr B, \Gamma_3 \sqcup \Delta_3}\parr$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2, C, \Gamma_3}{\vdash \Gamma_1, A \parr B, \Gamma_2, \underline{C}, \Gamma_3}\parr \qquad \vdash \Delta_1, \Delta_2, \underline{C^{\perp}}, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\,cut \quad \rightarrow_{\beta} \quad \dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2, \underline{C}, \Gamma_3 \qquad \vdash \Delta_1, \Delta_2, \underline{C^{\perp}}, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A, B, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\,cut}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\parr$$

$$\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, C^{\perp}, \Delta_2, A, \Delta_3 \qquad \vdash \Sigma_1, \Sigma_2, B, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, \underline{C^{\perp}}, \Delta_2 \sqcup \Sigma_2, A \otimes B, \Delta_3 \sqcup \Sigma_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, A \otimes B, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\,cut \quad \rightarrow_{\beta}$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \vdash \Delta_1, \underline{C^{\perp}}, \Delta_2, A, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, A, \Gamma_3 \sqcup \Delta_3}\,cut \qquad \vdash \Sigma_1, \Sigma_2, B, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, A \otimes B, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, \Delta_2, A, \Delta_3 \qquad \vdash \Sigma_1, C^{\perp}, \Sigma_2, B, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, \underline{C^{\perp}}, \Delta_2 \sqcup \Sigma_2, A \otimes B, \Delta_3 \sqcup \Sigma_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, A \otimes B, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\,cut \quad \rightarrow_{\beta}$$

$$\dfrac{\vdash \Delta_1, \Delta_2, A, \Delta_3 \qquad \dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \vdash \Sigma_1, \underline{C^{\perp}}, \Sigma_2, B, \Sigma_3}{\vdash \Gamma_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Sigma_2, B, \Gamma_3 \sqcup \Sigma_3}\,cut}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, A \otimes B, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, A, \Delta_2, C^{\perp}, \Delta_3 \qquad \vdash \Sigma_1, B, \Sigma_2, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, A \otimes B, \Delta_2 \sqcup \Sigma_2, \underline{C^{\perp}}, \Delta_3 \sqcup \Sigma_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\,cut \quad \rightarrow_{\beta}$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \vdash \Delta_1, A, \Delta_2, \underline{C^{\perp}}, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\,cut \qquad \vdash \Sigma_1, B, \Sigma_2, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, A, \Delta_2, \Delta_3 \qquad \vdash \Sigma_1, B, \Sigma_2, C^{\perp}, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, A \otimes B, \Delta_2 \sqcup \Sigma_2, \underline{C^{\perp}}, \Delta_3 \sqcup \Sigma_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\,cut \quad \rightarrow_{\beta}$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \vdash \Sigma_1, B, \Sigma_2, \underline{C^{\perp}}, \Sigma_3}{\vdash \Gamma_1 \sqcup \Sigma_1, B, \Gamma_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Sigma_3}\,cut \qquad \vdash \Delta_1, A, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, C, \Gamma_2, A, \Gamma_3 \qquad \vdash \Delta_1, \Delta_2, B, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \underline{C}, \Gamma_2 \sqcup \Delta_2, A \otimes B, \Gamma_3 \sqcup \Delta_3}\otimes \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, A \otimes B, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\,cut \quad \to_\beta$$

$$\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, A, \Gamma_3 \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2, \Sigma_3}{\vdash \Gamma_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Sigma_2, A, \Gamma_3 \sqcup \Sigma_3}\,cut \qquad \vdash \Delta_1, \Delta_2, B, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, A \otimes B, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, A, \Gamma_3 \qquad \vdash \Delta_1, C, \Delta_2, B, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \underline{C}, \Gamma_2 \sqcup \Delta_2, A \otimes B, \Gamma_3 \sqcup \Delta_3}\otimes \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, A \otimes B, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\,cut \quad \to_\beta$$

$$\dfrac{\vdash \Gamma_1, \Gamma_2, A, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, \underline{C}, \Delta_2, B, \Delta_3 \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, \Delta_2 \sqcup \Sigma_2, B, \Delta_3 \sqcup \Sigma_3}\,cut}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, A \otimes B, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2, C, \Gamma_3 \qquad \vdash \Delta_1, B, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, \underline{C}, \Gamma_3 \sqcup \Delta_3}\otimes \qquad \vdash \Sigma_1, \Sigma_2, \underline{C^\perp}, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\,cut \quad \to_\beta$$

$$\dfrac{\vdash \Gamma_1, A, \Gamma_2, \underline{C}, \Gamma_3 \qquad \vdash \Sigma_1, \Sigma_2, \underline{C^\perp}, \Sigma_3}{\vdash \Gamma_1 \sqcup \Sigma_1, A, \Gamma_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Sigma_3}\,cut \qquad \vdash \Delta_1, B, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \vdash \Delta_1, B, \Delta_2, C, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, \underline{C}, \Gamma_3 \sqcup \Delta_3}\otimes \qquad \vdash \Sigma_1, \Sigma_2, \underline{C^\perp}, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\,cut \quad \to_\beta$$

$$\dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, B, \Delta_2, \underline{C}, \Delta_3 \qquad \vdash \Sigma_1, \Sigma_2, \underline{C^\perp}, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, B, \Delta_2 \sqcup \Sigma_2, \Delta_3 \sqcup \Sigma_3}\,cut}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2 \qquad \dfrac{\vdash \Delta_1, C^\perp, \Delta_2 \qquad \vdash \Sigma_1, \Sigma_2}{\vdash \Delta_1 \sqcup \Sigma_1, \underline{C^\perp}, \Delta_2 \sqcup \Sigma_2}\,mix_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\,cut \quad \to_\beta$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2 \qquad \vdash \Delta_1, \underline{C^\perp}, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2}\,cut \qquad \vdash \Sigma_1, \Sigma_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\,mix_2$$

$$\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2 \qquad \dfrac{\vdash \Delta_1, \Delta_2 \qquad \vdash \Sigma_1, C^\perp, \Sigma_2}{\vdash \Delta_1 \sqcup \Sigma_1, \underline{C^\perp}, \Delta_2 \sqcup \Sigma_2}\,mix_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\,cut \quad \to_\beta$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2 \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2}{\vdash \Gamma_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Sigma_2}\,cut \qquad \vdash \Delta_1, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\,mix_2$$

$$\dfrac{\dfrac{\vdash \Gamma_1, C, \Gamma_2 \qquad \vdash \Delta_1, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, \underline{C}, \Gamma_2 \sqcup \Delta_2}\,mix_2 \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\,cut \quad \to_\beta$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2 \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2}{\vdash \Gamma_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Sigma_2}\, cut \qquad \vdash \Delta_1, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\, mix_2$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2 \qquad \vdash \Delta_1, C, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, \underline{C}, \Gamma_2 \sqcup \Delta_2}\, mix_2 \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\, cut \quad \rightarrow_\beta \quad \dfrac{\vdash \Gamma_1, \Gamma_2 \qquad \dfrac{\vdash \Delta_1, \underline{C}, \Delta_2 \qquad \vdash \Sigma_1, \underline{C^\perp}, \Sigma_2}{\vdash \Delta_1 \sqcup \Sigma_1, \Delta_2 \sqcup \Sigma_2}\, cut}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\, mix_2$$

C.2 Cut Elimination for Units in MLL$_{mix}$

Key cases

$$\dfrac{\dfrac{}{\vdash \underline{1}}\, 1 \qquad \dfrac{\vdash \Gamma, \Delta}{\vdash \Gamma, \underline{\perp}, \Delta}\, \perp}{\vdash \Gamma, \Delta}\, cut \quad \rightarrow_\beta \quad \vdash \Gamma, \Delta$$

$$\dfrac{\dfrac{\vdash \Gamma, \Delta}{\vdash \Gamma, \underline{\perp}, \Delta}\, \perp \qquad \dfrac{}{\vdash \underline{1}}\, 1}{\vdash \Gamma, \Delta}\, cut \quad \rightarrow_\beta \quad \vdash \Gamma, \Delta$$

Commutations

$$\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, C^\perp, \Delta_2, \Delta_3}{\vdash \Delta_1, \underline{C^\perp}, \Delta_2, \perp, \Delta_3}\, \perp}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, \perp, \Gamma_3 \sqcup \Delta_3}\, cut \quad \rightarrow_\beta \quad \dfrac{\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \vdash \Delta_1, \underline{C^\perp}, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, cut}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, \perp, \Gamma_3 \sqcup \Delta_3}\, \perp$$

$$\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, \Delta_2, C^\perp, \Delta_3}{\vdash \Delta_1, \perp, \Delta_2, \underline{C^\perp}, \Delta_3}\, \perp}{\vdash \Gamma_1 \sqcup \Delta_1, \perp, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, cut \quad \rightarrow_\beta \quad \dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \vdash \Delta_1, \Delta_2, \underline{C^\perp}, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, cut}{\vdash \Gamma_1 \sqcup \Delta_1, \perp, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, \perp$$

$$\dfrac{\dfrac{\vdash \Gamma_1, C, \Gamma_2, \Gamma_3}{\vdash \Gamma_1, \underline{C}, \Gamma_2, \perp, \Gamma_3}\, \perp \qquad \vdash \Delta_1, \underline{C^\perp}, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, \perp, \Gamma_3 \sqcup \Delta_3}\, cut \quad \rightarrow_\beta \quad \dfrac{\dfrac{\vdash \Gamma_1, \underline{C}, \Gamma_2, \Gamma_3 \qquad \vdash \Delta_1, \underline{C^\perp}, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, cut}{\vdash \Gamma_1 \sqcup \Delta_1, \perp, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, \perp$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, C, \Gamma_3}{\vdash \Gamma_1, \perp, \Gamma_2, \underline{C}, \Gamma_3}\, \perp \qquad \vdash \Delta_1, \Delta_2, \underline{C^\perp}, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \perp, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, cut \quad \rightarrow_\beta \quad \dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, \underline{C}, \Gamma_3 \qquad \vdash \Delta_1, \Delta_2, \underline{C^\perp}, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, cut}{\vdash \Gamma_1 \sqcup \Delta_1, \perp, \Gamma_2 \sqcup \Delta_2, \Gamma_3 \sqcup \Delta_3}\, \perp$$

C.3 Cut Elimination for Units in MLLM

Key case

$$\dfrac{\dfrac{}{\vdash \underline{\mathtt{I}}}\, \mathtt{I} \qquad \dfrac{}{\vdash \underline{\mathtt{I}}}\, \mathtt{I}}{\vdash}\, cut \quad \rightarrow_\beta \quad \dfrac{}{\vdash}\, mix_0$$

This is the only case where we generate a rule in the reduct which is not present in the redex.

Appendix D. Rule Permutations in MLLM

We give the equations defining the congruence $=_\sigma$ on MLLM proofs, coming from permutations of rules (we ignore permutations involving (*cut*) rules as it suffices for the results about cut-free proofs we need).

The first equation is not really a rule permutation but it is contained in $=_{\beta\rho}$ anyway (Lemma 18) and satisfied by the translation into proof nets (Section 2.1). We put it as it decreases the number of other equations to consider in $=_\sigma$ (thanks to the underlying left-right symmetry in (mix_2) rules).

$$\dfrac{\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \sqcup \Delta}\, mix_2 \qquad \vdash \Sigma}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2 \quad =_\sigma \quad \dfrac{\vdash \Gamma \qquad \dfrac{\vdash \Delta \qquad \vdash \Sigma}{\vdash \Delta \sqcup \Sigma}\, mix_2}{\vdash \Gamma \sqcup \Delta \sqcup \Sigma}\, mix_2$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2, C, D, \Gamma_3}{\vdash \Gamma_1, A \parr B, \Gamma_2, C, D, \Gamma_3}\parr}{\vdash \Gamma_1, A \parr B, \Gamma_2, C \parr D, \Gamma_3}\parr \quad =_\sigma \quad \dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2, C, D, \Gamma_3}{\vdash \Gamma_1, A, B, \Gamma_2, C \parr D, \Gamma_3}\parr}{\vdash \Gamma_1, A \parr B, \Gamma_2, C \parr D, \Gamma_3}\parr$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2 \qquad \vdash \Delta_1, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, A, B, \Gamma_2 \sqcup \Delta_2}\, mix_2}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2}\parr \quad =_\sigma \quad \dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2}{\vdash \Gamma_1, A \parr B, \Gamma_2}\parr \qquad \vdash \Delta_1, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2}\, mix_2$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2, C, \Gamma_3 \qquad \vdash \Delta_1, \Delta_2, D, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A, B, \Gamma_2 \sqcup \Delta_2, C \otimes D, \Gamma_3 \sqcup \Delta_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2, C \otimes D, \Gamma_3 \sqcup \Delta_3}\parr \quad =_\sigma \quad \dfrac{\dfrac{\vdash \Gamma_1, A, B, \Gamma_2, C, \Gamma_3}{\vdash \Gamma_1, A \parr B, \Gamma_2, C, \Gamma_3}\parr \qquad \vdash \Delta_1, \Delta_2, D, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2, C \otimes D, \Gamma_3 \sqcup \Delta_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, C, \Gamma_3 \qquad \vdash \Delta_1, A, B, \Delta_2, D, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A, B, \Gamma_2 \sqcup \Delta_2, C \otimes D, \Gamma_3 \sqcup \Delta_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2, C \otimes D, \Gamma_3 \sqcup \Delta_3}\parr \quad =_\sigma \quad \dfrac{\vdash \Gamma_1, \Gamma_2, C, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, A, B, \Delta_2, D, \Delta_3}{\vdash \Delta_1, A \parr B, \Delta_2, D, \Delta_3}\parr}{\vdash \Gamma_1 \sqcup \Delta_1, A \parr B, \Gamma_2 \sqcup \Delta_2, C \otimes D, \Gamma_3 \sqcup \Delta_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2, C, D, \Gamma_3 \qquad \vdash \Delta_1, B, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, C, D, \Gamma_3 \sqcup \Delta_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, C \parr D, \Gamma_3 \sqcup \Delta_3}\parr \quad =_\sigma \quad \dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2, C, D, \Gamma_3}{\vdash \Gamma_1, A, \Gamma_2, C \parr D, \Gamma_3}\parr \qquad \vdash \Delta_1, B, \Delta_2, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, C \parr D, \Gamma_3 \sqcup \Delta_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \vdash \Delta_1, B, \Delta_2, C, D, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, C, D, \Gamma_3 \sqcup \Delta_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, C \parr D, \Gamma_3 \sqcup \Delta_3}\parr \quad =_\sigma \quad \dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, B, \Delta_2, C, D, \Delta_3}{\vdash \Delta_1, B, \Delta_2, C \parr D, \Delta_3}\parr}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, C \parr D, \Gamma_3 \sqcup \Delta_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2, C, \Gamma_3 \qquad \vdash \Delta_1, \Delta_2, D, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A, \Gamma_2 \sqcup \Delta_2, C \otimes D, \Gamma_3 \sqcup \Delta_3}\otimes \qquad \vdash \Sigma_1, B, \Sigma_2, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, C \otimes D, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes \quad =_\sigma$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2, C, \Gamma_3 \qquad \vdash \Sigma_1, B, \Sigma_2, \Sigma_3}{\vdash \Gamma_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Sigma_2, C, \Gamma_3 \sqcup \Sigma_3}\otimes \qquad \vdash \Delta_1, \Delta_2, D, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, C \otimes D, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, \Gamma_2, C, \Gamma_3 \qquad \vdash \Delta_1, A, \Delta_2, D, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A, \Gamma_2 \sqcup \Delta_2, C \otimes D, \Gamma_3 \sqcup \Delta_3}\otimes \qquad \vdash \Sigma_1, B, \Sigma_2, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, C \otimes D, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes \quad =_\sigma$$

$$\dfrac{\vdash \Gamma_1, \Gamma_2, C, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, A, \Delta_2, D, \Delta_3 \qquad \vdash \Sigma_1, B, \Sigma_2, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, A \otimes B, \Delta_2 \sqcup \Sigma_2, D, \Delta_3 \sqcup \Sigma_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, C \otimes D, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, B, \Delta_2, C, \Delta_3 \qquad \vdash \Sigma_1, \Sigma_2, D, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, B, \Delta_2 \sqcup \Sigma_2, C \otimes D, \Delta_3 \sqcup \Sigma_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, C \otimes D, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes \quad =_\sigma$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \vdash \Delta_1, B, \Delta_2, C, \Delta_3}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2, C, \Gamma_3 \sqcup \Delta_3}\otimes \qquad \vdash \Sigma_1, \Sigma_2, D, \Sigma_3}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, C \otimes D, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Delta_1, \Delta_2, C, \Delta_3 \qquad \vdash \Sigma_1, B, \Sigma_2, D, \Sigma_3}{\vdash \Delta_1 \sqcup \Sigma_1, B, \Delta_2 \sqcup \Sigma_2, C \otimes D, \Delta_3 \sqcup \Sigma_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, C \otimes D, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes \quad =_\sigma$$

$$\dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \dfrac{\vdash \Gamma_1, A, \Gamma_2, \Gamma_3 \qquad \vdash \Sigma_1, B, \Sigma_2, D, \Sigma_3}{\vdash \Gamma_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Sigma_2, D, \Gamma_3 \sqcup \Sigma_3}\otimes}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2, C \otimes D, \Gamma_3 \sqcup \Delta_3 \sqcup \Sigma_3}\otimes$$

$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2 \qquad \vdash \Delta_1, B, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2}\otimes \qquad \vdash \Sigma_1, \Sigma_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\,mix_2 \quad =_\sigma \quad \dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2 \qquad \vdash \Sigma_1, \Sigma_2}{\vdash \Gamma_1 \sqcup \Sigma_1, A, \Gamma_2 \sqcup \Sigma_2}\,mix_2 \qquad \vdash \Delta_1, B, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\otimes$$

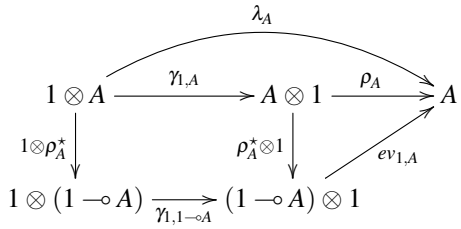
$$\dfrac{\dfrac{\vdash \Gamma_1, A, \Gamma_2 \qquad \vdash \Delta_1, B, \Delta_2}{\vdash \Gamma_1 \sqcup \Delta_1, A \otimes B, \Gamma_2 \sqcup \Delta_2}\otimes \qquad \vdash \Sigma_1, \Sigma_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\,mix_2 \quad =_\sigma \quad \dfrac{\vdash \Gamma_1, A, \Gamma_2 \qquad \dfrac{\vdash \Delta_1, B, \Delta_2 \qquad \vdash \Sigma_1, \Sigma_2}{\vdash \Delta_1 \sqcup \Sigma_1, B, \Delta_2 \sqcup \Sigma_2}\,mix_2}{\vdash \Gamma_1 \sqcup \Delta_1 \sqcup \Sigma_1, A \otimes B, \Gamma_2 \sqcup \Delta_2 \sqcup \Sigma_2}\otimes$$

Appendix E. Auxiliary Categorical Results

Lemma 52. In any symmetric monoidal closed category, we have:

$$\lambda_A = 1 \otimes \rho_A^\star \,;\, \gamma_{1,1\multimap A} \,;\, ev_{1,A} : 1 \otimes A \longrightarrow A$$

Proof.

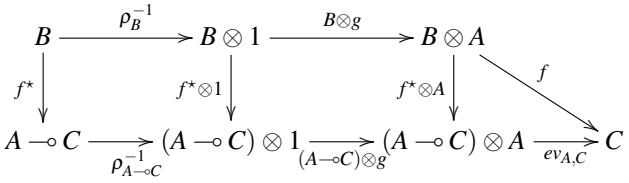


The top triangle comes from the axiom of symmetry in symmetric monoidal categories. The square is naturality of symmetry. The bottom-right triangle comes from the definition of curryfication. □

Lemma 53 (Partial Evaluation)**.** Given a symmetric monoidal closed category, A, B and C three objects, and $f : B \otimes A \longrightarrow C$ and $g : 1 \longrightarrow A$ two morphisms, we have:

$$\rho_B^{-1} ; B \otimes g ; f = f^\star ; \rho_{A \multimap C}^{-1} ; (A \multimap C) \otimes g ; ev_{A,C} : B \longrightarrow C$$

Proof.



The first square is naturality of ρ^{-1}. The second square is bifunctoriality of $\otimes$. The triangle comes from the definition of curryfication. □