

Rocq the Sequents

a Rocq-band talk

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<https://github.com/olaure01/rocq-sequents>

Bounded Lattices = IALL

Formulas and Sequents

$$A ::= X \mid A \wedge A \mid \top \mid A \vee A \mid \perp \qquad A \vdash B$$

Rules [Whitman 41]

$$\frac{}{X \vdash X}$$

$$\frac{C \vdash A \quad C \vdash B}{C \vdash A \wedge B}$$

$$\frac{A \vdash C}{A \wedge B \vdash C}$$

$$\frac{B \vdash C}{A \wedge B \vdash C}$$

$$\frac{}{C \vdash \top}$$

$$\frac{C \vdash A}{C \vdash A \vee B}$$

$$\frac{C \vdash B}{C \vdash A \vee B}$$

$$\frac{A \vdash C \quad B \vdash C}{A \vee B \vdash C}$$

$$\frac{}{\perp \vdash C}$$

Admissible Rules

$$\frac{}{A \vdash A}$$

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

```
Inductive iall :  
  formula -> formula -> Prop := ...
```

Rocq it ... in `Prop`

```
Inductive iall :  
  formula -> formula -> Prop := ...
```

```
Inductive iall_s :  
  nat -> formula -> formula -> Prop := ...
```

Rocq it again ... in `Type`

```
Inductive iall :  
  formula -> formula -> Type := ...
```

Rocq it again ... in `Type`

```
Inductive iall :  
  formula -> formula -> Type := ...
```

```
Fixpoint pweight A B (pi : ⊢ A, B) := S  
match pi with  
| ax | t => 0  
| w pi1 pi2 => max (pweight pi1) (pweight pi2)  
| ex pi1 | v1 pi1 | v2 pi1 => pweight pi1  
end.
```

Involutive Lattices = ALL

Definition

$$\neg\neg A = A$$

$$A \leq B \implies \neg B \leq \neg A$$

Formulas and Sequents

$$A ::= X \mid \bar{X} \mid A \wedge A \mid \top \mid A \vee A \mid \perp \quad \vdash A, B$$

$$\neg X = \bar{X} \quad \neg \bar{X} = X \quad \neg(A \wedge B) = \neg A \vee \neg B \quad \neg \top = \perp \quad \dots$$

Rules

$$\frac{}{\vdash \bar{X}, X} \quad \frac{\vdash A, B}{\vdash B, A}$$

$$\frac{\vdash A, C \quad \vdash B, C}{\vdash A \wedge B, C}$$

$$\frac{}{\vdash \top, C}$$

$$\frac{\vdash A, C}{\vdash A \vee B, C}$$

$$\frac{\vdash B, C}{\vdash A \vee B, C}$$

Admissible Rules

$$\frac{}{\vdash \neg A, A}$$

$$\frac{\vdash A, \neg B \quad \vdash B, C}{\vdash A, C}$$

```
Inductive formula :=  
| var (_ : bool) (_ : Atom)  
| bin (_ : bool) (_ _ : formula)  
| nul (_ : bool).
```

```
Fixpoint neg A :=  
match A with  
| var b X => var (negb b) X  
| bin b B C => bin (negb b) (neg B) (neg C)  
| nul b => nul (negb b)  
end.
```

OrthoLattices = OL

Definition

$$A \vee \neg A = \top$$

Formulas and Sequents

$$A ::= X \mid \bar{X} \mid A \wedge A \mid \top \mid A \vee A \mid \perp \quad \vdash A, B$$
$$\neg X = \bar{X} \quad \neg \bar{X} = X \quad \neg(A \wedge B) = \neg A \vee \neg B \quad \neg \top = \perp \quad \dots$$

Rules

$$\frac{}{\vdash \bar{X}, X} \quad \frac{\vdash A, B}{\vdash B, A} \quad \frac{\vdash A, A}{\vdash A, C}$$
$$\frac{\vdash A, C \quad \vdash B, C}{\vdash A \wedge B, C} \quad \frac{}{\vdash \top, C} \quad \frac{\vdash A, C}{\vdash A \vee B, C} \quad \frac{\vdash B, C}{\vdash A \vee B, C}$$

Admissible Rules

$$\frac{}{\frac{}{\vdash \neg A, A}} \quad \frac{\vdash A, \neg B \quad \vdash B, C}{\vdash A, C}$$

... later ...

Residuated Lattices = Lambek Calculus

Formulas and Sequents

$$A ::= X \mid A \wedge A \mid \top \mid A \vee A \mid \perp \mid A \otimes A \mid 1 \mid A \multimap A \mid A \circ A$$
$$\Gamma \vdash A$$

Rules (1/2)

$$\overline{X \vdash X}$$
$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$
$$\frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C}$$
$$\frac{\Gamma, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C}$$
$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$
$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$
$$\frac{\Gamma, A, \Delta \vdash C \quad \Gamma, B, \Delta \vdash C}{\Gamma, A \vee B, \Delta \vdash C}$$
$$\overline{\Gamma \vdash \top}$$
$$\overline{\Gamma, \perp, \Delta \vdash C}$$

Residuated Lattices = Lambek Calculus

Rules (2/2)

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C}$$

$$\frac{}{\vdash 1} \quad \frac{\Gamma, \Delta \vdash C}{\Gamma, 1, \Delta \vdash C}$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}$$

$$\frac{\Delta \vdash A \quad \Gamma, B, \Sigma \vdash C}{\Gamma, \Delta, A \multimap B, \Sigma \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B \multimap A}$$

$$\frac{\Delta \vdash A \quad \Gamma, B, \Sigma \vdash C}{\Gamma, B \multimap A, \Delta, \Sigma \vdash C}$$

Admissible Rules

$$\frac{}{A \vdash A}$$

$$\frac{\Delta \vdash A \quad \Gamma, A, \Sigma \vdash B}{\Gamma, \Delta, \Sigma \vdash B}$$

Two approaches to cut admissibility:

- double induction using also size of cut formula;
- constraint on the size of the output proof.

Required because of the multiplicative connectives: *stacks of cuts*.

$$\begin{array}{c}
 \frac{\frac{\frac{\pi_1}{\Delta \vdash A} \quad \frac{\pi_2}{\Sigma \vdash B}}{\Delta, \Sigma \vdash A \otimes B} \quad \frac{\pi_3}{\Gamma, A, B, \Xi \vdash C}}{\Gamma, A \otimes B, \Xi \vdash C}}{\Gamma, \Delta, \Sigma, \Xi \vdash C} \\
 \downarrow \\
 \frac{\frac{\pi_1}{\Delta \vdash A} \quad \frac{\frac{\pi_2}{\Sigma \vdash B} \quad \frac{\pi_3}{\Gamma, A, B, \Xi \vdash C}}{\Gamma, A, \Sigma, \Xi \vdash C}}{\Gamma, \Delta, \Sigma, \Xi \vdash C}
 \end{array}$$

Commutative Residuated Lattices = IMALL

Formulas and Sequents

$$A ::= X \mid A \wedge A \mid \top \mid A \vee A \mid \perp \mid A \otimes A \mid 1 \mid A \multimap A$$
$$\Gamma \vdash A$$

Rules (1/2)

$$\frac{}{X \vdash X} \qquad \frac{\Gamma \vdash C}{\Gamma \sigma \vdash C}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C} \qquad \frac{\Gamma, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma, A, \Delta \vdash C \quad \Gamma, B, \Delta \vdash C}{\Gamma, A \vee B, \Delta \vdash C}$$

$$\frac{}{\Gamma \vdash \top} \qquad \frac{}{\Gamma, \perp, \Delta \vdash C}$$

Commutative Residuated Lattices = IMALL

Rules (2/2)

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \quad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C} \quad \frac{}{\vdash 1} \quad \frac{\Gamma, \Delta \vdash C}{\Gamma, 1, \Delta \vdash C}$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B} \quad \frac{\Delta \vdash A \quad \Gamma, B, \Sigma \vdash C}{\Gamma, \Delta, A \multimap B, \Sigma \vdash C}$$

Admissible Rules

$$\frac{}{A \vdash A}$$

$$\frac{\Delta \vdash A \quad \Gamma, A, \Sigma \vdash B}{\Gamma, \Delta, \Sigma \vdash B}$$

Dancing with permutations ... in `Type`

```
Inductive PermutationT : crelation (list A):= ...
```

```
Instance imall_PermutationT :
```

```
  Proper (@PermutationT _ ==> eq ==> iffT) imall.
```

Heyting Algebras = LJ

Formulas and Sequents

$$A ::= X \mid A \wedge A \mid \top \mid A \vee A \mid \perp \mid A \rightarrow A$$

$$\Gamma \vdash A$$

Rules (1/2)

$$\frac{}{X \vdash X}$$

$$\frac{\Gamma \vdash C}{\Gamma \sigma \vdash C}$$

$$\frac{\Gamma, A, A, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}$$

$$\frac{\Gamma, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C}$$

$$\frac{\Gamma, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$\frac{\Gamma, A, \Delta \vdash C \quad \Gamma, B, \Delta \vdash C}{\Gamma, A \vee B, \Delta \vdash C}$$

$$\frac{}{\Gamma \vdash \top}$$

$$\frac{}{\Gamma, \perp, \Delta \vdash C}$$

Heyting Algebras = LJ

Rules (2/2)

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \quad \frac{\Delta \vdash A \quad \Gamma, B, \Sigma \vdash C}{\Gamma, \Delta, A \rightarrow B, \Sigma \vdash C}$$

Admissible Rules

$$\frac{\Gamma, \Delta, \dots, \Delta \vdash C}{\Gamma, \Delta \vdash C} \quad \frac{}{A \vdash A} \quad \frac{\Delta \vdash A \quad \Gamma, A, \Sigma \vdash B}{\Gamma, \Delta, \Sigma \vdash B}$$

Multi-cut up to permutation:

$$\frac{\Gamma \vdash A \quad \Delta \vdash B \quad \Delta \sim A, \dots, A, \Delta'}{\Gamma, \Delta' \vdash B}$$

Definably-Complete Meet-Semi-Lattices = I&LL₂

Formulas and Sequents

$$A ::= X \mid A \wedge A \mid \top \mid \forall X.A \qquad A \vdash B$$

Rules

$$\frac{}{X \vdash X}$$
$$\frac{C \vdash A \quad C \vdash B}{C \vdash A \wedge B} \qquad \frac{A \vdash C}{A \wedge B \vdash C} \qquad \frac{B \vdash C}{A \wedge B \vdash C} \qquad \frac{}{C \vdash \top}$$
$$\frac{C \vdash A}{C \vdash \forall X.A} \quad X \notin C \qquad \frac{A[B/x] \vdash C}{\forall X.A \vdash C}$$

Admissible Rules

$$\frac{}{\frac{}{A \vdash A}} \qquad \frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

Anti-locally-nameless representation:

$$\frac{C\uparrow \vdash A\uparrow[0/x]}{C \vdash \forall X.A} \quad \frac{A[B/x] \vdash C}{\forall X.A \vdash C} \text{ } B \text{ closed}$$

Anti-locally-nameless representation:

$$\frac{C \uparrow \vdash A \uparrow [0/x]}{C \vdash \forall X.A} \quad \frac{A [B/x] \vdash C}{\forall X.A \vdash C} \quad B \text{ closed}$$

Cut reduction:

$$\frac{\frac{\pi_1}{C \uparrow \vdash A \uparrow [0/x]} \quad \frac{\pi_2}{A [B/x] \vdash D}}{C \vdash \forall X.A \quad \forall X.A \vdash D} \quad B \text{ closed}}{C \vdash D}$$

$$\downarrow$$

$$\frac{\pi_1 [B/\downarrow_0] \quad \pi_2}{C \vdash A [B/x] \quad A [B/x] \vdash D}}{C \vdash D}$$
