HW III: All of the basics (due Feb 28th, before tutorial)

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triple and let S be a set and for every $s \in S$, $A_s \in \mathcal{F}$ is an event with $\mathbf{P}\{A_s\} = 1$. Moreover, assume that $\cap_{s\in S} A_s \in \mathcal{F}$. Does it imply that $\mathbf{P}\{\cap_{s\in S} A_s\} = 1$? Answer this question (with justification) in the three cases: S is finite, S is countable and $S = [0, 1]$.

2. Suppose you have a device that generates random bits that are guaranteed to be independent and have the same $\mathsf{Bernouilli}(p)$ distribution, except that you do not know the value of $p \in]0, 1[$. Design an algorithm that uses this source to produce a uniform bit and analyze the expected number of uses of the device that are needed to generate one uniform bit.

   Now suppose I want to generate n random bits using this strategy. I want to make sure that the probability of using the device more than tn times is at most $\frac{1}{100}$. Give a value of t for which this is the case (of course, you should try to make it as small as you can).

3. Recall the coupon collector problem. Let X be the number of boxes that are bought before having at least one of each coupon. Show that

$$\mathbf{P}\{X \geq n \ln n + cn\} \leq e^{-c} .$$

   In class we proved a similar bound using Chebychev's inequality. Here you are asked to prove this better bound in an elementary way.