**HW IV: Chernoff** (due March 7th before tutorial)

1. Let $X$ be an arbitrary random variable with $0 \leq X \leq 1$ and $\mathbf{E}\{X\} = p$. Consider the random variable $Y \in \{0, 1\}$ with $\mathbf{P}\{Y = 1\} = p$. Show that for any $\lambda > 0$, $\mathbf{E}\left\{e^{\lambda X}\right\} \leq \mathbf{E}\left\{e^{\lambda Y}\right\}$.

   Using this fact, show that the Chernoff bound we saw in class still holds if we replace the condition $X_i \in \{0, 1\}$ by $X_i \in [0, 1]$.

2. Suppose you are given a randomized polynomial-time algorithm $\mathcal{A}$ for deciding whether $x \in \{0, 1\}^*$ is in the language $L$ or not. Suppose it has the following property. If $x \in L$, then $\mathbf{P}\{\mathcal{A}(x) = 0\} \leq 1/4$ and if $x \notin L$, then $\mathbf{P}\{\mathcal{A}(x) = 1\} \leq 1/3$. Note that the probability here is taken over the randomness used by the algorithm $\mathcal{A}$ and *not* over the input $x$. Construct a randomized polynomial-time algorithm $\mathcal{B}$ that is allowed to make independent calls to $\mathcal{A}$ such that for all inputs $x \in \{0, 1\}^*$, we have $\mathbf{P}\{\mathcal{B}(x) = \mathbf{1}_{x \in L}\} \geq 1 - 2^{-|x|}$. Here $\mathbf{1}_{x \in L} = 1$ if $x \in L$ and $0$ otherwise, and $|x|$ denotes the length of the bitstring $x$.