HW IX: Markov chain mixing time (due May 9th before tutorial)

Let P be the transition matrix of a Markov chain on a finite state space S, with stationary distribution $\{\pi_i\}_{i\in S}$. Our objective here is to give a method to obtain bounds on the time for the Markov chain to converge to the stationary distribution π.

1. First, we need to define a distance between probability distributions. For two distributions μ and ν on S, we define
$$\|\mu - \nu\|_{TV} = \max_{A\subset S} |\mu(A) - \nu(A)| \ .$$
Prove that $\|\mu - \nu\|_{TV} = \frac{1}{2}\sum_{i\in S} |\mu(i) - \nu(i)|$.

2. Let X, Y be random variables taking values in S (and living in the same probability space) such that X has distribution μ and Y has distribution ν. Show that $\|\mu - \nu\|_{TV} \le \mathbf{P}\{X \neq Y\}$.

   Bonus: Prove that for any μ and ν, there exists a coupling of X and Y, i.e., a joint probability space, such that $X \sim \mu$ and $Y \sim \nu$ and we have $\|\mu - \nu\|_{TV} = \mathbf{P}\{X \neq Y\}$.

3. The convergence theorem we showed (or will show) in class amounts to saying that for any starting state i, we have $\|P^n_{i,\cdot} - \pi\|_{TV} \to 0$ as $n \to \infty$. Here $P^n_{i,\cdot}$ is the probability distribution of the state of the chain at time n starting at state i. We now use question 2 to obtain a bound on the mixing time of a specific Markov chain.

   We consider a Markov chain P with state space $\mathcal{S}_n$ the set of permutations of $\{1,\ldots,n\}$. When we are at state σ, we choose $i \in \{1,\ldots,n\}$ uniformly at random and the next state is given by $\sigma' = \sigma \cdot (i \ i-1 \ldots 1)$, i.e., the composition of the permutation σ and the cycle permutation $1 \mapsto 2, 2 \mapsto 3, \ldots, i \mapsto 1$. In words, think of $1,\ldots,n$ as cards arranged in some order given by σ ($\sigma(i)$ is the number of the i-th card) and we shuffle the set of cards by taking the top card and mapping it to a random position i, getting a new arrangement of cards given by σ'.

   (a) Show that the uniform distribution u over $\mathcal{S}_n$ is a stationary distribution.

   (b) Let $\{X_t\}$ and $\{Y_t\}$ be Markov chains with transition matrix P starting at $X_0 = \sigma$ and Y_0 having distribution u. Show that
   $$\|P^t_{\sigma,\cdot} - u\|_{TV} \le \mathbf{P}\{X_t \neq Y_t\} \ .$$

   (c) Choose a coupling, i.e., a joint probability space for $\{X_t\}$ and $\{Y_t\}$ in such a way to have $\mathbf{P}\{X_t \neq Y_t\} \le \frac{1}{10}$ for $t = O(n \log n)$.