Moore and Seiberg equations, Topological Field Theories, and Galois Theory

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Abstract

The following text is an outline of a review paper on Moore and Seiberg's equations, Topological (Projective) Field Theories in three dimensions and their relationship with Grothendieck's second paragraph of the "Esquisse d'un Programme". Because of schedule and length problems this review paper has not been included in this volume[1].

First of all, we recall the construction of projective topological field theories in three dimensions from solutions to Moore and Seiberg's equations. We discuss the possible relation between this result and the reconstruction conjecture of the Teichműller tower from its two first floors. Finally, we suggest an explicit translation of the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into an action on a wide class or three dimensional topological field theories arising from rational conformal field theories in two dimensions.

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1 Summary

Our aim is to point out some relationships between recent developments in Topological Field Theories, the classification program of Rational Conformal Field Theories and deep ideas expressed by A. Grothendieck in the “Esquisse d’un Programme” [15].

First of all, we would like to stress that our present knowledge does not pretend to be a definitive and complete mathematical theory since most of this wonderful story is still to be discovered. We would like to point out why, in our opinion, there is a deep connection between the world of Rational Conformal Field Theory and Grothendieck’s one. In the end, the best advice we can give to the reader is to read the wonderful text by Grothendieck [15] and make up his own mind. Moreover, the present text is nothing else than a detailed introduction to theses matters. We refer the reader to [6] in order to obtain a more detailed account of the subject. For convenience, the contents of [6] are described in section 2.

Conformal Field Theory

Conformal field theory was originally studied for a systematic description of isotropic universality classes in two dimensions [2]. A few years after their discovery, it became apparent that these theories were a prototype for the so-called geometrical quantum field theories [27][1]. A special class of them, called Rational Conformal Field Theories (RCFT), attracted special attention during the late eighties. It turned out that RCFTs provided very interesting representations of various modular groups. This was discovered firstly in genus one [3], and then in genus zero [29]. The important discovery of Verlinde [30] drew attention to this structure. Moore and Seiberg then produced an important synthesis of this subject [21][23][22]. In this work, they showed the importance of a few matrices associated with each Rational conformal field theory. These matrices have to satisfy polynomial equations, called the Moore and Seiberg’s equations. It must be mentioned that these matrices can be computed as monodromy matrices of some holomorphic multivalued functions on moduli space: see [3] for the genus one case and [8][9] for the some examples in genus zero. In passing, one notices that the Moore and Seiberg matrices represent endomorphisms of spaces associated with the following values of (g, n):

$$(0,3)\ (0,4)\ (1,1)$$

and that Moore and Seiberg’s equations involve endomorphisms of spaces associated with

$$(0,5)\ (1,2).$$

Other authors [14][25][11] also discovered independently the same structure but in a completely different context.

Topological Field Theories in three dimensions

At the same time, Witten discovered from the point of view of Chern-Simons theory, a deep connection between Moore and Seiberg’s data associated with any RCFT and three-dimensional topological theories [31]. More precisely, Chern-Simons theory associated with a compact, connected, Lie group G can be "solved"[2] using Moore and Seiberg's data associated with the Wess-Zumino-Witten model based on G. This mapping has been made more precise by many authors, for example [12][13][7]. It also became clear that Moore and Seiberg's equations could be obtained from the requirement of topological invariance [31][24]. In fact, this result can be proved partially: one has to impose a few hypotheses and to consider only *non projective* topological field theories. In this case, only solutions to Moore and Seiberg's equations with $c \equiv 0 \pmod 8$ can be recovered (see [5, Chapter 5]). On the other hand, it was expected that one could reconstruct a 3D TFT from any solution to Moore and Seiberg's equations. For example, topological invariants were defined by Kontsevitch in the case of undecorated closed manifolds [18] and also by Crane using Heegaard decompositions. The latter technique was used also by Kohno [17] with some explicit solutions to Moore and Seiberg's equation coming from the WZW model based on $SU(2)$. It was shown in [4] how to reconstruct a projective topological field theory from any solution to Moore and Seiberg's equations.

In a slightly different context, Reshetikhin and Turaev [26] defined Topological Field Theories (TFT) using Kirby's calculus and quantum groups. The quantum group is an example of a modular Hopf algebra, the representation theory of which provides us with a solution to Moore and Seiberg's equations. Other works were also based on the same point of view: [19][20].

Grothendieck's "Teichmüller tower"

Besides this, already, widely spread work, Grothendieck developed between 1981 and 1985 an extremely ambitious research program summarized in [15]. One of the main proposal of this program was to develop a new understanding of the absolute Galois group of the field $\mathbb{Q}$ (*i.e.* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) by interpreting it as a group of transformations of an appropriate combinatorial object. The third paragraph of [15] explains how this group acts on the set of all "children's drawings" which are widely discussed in this volume. This is a first combinatorial approach to this description of the Galois group.

On the other hand, the second paragraph suggests that one should consider an important notion, called the Teichmüller tower. It is formed by the system of all moduli spaces[3] $\mathcal{M}_{g,n}$ of Riemann surfaces of any genus and with any number of punctures, together with a few fundamental operations such as the "sewing of surfaces", the "forgetting of marked points" and so on... As explained by Grothendieck, all this structure is reflected on suitable families of fundamental groupoids (with respects to suitable families of base points).

Two fundamental conjectures appear in [15, Paragraph 2]:

- The reconstruction conjecture: the whole structure of the tower can be

[2] That is to say, any partition function, or any correlation function of any observable can be explicitly computed.

[3] In algebraic geometry, the relevant concept is the one of algebraic multiplicity which are not schemes but algebraic stacks. This is due to the existence of Riemann surfaces with non generic automorphism group.

reconstructed from the two first floors (the floors are indexed by $3g-3+n$, which is the complex dimension of the corresponding moduli space). The first floor provides a "system of generators" and the second one, a "system of relations". This gives the following values of (g, n):

$$\begin{cases} \text{Generators}: & (0,4)\ (1,1) \\ \text{Relations}: & (0,5)\ (1,2) \end{cases}$$

- The Galois action conjecture: The structure of the tower is rigid enough for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to act on its profinite completion, preserving all relations between the corresponding profinite groupoids.

  Grothendieck then suggested that one should parametrize each element of the Galois group by one or several elements of the profinite completion of the free group with two generators[4], subjected to certain relations. It is extremely important to find necessary and sufficient conditions conditions for such elements to arise from the action of the absolute Galois group.

To our knowledge, these results remain conjectural, although some evidence for their validity exists.

Relationships between Conformal and Topological Field Theories and the "Esquisse d'un programme"

Finally, reading the Esquisse made it clear that there is a deep relationship between Grothendieck's unpublished work and Rational Conformal Field theory. In fact, this relationship is far from being established with all the rigor and precision suitable for this subject. The central object considered by Grothendieck – *i.e.* the Teichműller tower – has, up to now, not been constructed[5]. Hence, none of its properties have been proved. Our purpose will be to explain or suggest how this story should go. A great deal of work will be necessary before this "philosophy" turns into a clean mathematical theory.

- For us, the starting point was noticing that Grothendieck's values for (g, n) in his reconstruction conjecture for the tower were exactly the values relevant in Moore and Seiberg's work. From this emerged the idea that solutions to Moore and Seiberg's equations define projective representations of the Teichműller tower. We can refine the conjecture: Grothendick pretends that the Teichműller tower can be constructed using various systems of base points. In particular, he mentioned the "small box" in which all base points arise from pants (*i.e.* surfaces of topological type $(0, 3)$). We conjecture that Moore and Seiberg's matrices represent generators of the tower between such base points.

[4] This is nothing but the algebraic fundamental group of $P_1(\mathbb{C}) \setminus \{\nvdash, \nVdash, \infty\}$ with respect to some base point, which is the moduli space for Riemann surfaces of genus zero with four ordered points on it.

[5] It is likely that various versions of the notion exist, depending on the framework – algebraic geometry, differential geometry, topology, combinatorics ... – considered...

My opinion is that Moore and Seiberg's work needs to be settled on a firmer basis. A possible way of performing this would be to define the Teichműller tower, then study its projective representations, and produce Moore and Seiberg's data from such representations. The so called completeness theorem [22, Appendix B] of Moore and Seiberg should then be the expression, in representation theory, of the reconstruction conjecture of Grothendieck [15, Paragraph 2].

Finally, starting from an axiomatic definition of a conformal field theory *à la* Segal, and an *intrinsic* definition – still to be found – of what a chiral algebra is, one should be able, first to define RCFTs, then to be able to *prove* that any RCFT should provide a projective representation of the Teichműller tower. All these steps being completed, Moore and Seiberg's work could be considered as rigorously based.

- In the "Esquisse d'un programme", Grothendieck explained that elements of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act as outer automorphisms of the tower itself. We were led to conjecture the existence of an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on solutions to Moore and Seiberg's equations, or equivalently, on three dimensional topological field theories. More precisely, since Moore and Seiberg matrices can be computed from monodromies of "conformal blocks" along paths in Poincaré's half plane or in $P_1(\mathbb{C})\setminus\{\nvdash,\nVdash,\infty\}$ between $\mathbb{Q}$-rational points, one should expect an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on these data. Our claim is that, under certain hypotheses, this action is nothing but the action of Galois coefficients of Moore and Seiberg matrices.

  For example, in genus one, the so called "conformal blocks" are nothing but the characters of the chiral algebras. These are holomorphic functions on Poincaré's half plane $\mathfrak{H}$. Their Puiseux expansion in terms of $q = \exp(2\pi i\tau)$ (where $\tau \in \mathfrak{H}$) is of the form:

$$\chi_j(q) = q^{h_j - c/24} \sum_{n\geq 0} a_j(n)\, q^n \tag{1}$$

  where each $a_j(n)$ is an integer since it is the dimension of a finite dimensional vector space. The h_j's and c are rational numbers arising from the underlying RCFT. Let me define $s.\,\chi_j$ the analytic continuation of χ_j along the path $t \in]0,1[\mapsto t \in \mathfrak{H}$ that interpolate between Deligne's tangential base point $\overrightarrow{01}$ and $\overrightarrow{10}$ (here, we have mapped $\mathfrak{H}$ onto the open unit disk using $\tau \mapsto q(\tau)$), then

$$(s.\,\chi_j) = \sum_k {S_j}^k\, \chi_k \tag{2}$$

  Let us consider $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and compute its action, in the fashion of

Y. Ihara [16]:

$$\begin{aligned}
(\sigma^{-1}.\,\chi_j)(q) &= \chi_j(q) \\
s.\,(\sigma^{-1}.\,\chi_j)(q) &= \sum_k S_j{}^k \chi_k(q) \\
\sigma.\,(s.\,[\sigma^{-1}.\,\chi_j])(q) &= \sum_k \sigma(S_j{}^k)\,\chi_k(q)
\end{aligned}$$

In this computation, the rationality of the $a_j(n)$s is used. In the end, we find that the action of σ, computed following Y. Ihara's prescription's, transforms S into $\sigma(S)$.

Similar reasoning on conformal blocks for the Riemann sphere with four marked points led us to our conjecture. We refer the reader to [6] for more details.

Of course, what remains to be done is to explore the consequences of this program for the study of three-dimensional geometry.

2 Further investigations

Extra information can be obtained from [6]. In this section, we describe the contents of this paper.

In the first section, we recall the axiomatic formulation of topological field theory in the spirit of Atiyah [1], Segal [27][28] and [7]. Our presentation is a refined version of [5, Chapter 1] and [4] suitable for dealing with other ground fields than $\mathbb{C}$. In a second section, we describe Moore and Seiberg's equations. We have tried to present this subject in a more intrinsic way than in the original papers [22]. Nevertheless, our presentation is far from being satisfactory... Nevertheless, we have tried to describe an axiomatization of the basic object handled by Moore and Seiberg, namely a certain 2-complex the vertices of which are trivalent graph with circularized vertices.

Then, we review the construction of a three-dimensional topological projective field theory [4] from solutions to Moore and Seiberg's equations. We've put the emphasis on representations of the modular groups that arise from these topological field theory. The proof of topological invariance using Kirby's calculus is also recalled.

The last section is devoted to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a certain class of topological field theories. We inform the reader that it requires some familiarity with Conformal Field Theory... As explained above, we *suggest* that the translation on 3D TFTs of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ discovered by Grothendieck [15] is nothing other than the number theory action on the matrix elements of the operators in the 3D TFT. Our reasoning is based on the computation of Moore and Seiberg's matrices from conformal blocks in RCFTs. The example of the S matrix described in the previous section contains the basic idea. As we explained before, in the case of conformal blocks for the Riemann sphere with four marked points, we have to rely on some hypotheses:

- Conformal blocks for the four-punctured sphere must be algebraic functions of the anharmonic quotient of the four points.
- Conformal blocks on the four-punctured sphere must have a Puiseux expansion near zero of a specific form: these blocks depend on the anharmonic quotient of the four points and the Puiseux expansion is assumed to have rational coefficients. We show that this hypothesis is satisfied by minimal models with respect to the Virasoro algebra or any non twisted Kac-Moody algebra associated with a finite dimensional simple Lie algebra over $\mathbb{C}$.

Let us mention that since no definition of a chiral algebra is available, we still do not know any good definition of RCFTs and therefore, we are not able to justify these hypotheses in a general framework!

Finally, we recall that such a Galois action has been considered in a slightly different context by Drinfel'd [10]. In his work, Drinfel'd described this Galois action by a pair $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F_2}$ satisfying particular conditions[6]. Equivalent results were also obtained by Y. Ihara in [16]. These approaches follow Grothendieck's insight of describing elements of the absolute Galois group by outer automorphisms of the Teichműller tower. Since a precise definition of the Teichműller tower is still lacking, our strategy will be to rely on what is conjectured to be its representation theory – that is TFTs in 3D – and to try to translate this Galois action on the tower onto its representations. The surprise is that our final result is not expressed in terms of a pair $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F_2}$. We find instead the number theory action on matrix elements of operators representing elements of the various modular groups. An important question is to understand the implications of this phenomenon. In our opinion, a (good) definition of the Teichműller tower is necessary in order to firstly formulate Grothendieck's questions in a precise way, and then secondly to understand the connection between the various approaches.

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In December 1993, I also had the opportunity to present these matters in Paris 7. I would like to thank the organizers and the audience of the seminar on Quantum Groups and Geometrical Theories for their advices and remarks.

[6]See equations (4.3), (4.4) and (4.10) of [10]

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