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Wake pattern of a boat

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Summary

A ship moving over the surface of undisturbed water sets up waves emanating from the bow and stern of the ship. The waves created by the ship consist of a V shaped wake pattern and transverse waves between the wake lines. The pattern can be explained by understanding the dispersive nature of water surface waves. These waves were first studied by Lord Kelvin but his derivations are rigorous and difficult. The following derivation is less rigorous and easier, but gives essentially the same results.

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1 Introduction

When an object, that is the source of waves (sound, light or other), moves at speeds greater than the speed of waves, a dramatic effect known as the shock wave occurs. In this case the source is actually outrunning the waves it produces. If the source is moving at the speed of waves, the wave fronts it emits pile up directly in front of it. If the source is moving faster than the waves, wave fronts pile up on one another along the sides. The different wave crests overlap one another and form a single very large crest which is named the shock wave. Behind this very large crest there is usually a very large trough. A shock wave is a result of the interference of a very large number of wave fronts. In three dimensional spaces the piled up wave fronts would form a shape of a cone, while in two dimensional spaces they would form a shape of a letter V.

When one hears of a shock wave he instantly thinks of an aircraft traveling at supersonic speed or of the bow wave of a boat traveling faster than the speed of the water waves it produces. One is correct in the first case, but wrong in the second as will be explained in this seminar. In fact explaining phenomena of a shock wave on water surface as a constructive interference of wave fronts is a general misconception found in many text books[7].

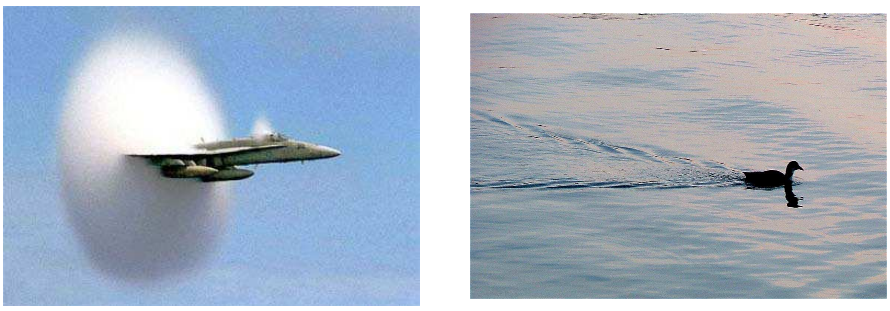

Figure 1: Condensation cloud in a low pressure region in a form of a shock wave cone[6] and a V shape wake behind a duck[8].

If we're discussing sound, the explanation of the shock wave is simple but if we're discussing waves on water surface we must take in account dispersion. The angle at the top of the V shape in this case is much smaller as it would be without this effect.[1]

The picture (2) shows a complex wake pattern of two boats traveling at uniform velocity. The pattern has two distinctive patterns. One component consists of two wake lines that together form a shape of the letter V with the boat at the point. The wake line itself is complex as it is not straight but more feathery in appearance. Individual waves that form the wake line do not propagate normal to the wake but travel more forward in the direction of the boat.

Figure 2: Wake pattern of a boat

Another interesting fact of the feathery wake line is that it has a limited width. The individual wave has a limited extension along the wake line. The entire pattern is moving at the same speed as the boat, in fact it looks like it was attached to the boat. At sufficient velocity of the boat the wake pattern is nearly independent of the boat speed. Each arm of the wake pattern makes an angle of around 19° to the boat trajectory and the featherlet waves that form the arms are at the angle of about 55° in respect to the trajectory of the boat (about 35° to the arm of the wake). Only the wavelength of the crests is dependant to the boat velocity.

The second component of the wake pattern consists of transverse waves, each of which can be approximated by an arch of a circle. They can be found in between the V shaped arms and they even extend to the outside of that area. They follow the boat with the same velocity as the boat travels so that it seems as this pattern to is attached to the boat. The radius of curvature of each crest is the same as the distance of that crest from the boat. If we name that distance L, the center of the radius of curvature for the mentioned crest lies 2L away from the boat. The wavelength of this waves is bigger than the wavelength of the waves on the V arms.

The complex pattern of the wake can be explained by taking in account the dispersive nature of the water surface waves. The group velocity of water surface waves is half the phase velocity of those waves, which itself follows from the fact that the phase velocity for a given wavelength is proportional to the square root of the wavelength if we assume deep water and neglect surface tension [1]. Lord Kelvin[2] was the first to give a sophisticated and rigorous explanation of the complex pattern. The following derivation is less rigorous

[1] Surface tension is only important for the wavelengths much shorter than we consider here.

[2] William Thompson, (26 June 1824 17 December 1907) an Irish-born British mathematical physicist and engineer.

and less difficult but gives essentially the same results.

2 Nondispersive waves

The simple wake we can observe in a nondispersive medium is produced by a point source traveling at speed greater than the phase speed of waves. In such medium all waves have uniform speed, nondependent of the wavelength. As a result the phase velocity and the group velocity are the same (sound, electromagnetic wave in vacuum).

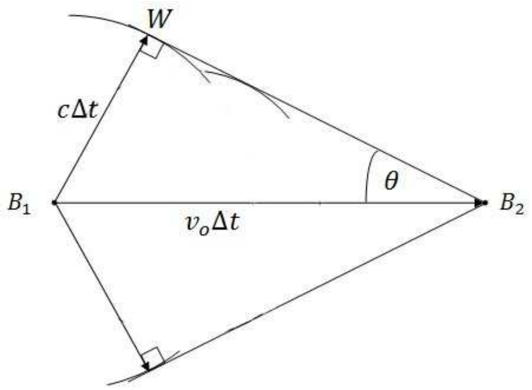

Figure 3: Wake in a nondispersive medium: The source generates a wave at point B_1 at time $t = 0$ and then travels the distance to the point B_2 in time Δt. The wave generated at B_1 at $t = 0$ in that time travels the distance $c\Delta t$ and it reaches the point W.

In three dimensional the wake is a cone and in two dimensions it is a V shaped pair of lines. If we consider the wake as a group of piled up front waves we can easily see (figure (3)) the connection between the velocity of the source and the top angle of the cone θ (or V shaped lines) as

$$c = v_0 \sin\theta, \tag{1}$$

where c is the phase velocity and v_0 the velocity of the source. If the phase velocity were equal to the velocity of the source the angle θ would be 90° and the wake would follow the source as a straight wave attached to the source. If the source velocity were less than phase velocity there would be no wake. In that case the pattern is formed of circular waves which are piled up toward the source - the smaller the wave, the closer the center of it is to the source.

3 Waves on water surface

3.1 Speed of waves in deep water

Our objective in this subsection is to derive the formula for the speed c of an ocean wave. We assume that a particle at the surface executes uniform circular motion. The centripetal acceleration of a particle in circular motion is $\omega^2 R$ and it is always directed to the center where the speed is ωR and is always tangent to the circular path. We consider a small volume of fluid on the surface where the wave is traveling to the right. At the crest of the wave, the water is moving forward, while the acceleration points downward. The velocity of the small volume on the top of the crest is $c + \omega R$. Half-wavelength to the right the particle is moving backwards while the wave still travels forwards at the same speed. The velocity of the small volume is therefore $c - \omega R$. By using Bernoulli's principle we derive an equation

$$\frac{1}{2}\rho(c + \omega R)^2 = \frac{1}{2}\rho(c - \omega R)^2 + \rho g 2R,$$

which gives the desired result

$$\omega = \frac{g}{c} = \sqrt{gk}. \tag{2}$$

The speed of the wave depends only on the acceleration of gravity g and on the frequency of the wave. The formula (2) can be written in the terms of the wavelength λ. Noting that the wavelength is the distance traveled in one period $2\pi/\omega$, the wavelength is $2\pi c/\omega$. This leads to a more familiar equation for phase velocity

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{g}{k}} \tag{3}$$

The equation (2) is only valid for the particles on the water surface. Only the top layers move in circles; the lower layers move in more and more flatter ellipses. If we wanted to take in account all that,

> we should start further back, from the equation of continuity for an incompressible fluid. Taking second derivatives of the horizontal and vertical displacements with respect to the space and time we would obtain a wave equation. The dependence of the amplitude on the vertical coordinate involves hyperbolic functions.[2]

3.1.1 Group velocity for deep water waves

Equation (2) gives the dispersion relation for deep water small amplitude gravity-dominated sinusoidal waves. Waves of greater wavelength travel faster than the waves of smaller wavelength. By differentiating equation (3) we easily found the formula for group velocity

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c \tag{4}$$

The group velocity for deep water gravity waves is half the phase velocity. As it will be explained, the entire wake pattern follows from equation (4).

3.2 Dispersion relation for water waves in general

The velocity of idealized traveling waves on the ocean is wavelength dependent and for shallow enough depths, it also depends upon the depth of the water. The wave speed is given by

$$c^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda} \tag{5}$$

Deep and shallow water waves have different dispersion relations even though they look roughly the same propagating on the water surface. The reason is the difference in their flow pattern. For deep water waves the amplitude of horizontal component of motion of the water droplets decreases exponentially with the equilibrium depth h below surface, becoming negligible for depths greater than λ.

> The amplitude is proportional to $\exp -kh$; at $h = \lambda$ it has decreased by factor $\exp -2\pi = 0,002$. [3]

On the other hand all the droplets of different equilibrium depth in shallow water move with the same horizontal motion (The friction on bottom is here neglected).
The water is deep if the depth is greater than the half of wavelength while we consider it shallow if it's 1/20 of the wavelength deep.

In deep water, the hyperbolic tangent in the expression approaches 1, so the first term in the square root determines the deep water speed. The limits on the tanh function are

$$\begin{aligned} \tanh x &\approx 1; \quad \text{for large x,} \\ \tanh x &\approx x; \quad \text{for small x,} \end{aligned}$$

so the limiting cases for the velocity expression are

$$\begin{aligned} c &= \sqrt{\frac{g\lambda}{2\pi}}; \quad \text{for deep water } h > \frac{\lambda}{2} \\ c &= \sqrt{gh}: \quad \text{for shallow water } h < \frac{\lambda}{20} \end{aligned} \tag{6}$$

While this wave speed equation may be a good approximation of the experimental wave speed, it cannot be depended upon as a precise description of the speed. It presumes an ideal fluid, level bottom, idealized wave shape, etc. It is also the speed of a progressive wave with respect to the liquid and therefore does not include any current speed of the water.

As seen by the (6) the waves in shallow water are nondespersive. The phase velocity and the group velocity are the same and the generation of a wake pattern of a ship or a duck would be easily explained by equation (1) in the 2nd section. What is the wake pattern of a boat like if the dispersion has to be taken in the account will be explained in the next section.

4 The V shaped wake pattern

To simplify the problem we consider a point source traveling at uniform velocity v_0. The source generates a succession of circular waves of broad wavelength spectrum. We consider one wavelength for which the phase velocity is given by equation (3). The phase velocities of the wavelengths we consider must all be under the velocity of the source (less than v_0) to obtain a V-shaped pattern as seen on the photo (2).

The wave train to which the considered wavelength pertains must be very long. The considered wavelength travels with the exact velocity, while those with slightly smaller wavelengths travel a bit slower, those with greater wavelengths travel a bit faster. We have not only the crest that passes through the boat as seen on picture (3) but also a large number of crests that are slightly ahead or slightly behind of the one shown on picture (3).

In every moment the boat is generating an expanding circular wave right at the boat. That crest passes through boat position and travels with the boat. This crest is called "the canonical crest"[1]. It starts at B_2 and passes through the point W. It is the wake we would have if water was a nondispersive medium. (The possibility of phase shift that might cause the canonical crest to be slightly displaced with respect to the boat is here neglected.)

We consider a narrow band of wavelengths centered on the chosen wavelength. All the waves in the band propagate in nearly the same direction; they have almost the same velocity and the same wavelength, but not totally the same. Those with slightly longer wavelength are overtaking the slower ones. The band we consider was generated at the point B_1 (figure 4) when the boat was there and it than moved towards point W. The canonical crest of the central wavelength arrives at the point W when the boat arrives at the point B_2. The crests of slightly longer or shorter wavelengths arrive at W slightly sooner or later and the result is destructive interference in point W.

The group velocity is half the phase velocity as it was shown by equation (4). The group velocity is the velocity of the location where the narrow band of wavelengths continues to give a constructive interference. The major disturbance is therefore found half way between points B_1 and W and not in W as it would be in a nondispersive medium.

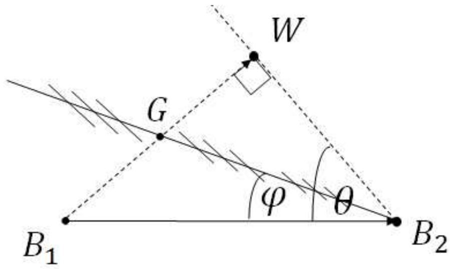

Figure 4: Wake produced by a narrow band of wavelengths. Destructive interference within the band makes all the wave crests invisible except the ones that pass the B_2G line.

The construction of the V shaped wake pattern is now simple to understand. By drawing points B_1 and B_2 the distance between boat positions is given and by the help of equation (1) we construct a canonical crest line for the correct θ angle. Half way between points B_1 and W we draw the point G (for group) which gives the location of the wave group when the boat is at point B_2. By connecting B_2 and G we get a straight line representing the wake line. All the wave crests of the central wavelength are invisible for all points between B_1 and W and become visible only on the wake line. This is shown on the figure (4).

4.1 Dominant angle of the wake pattern

Phase and group velocity are given for every wavelength (eq. 3) so each narrow band of wavelengths will give a wake crest similar to the one described in previous section. As it is obvious that the boat generates a broad specter of wavelengths as it travels forward, the question arises why there is only one dominant wake crest visible. Every wavelength corresponds to its phase and group velocity and therefore its wake crest angle. So why only one is visible?

The reader will get the answer by making a simple construction as it was made on figure (5). Draw the boat line B_1B_2 and choose several different values for θ angle. Different θ corresponds to different phase velocity, therefore to different wavelengths. It is best to choose angles 40°, 55° and 70°. Construct points W and G but be sure to obey the (1) equation. By drawing the B_2G lines you will get the φ angles that are all very similar. The biggest one we get corresponds to θ = 55° with a value around 19°.

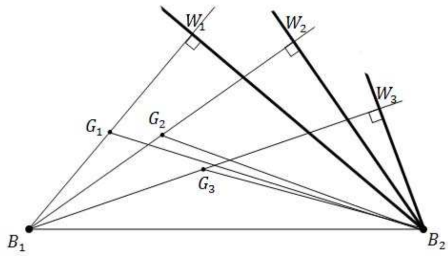

Figure 5: Wake angle construction for three different angles coresponding to three different phase velocities. The angles choosen are $\theta_1 = 40°, \theta_2 = 55°$ and $\theta_3 = 70°$

The maximum wake angle can be calculated by choosing units for the distances on figure (4) as $B_2W = 1$ and $B_1G = GW = a$. In those units the angles are

$$\theta = \arctan 2a$$

and

$$\theta - \varphi = \arctan a.$$

To find maximum value of ϕ we differentiate with respect to a and set result to zero.

$$\frac{d\varphi}{da} = \frac{2}{1+a^2} - \frac{1}{1+a^2} = 0,$$

and as result we get $a = 0,71$ which coresponds to $\varphi = 19,47°$ and $\theta = 54,74°$.

According to equation (1) the $\theta = 54,74°$ corresponds to the phase velocity of $0,82v_0$ and group velocity $0,41v_0$. The calculated wavelength λ_0 for that angle is $0,43\frac{s^2}{m}v_0^2$ (eq. (3)) In the wavelength region near λ_0 a wide band of wavelengths basically give the same wake crest angle that is near 19° and that is the wake line we see. The dominant wake crest angle does not depend on source (boat) velocity as it was demonstrated. The only dependence on boat speed is in the wavelength. The faster the boat is, the greater will be the wavelength of feathery crests on the wake line, but the angle between the V lines will stay the same.

5 The transverse wake

Between the V shaped wake lines lies another interesting pattern that consists of circular waves that seem to travel attached to the boat. These waves are generated by point disturbances the boat makes as it travels along its path. Each of those disturbances produces a circular wave. Far from the point of its origin the circular wave can be approximated by a straight line on a small section of the circle. The wave crossing the boats trajectory can be assumed as a straight wave propagating in the direction of the boat. As it can be seen that each wave follows the boat at the unchanging distance, we can easily agree that the phase velocity of those waves is the same as the speed of the boat. It looks as if the whole pattern was attached to the boat because it does not change in time. The most important contributions to the transverse wake pattern must be wavelengths in a narrow band centered around λ_1 that corresponds to the phase velocity the same as the speed of the boat. It looks as if the whole pattern was attached to the boat because it does not change in time. The most important contributions to the transverse wake pattern must be wavelengths in a narrow band centered around λ_1 that corresponds to the phase velocity the same as the speed of the boat. Those wavelengths are (again by eq. 3) given by

$$\lambda_1 = \frac{2\pi v_0^2}{g}.$$

As the boat moves along its path it sets the individual points it crosses oscillating. We consider one of this points and we name it B_1. As the wavelength is set, the frequency of the oscillating parts of water surface is also given as

$$\nu_1 = \frac{\omega}{2\pi} = \frac{v_0}{\lambda} = \frac{g}{2\pi v_0}.$$

For a boat that travels 15 km/h the calculated wavelength is 7,47 m and corresponding frequency is 0,37 Hz, meaning that B_1 makes one oscillation in 2,7 s. As the boat moves on it leaves B_1 oscillating at frequency ν_1 and emitting circular waves with the wavelength λ_1.

Later in time the boat is at B_2 that is a distance $2L$ away from its starting point in B_1. The crest that was emitted in B_1 when the boat was there, is now $2L$ away from B_1 as it has v_0 phase velocity. It is called "canonical circular wave"[1] because his arch passes B_2 when the boat is there. We have many circular waves between B_1 and circular crest propagating from B_1. Those crests have slightly shorter or longer wavelength and therefore cause destructive interference on the B_1B_2 line. The canonical crests and other crests are all invisible because of the interference within the wave band leaving the only maximum half way between those points. Because the group velocity is half of phase velocity the group of waves is found L distance away from the boat (figure 6).

> Thus we have the rule that any curved wave you see following the boat and lagging behind the boat a distance L has as its most important parent (at the moment) a point B_1 located a distance $2L$ behind the boat. Therefore the wave has radius of curvature R given by $R = L$. (In the more rigorous theory the curved waves are more complicated than arches of circles, but their curvature at the boat trajectory agrees with our result $R = L$.)[1]

The wake we see lagging a distance L behind the boat is therefore generated by an oscillating point $2L$ away from the boat where his "most important parent" can be found. As the boat moves forwards the crest is continuously being regenerated, its "parent" is now another disturbance left behind the boat.

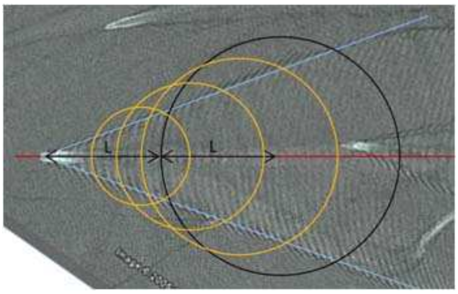

Figure 6: The waves between the V lines are circular. The curve radius of the given wave is the same as the distance of the wave from the boat.[8]

6 Comparison with experiment

In this last section I would like to compare the theory given above to the experiment. To do so, I had to find a photo where the boat and its wake were photographed from above. The second problem was finding one with the transverse pattern that was actually seen. The transverse wave crests amplitudes are much smaller than the amplitudes of the feathery crests along the wake V lines and are therefore much harder to photograph. The best photo I was able to find was one showing several boats 7 .

Figure 7: There are more than 7 wakes of different boats on this photo. The wakes in the lower right part of the photo are all consistent with the theory in this seminar, while the ones in the upper right corner are not. The reason could be the depth of the water since all the smaller wakes are found very close together.[8]

I have analyzed the biggest wake on the photograph (7). By enlarging that part and rotating it I was able to measure the angle of the V shaped wake pattern, simply by using the tan function. The calculated angle is $19,48°$(8)

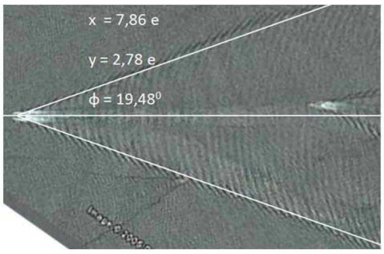

Figure 8: The analyze of the V pattern angle[8]

By choosing one wave in the transverse wake I was able to measure the distance between the wave and the boat as $L = 2,43$ (The unites here are not important) and the circle with radius L fitted nicely to the wave. I measured the chord to be 1.87 unites (C figure 9) and sagitaee as 0.15 unites (S figure 9) and by using equation

$$R = \frac{C^2}{8S} + \frac{S}{2}$$

I was able to calculate the radius of the circle to be 2.99 units.

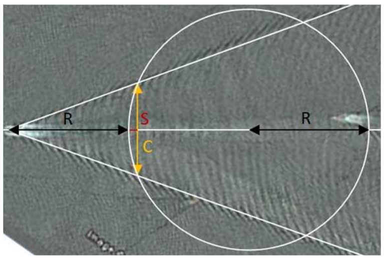

Figure 9: The analyze of the transverse pattern.[8].

The theory[1] suggests that the wavelength ratio of the wavelength of the transverse part and the wavelength on the V pattern is around 1.5. By measuring those wavelengths I was able to confirm that assumption (figure 10).

The last analyze concerns the width of the featherlet waves. The question is equivalent to the question, how many featherlet waves do you cross near G if you follow the line from B_1 to W (figure4). The number of the featherlets crossed should be $1.5\sqrt{N}$, where N is the wavelet number, starting at the boat and counting featherlets crossed as you progress back along the wake line.

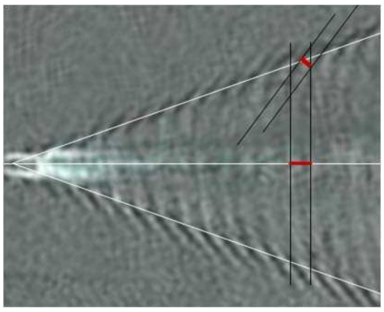

Figure 10: The analyze of the wavelengths ratio.[8]

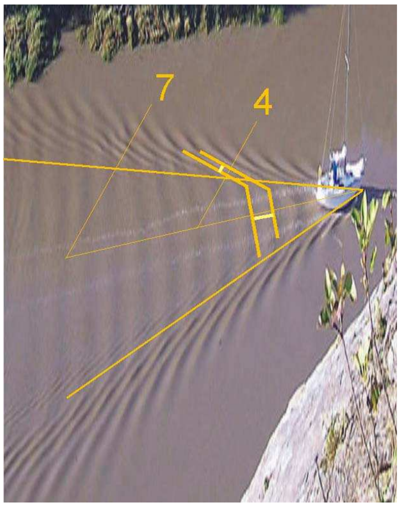

Figure 11: The analyze of the width of featherlet waves.

Counting back 6 featherlets I was able to count 4 waves along the B_1G line (theory sais 3,7) and counting back 12 featherlets i found between 6 or 7 waves (theory sais 5,2) (figure 11).

References

[1] Frank S. Crawford, **Elimentary derivation of the wake pattern of a boat** Am. J. Phys. 52(9), September 1984

[2] Stefan Machlup, **Speed of waves in deep water: elementary derivation** Am. J. Phys. 52, 1147-1148(1984)

[3] Frank S. Crawford, **A simple model for water-wave dispersion relations** Am. J. Phys., Vol. 55, No. 2, February 1987

[4] http://athome.harvard.edu/programs/stone/imageframe/04.html

[5] http://www.wikiwaves.org/index.php/Ship Kelvin Wake

[6] http://www.chrisgood.com/aircraft/

[7] Giancoli Duglas C. **Physics for scientists and engineers** Prentice Hall 3rd edition, ISBN 0-13-021517-1

[8] The photograph was found via http://images.google.si