Intrinsic nonlinear effects in vibrating strings

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The two perpendicular polarizations of transverse oscillation in stretched strings are shown to be parametrically coupled so that energy is spontaneously exchanged between the perpendicular modes. The approximate solution of the equations of motion shows that the trajectory is an ellipse with a reverse precession. The effect is commonly observed in the teaching laboratory, and may be important in the performance of musical instruments.

INTRODUCTION

The freely vibrating stretched wire is commonplace in both the scientific world and in the musical world. Every elementary course in vibrations and waves draws attention to the sinusoidal oscillation and the harmonic structure of the overtones. More detailed nonlinear effects have been observed in real musical instruments,[1,2] having their origin in the finite stiffness of the wire and in the effect of the bridges supporting the wire, leading to departures from harmonicity.

The parametric excitation of vibrations was first demonstrated by Melde and treated theoretically by Rayleigh[3]: if the tension of the wire is varied periodically at twice one of the mode frequencies of the wire, then the system is unstable and oscillations grow. The purpose of this paper is to describe the existence of intrinsic parametric excitation in the simple stretched wire, and to explain why the behavior of such wires when plucked or struck is usually not that expected on the simple linear treatment, which suggests that a steady planar motion should ensue.

Transverse vibrations are possible in two perpendicular polarizations. For a perfectly symmetrical system, these two perpendicular sets of modes will be degenerate. The degeneracy tends to be lifted in practical systems by the asymmetry of the bridge mountings. It has long been appreciated that the two perpendicular modes decay at different rates,[4,5] since the coupling to the soundboard via the bridge is different for the two polarizations. We shall first consider a string that is perfectly symmetrically supported, and we shall ignore stiffness effects, being concerned only with nonlinear effects intrinsic to the geometry of the system.

A string vibrating at frequency p in, say, the x direction must have a tension T which varies slightly at twice the frequency. This will give rise to the generation of the third harmonic, for we have a nonlinear oscillation if T is not constant. However, that oscillating tension can parametrically excite the perpendicular mode in the y direction, which will therefore grow, and eventually it is possible for most (but not all) of the energy to transfer from the initial x mode to the parametrically excited y mode. The simple stretched string should, then, properly be considered as two nonlinear oscillators parametrically coupled.

EQUATIONS OF MOTION

Consider an element of the string of length l_0 at rest (see Fig. 1) and tension T_0. When the string is displaced during oscillation, the length changes to l and the tension to $T = T_0 + YA(l - l_0)/l_0$, where Y is Young's modulus and A is the area of cross section of the wire. The work done in stretching the wire from l_0 to l_1 is

$$W = \int_{l_0}^{l_1} T\,dl = T_0(l_1 - l_0) + \frac{YA(l_1 - l_0)^2}{2l_0}. \quad (1)$$

From Fig. 1, $l_1^2 = l_0^2 + \delta r^2$, therefore, $l_1^2 - l_0^2 \approx (l_1 - l_0)2l_0 = \delta r^2$.

Therefore,

$$l_1 - l_0 = \left(\frac{\delta r}{l_0}\right)^2 \frac{l_0}{2} = \left(\frac{\partial r}{\partial z}\right)^2 \frac{\delta z}{2}, \quad (2)$$

where we have let $l_0 \to \delta z$.

Substituting (2) into (1) we obtain for the potential energy of the string

$$W = T_0 \left(\frac{\partial r}{\partial z}\right)^2 \frac{\delta z}{2} + YA\left(\frac{\partial r}{\partial z}\right)^4 \frac{\partial z}{8}, \quad (3)$$

where r is the radial displacement, hence $\delta r^2 = \delta x^2 + \delta y^2$ and

$$W = T_0\delta z \frac{1}{2}\left[\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2\right] + YA\delta z \frac{1}{8}\left[\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2\right]^2. \quad (4)$$

The kinetic energy (KE) of the element δz is

$$\mathrm{KE} = \mu\delta z(\dot{\mathbf{r}})^2/2 = \mu\delta z(\dot{x}^2 + \dot{y}^2)/2, \quad (5)$$

where μ is the linear mass density. We now let

$$x(z,t) = x(t)\sin kz,$$

$$y(z,t) = y(t)\sin kz$$

and substitute, and integrate over z, and obtain the Lagrangian for the complete string:

$$L = \left(\frac{n\pi}{4k}\right)\mu(\dot{x}^2 + \dot{y}^2) - T_0\frac{n\pi k}{4}(x^2 + y^2) \times \left(1 + \frac{1}{2}\sigma k^2(x^2 + y^2)\right), \quad (6)$$

where $\sigma = 3YA/8T_0$. Here n is the number of the harmonic mode: for the fundamental $n = 1$. We have assumed that $k_x = k_y$: parametric coupling will not occur between modes with different wave numbers.

From Lagrange's equations, we obtain the equations of motion for the string. To simplify their form, we use di-

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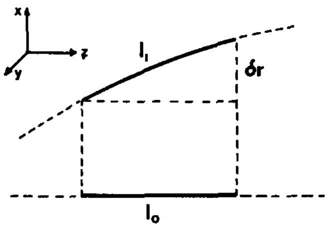

Fig. 1. Segment of the vibrating string, showing the coordinate system.

mensionless variables $X = kx$ and $Y = ky$, and put $\omega^2 = T_0k^2/\mu$:

$$\ddot{X} + \omega^2 X[1 + \sigma(X^2 + Y^2)] = 0, \quad (7)$$

$$\ddot{Y} + \omega^2 Y[1 + \sigma(X^2 + Y^2)] = 0. \quad (8)$$

These symmetrical equations describe two oscillators with cubic nonlinearities, coupled via a nonlinearity of the same order. In the limiting case of one mode predominating, i.e., $Y \ll X$, Eq. (7) reduces to

$$\ddot{X} + \omega^2 X + \sigma\omega^2 X^3 = 0, \quad (9)$$

which may be solved by standard perturbation methods to yield sinusoidal oscillations of amplitude a at the shifted frequency $p = \omega(1 + \sigma a^2/2)$ plus additional third harmonics.

Equation (8) reduces to

$$\ddot{Y} + \omega^2 Y(1 + \sigma X^2) = 0. \quad (10)$$

We let $X = a\cos pt$, being the first-order solution for X, and obtain

$$\ddot{Y} + \omega^2 Y[1 + (\sigma a^2/2)(1 + \cos 2pt)] = 0, \quad (11)$$

which is an equation of the Mathieu type, describing possible parametric excitation, but at a frequency somewhat greater than ω. The symmetry of the system leads us to expect that the Y motion be frequency shifted by the same amount as the X motion since the average string tension will be the same for each mode.

We note by inspection that the simplest exact solutions of equations (7) and (8) are circular motions: let $X = a\cos pt$ and $Y = a\sin pt$. Each equation yields

$$p^2 = \omega^2(1 + \sigma a^2). \quad (12)$$

The surprising result will appear that the general motion of the vibrating string is, to good approximation, a linear sum of two circular motions with different amplitudes and frequencies. The effect of the nonlinearity is to make those frequencies different from those given by Eq. (12).

APPROXIMATE SOLUTION

Mathematically, the system is now similar to the spherical pendulum described by Olson,[6] who gives an elegant treatment which will be followed here.

Observation of real strings shows that the motion essentially follows an elliptical figure precessing about its axis of symmetry. We therefore transform the Lagrangian (6) into a frame rotating at angular velocity Ω:

$$X = u\cos\Omega t - v\sin\Omega t$$
$$Y = u\sin\Omega t + v\cos\Omega t \quad (13)$$

where u, v are the orthogonal displacements of the string in the rotating frame (Fig. 2). Simple substitution yields the new Lagrangian

$$L' = (n\pi\mu/4k^3)[\dot{u}^2 + \dot{v}^2 + 2\Omega(u\dot{v} - \dot{u}v) + \Omega^2(u^2 + v^2) - \omega^2(u^2 + v^2) - \omega^2\sigma(u^2 + v^2)^2/2]. \quad (14)$$

The equations of motion follow:

$$\ddot{u} + \omega^2 u[1 - \Omega^2/\omega^2 + \sigma(u^2 + v^2)] - 2\Omega\dot{v} = 0, \quad (15)$$

$$\ddot{v} + \omega^2 v[1 - \Omega^2/\omega^2 + \sigma(u^2 + v^2)] + 2\Omega\dot{u} = 0. \quad (16)$$

Equation (15) can be written

$$\ddot{u} + p^2 u = (p^2 - \omega^2 + \Omega^2)u + 2\Omega\dot{v} - \omega^2\sigma u(u^2 + v^2) \quad (17)$$

We expect elliptical solutions: $u = a\cos pt$, $v = b\sin pt$. Substituting in the right-hand side of (17) and performing the trigonometric separation, we obtain

$$\ddot{u} + p^2 u = [a(p^2 - \omega^2 + \Omega^2) + 2\Omega bp - 3\omega^2\sigma a^3/4 - \omega^2\sigma ab^2/4]\cos pt + (\omega^2\sigma a/4)(b^2 - a^2)\cos 3pt. \quad (18)$$

For a steady-state solution to exist at frequency p, the coefficient of $\cos pt$ must be zero. We may also neglect the Ω^2 term, and assume $p \approx \omega$, writing $(p^2 - \omega^2) \approx 2\omega\delta\omega$, where $\delta\omega = p - \omega$. We thus obtain the following condition for a steady-state solution, and a corresponding condition similarly derived from Eq. (16):

$$\begin{aligned} 2b\delta\omega + 2\Omega a - 3\omega\sigma b^3/4 - \omega\sigma a^2 b/4 = 0, \\ 2a\delta\omega + 2\Omega b - 3\omega\sigma a^3/4 - \omega\sigma ab^2/4 = 0. \end{aligned} \quad (19)$$

These solve simultaneously to give expressions for the frequency shift $\delta\omega$ in the rotating frame, and the precession frequency Ω:

$$\delta\omega = 3\omega\sigma(a^2 + b^2)/8, \qquad \Omega = -\omega\sigma ab/4. \quad (20)$$

We note that for $a = b$ we have circular motion, and the frequency shift in the laboratory frame is $\delta\omega + \Omega = \omega\sigma a^2/2$ in agreement with the exact solution.

DISCUSSION

We now see that a strictly planar oscillation of the wire, with $b = 0$, remains planar since $\Omega = 0$; we observe simply a frequency shift due to the finite amplitude a. However, a small initial transverse component b, in quadrature, causes a slight ellipticity of motion which then precesses, thus ultimately transferring the energy substantially from the initial X mode to the Y mode and then back again. (We have, of course, neglected dissipative damping.) The rate of precession is proportional to the area of the elliptical trajectory. The precessive motion is the manifestation of the parametric excitation of the perpendicular mode. A finite amplitude b is required, since all parametric excitations require "seeding." Symmetry demands, of course, that once the motion is predominantly in the Y mode, parametric excitation of the X mode will follow; the process continues indefinitely.

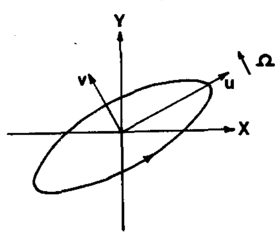

Fig. 2. Elliptical trajectory, showing the two-coordinate systems.

Transforming the elliptical motion back into the laboratory frame, we obtain

$$X = (a+b)/2[\cos(p+\Omega)t] + (a-b)/2[\cos(p-\Omega)t],$$
$$Y = (a+b)/2[\sin(p+\Omega)t] - (a-b)/2[\sin(p-\Omega)t], \quad (21)$$

which represents a linear superposition of two circular motions in opposite senses at frequencies $p+\Omega$, $p-\Omega$. The frequencies and amplitudes, however, do not correspond in the manner of Eq. (12).

NUMERICAL EXAMPLE

If l is the length of the wire, ρ the density of the material and $\nu = \omega/2\pi$, we may write $\sigma = 3YA/8T_0 = 3Y/(32l^2\rho\nu^2)$. For $l = 1$ m, $Y = 210 \times 10^9\ \mathrm{Nm^{-2}}$, $\rho = 7.8 \times 10^3\ \mathrm{kg\,m^{-3}}$ (piano wire), $\nu = 100$ Hz, we obtain $\sigma = 2.5$. If we further assume X and Y amplitudes of 10 and 1 mm, respectively, we obtain $\Omega/2\pi = \nu$ (precess) = 0.25 Hz, i.e., it will take 1 sec for the energy to transfer from being predominantly in the X mode to predominantly in the Y mode.

EFFECT OF ASYMMETRY

It is difficult to include these effects analytically. If the wire is not suspended symmetrically the X and Y motions have slightly different frequencies and the phase of the Y motion will drift with respect to that of the X motion. A nonprecessing elliptical trajectory would then possess a phase drift between two linear extremes, and describe simple Lissajous figures.

In practice two limits can be identified:

(i) If $(p_x - p_y) \ll \Omega$, the effect of asymmetry is barely noticeable due to an averaging effect. When the major axis is parallel to X (see Fig. 2) the major axis oscillation gains on the minor. After 90° precessive rotation the major axis is parallel to Y and the minor axis oscillation gains on the major, and the ellipse will revert to its original shape. The observer will see the precessive motion dominate; the precession rate Ω will oscillate slightly as the area of the ellipse varies.

(ii) If $(p_x - p_y) \gg \Omega$, the phase drift of the ellipse is rapid compared with its precessional motion. As the ellipse passes through its linear phase, when the X, Y motions are in phase, Ω reverses sign. The precessive motion therefore causes the ellipse to rock back and forth between its two linear extremes of phase. This effect can be readily observed in the conventional monochord of the teaching laboratory.

The intermediate case is difficult to visualize, but experiment indicates that it involves a critical condition in which the coordinate inversion by precession just outstrips the phase drift due to the asymmetry.

One important consequence of asymmetry is that an initially purely planar motion, such as might be expected in, for example, the struck piano string, can still lead to precessive motion and the excitation of the transverse mode.

CONCLUSION

We have shown that the intrinsic geometrical nonlinearity in the vibrating string causes its general oscillation to be precessive in nature, involving a form of parametric excitation of the mode perpendicular to that initially excited. In musical instruments these two modes are known to couple differently to the soundboard and radiating areas, having different decay rates and radiating efficiencies. This intrinsically nonlinear effect may therefore be relevant to understanding the development of the sound from stringed instruments.

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