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Can continuous-time stationary stable processes have discrete linear representations?☆

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Abstract

We show that a non-trivial continuous-time strictly α-stable, $\alpha \in (0,2)$, stationary process cannot be represented in distribution as a discrete linear process

$$\sum_{n=-\infty}^{\infty} f_t(n)\varepsilon_n, \qquad t \in \mathbb{R},$$

where $\{f_t\}_{t\in\mathbb{R}}$ is a collection of deterministic functions and $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ are independent strictly α-stable random variables. Analogous results hold for self-similar strictly α-stable processes and for strictly α-stable processes with stationary increments. As a consequence, the usual wavelet decomposition of Gaussian self-similar processes cannot be extended to the α-stable, $\alpha < 2$ case.

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1. Introduction and main results

Suppose that $X = \{X(t)\}_{t\in\mathbb{R}}$ is a *continuous-time* strictly α-stable, $\alpha \in (0,2)$, process which is *stationary*, that is, for all $h \in \mathbb{R}$, the processes $\{X(t+h)\}_{t\in\mathbb{R}}$ and $\{X(t)\}_{t\in\mathbb{R}}$ have the same

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finite-dimensional distributions. We want to know whether the process X can be represented in the sense of the finite-dimensional distributions as a linear process

$$\sum_{n=-\infty}^{\infty} f_t(n)\varepsilon_n, \quad t \in \mathbb{R}, \tag{1.1}$$

where the functions $f_t(n): \mathbb{R} \mapsto \mathbb{R}$ are measurable and deterministic for all $n \in \mathbb{Z}$ and $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ are independent strictly α-stable random variables. Recall that a random variable ξ is strictly α-stable with $\alpha \in (0,2)$ and $\alpha \neq 1$ if its characteristic function satisfies

$$E\exp\{\mathrm{i}\theta\xi\} = \exp\left\{-\sigma^\alpha|\theta|^\alpha\left(1 - \mathrm{i}\beta\,\mathrm{sign}(\theta)\tan\frac{\pi\alpha}{2}\right)\right\}, \tag{1.2}$$

for all $\theta \in \mathbb{R}$ and for some parameters $\sigma > 0$ and $\beta \in [-1,1]$, called scale and skewness parameters, respectively (see, for example, Samorodnitsky and Taqqu, 1994). When $\alpha = 1$, ξ is strictly 1-stable if $\beta = 0$ (Property 1.2.8 in Samorodnitsky and Taqqu, 1994) and, in view of (1.2), we have $E\exp\{\mathrm{i}\theta\xi\} = \exp\{-\sigma|\theta|\}$, that is, there is no shift parameter. A process $X = \{X(t)\}_{t\in\mathbb{R}}$ is called strictly α-stable with $\alpha \in (0,2)$ if all its linear combinations $\sum_{k=1}^n \theta_k X(t_k), \theta_k, t_k \in \mathbb{R}$, are strictly α-stable random variables. When $\beta = 0$ in (1.2), a strictly α-stable random variable or process is called symmetric α-stable. Let us also note that the case $\alpha = 2$, not considered here, corresponds to centered Gaussian distributions or processes.

We shall now state our main result, provide some insight about it and discuss its implications.

Theorem 1.1. *A continuous-time strictly α-stable stationary process $X = \{X(t)\}_{t\in\mathbb{R}}$ with $\alpha \in (0,2)$ has a representation* (1.1) *if and only if it is a trivial stationary process, that is, there is a strictly α-stable random variable Z such that, for all $t \in \mathbb{R}$, $X(t) = Z$ a.s.*

Theorem 1.1 is proved in Section 2. The theorem implies that if $\{X(t)\}_{t\in\mathbb{R}}$ is stationary and non-trivial, then there is at least one finite-dimensional distribution $(X(t_1),\ldots,X(t_k))$ which is different from $(\sum_n f_{t_1}(n)\varepsilon_n,\ldots,\sum_n f_{t_k}(n)\varepsilon_n)$. The proof relies fundamentally on the assumptions of continuous time and strictly α-stable, $\alpha \in (0,2)$, distributions.

Theorem 1.1 holds for $\alpha < 2$. The Gaussian case $\alpha = 2$ is very different. Gaussian stationary processes have *discrete* linear representations (1.1) under very weak assumptions. In the Gaussian case, one can use series expansions in L^2-orthogonal function bases and that Gaussian integrals over L^2-orthogonal functions are independent. Kwapień and Woyczyński (1992, p. 52), for example, provide a Karhunen–Loève representation for Gaussian processes with paths in a Banach space. Theorem 1.1 states that there is no such thing in the stable case $\alpha \in (0,2)$.

Recall that the process $X = \{X(t)\}_{t\in\mathbb{R}}$ has *stationary increments* if, for all $h \in \mathbb{R}$, the processes $\{X(t+h) - X(s+h)\}_{s,t\in\mathbb{R}}$ and $\{X(t) - X(s)\}_{s,t\in\mathbb{R}}$ have the same finite-dimensional distributions. If, for all $c > 0$ and some $H > 0$, the processes $\{X(ct)\}_{t\in\mathbb{R}}$ and $\{c^H X(t)\}_{t\in\mathbb{R}}$ have the same finite-dimensional distributions, then X is called *self-similar* (or H-self-similar). The following result, proved in Section 2, extends Theorem 1.1 to self-similar strictly stable processes and strictly stable processes with stationary increments. It follows from Theorem 1.1 by using one-to-one transformations between self-similar or stationary increments processes and stationary processes.

Corollary 1.1. *Suppose that $X = \{X(t)\}_{t\in\mathbb{R}}$ is a strictly α-stable, $\alpha \in (0,2)$, process having a representation (1.1).*

(i) *$\{X(t)\}_{t\in\mathbb{R}}$ is H-self-similar if and only if there are two strictly α-stable random variables Z_1 and Z_2 such that for $t > 0$,*

$$X(t) = t^H Z_1 \quad \text{a.s.}$$

and for $t < 0$,

$$X(t) = |t|^H Z_2 \quad \text{a.s.}$$

(ii) *Suppose also that $\int_{-\infty}^{0} \mathrm{e}^s |X(s)|\,\mathrm{d}s < \infty$ a.s. Then, $\{X(t)\}_{t\in\mathbb{R}}$ has stationary increments if and only if there is a strictly α-stable random variable Z such that, for all $t \in \mathbb{R}$,*

$$X(t) - X(0) = tZ \quad \text{a.s.}$$

(iii) *Supposing $\int_{-\infty}^{0} \mathrm{e}^s |X(s)|\,\mathrm{d}s < \infty$ a.s., $\{X(t)\}_{t\in\mathbb{R}}$ is H-self-similar and has stationary increments if and only if*

$$H = 1 \quad \textit{and} \quad X(t) = tZ \quad \text{a.s.}$$

for some strictly α-stable random variable Z.

Remark. The technical condition $\int_{-\infty}^{0} \mathrm{e}^s |X(s)|\,\mathrm{d}s < \infty$ a.s. allows the use of a one-to-one transformation between stationary increments and stationary processes. It always holds if the process $\{X(t)\}_{t\in\mathbb{R}}$ is a strictly α-stable process, continuous in probability, with stationary increments (see the bottom of page 307 in Cambanis and Maejima, 1989). In the case of stationary increments processes, continuity in probability follows from measurability (see Proposition 2.1 in Surgailis et al., 1998).

Corollary 1.1, in particular its part (iii), should be contrasted to the Gaussian case $\alpha = 2$. In the Gaussian case, self-similarity and stationarity of the increments properties characterize the distribution of a Gaussian process up to a multiplicative constant: the only H-self-similar Gaussian process with stationary increments is fractional Brownian motion. It is defined for any $H \in (0,1)$ and, when $H = \frac{1}{2}$, it coincides with the usual Brownian motion (see, for example, Samorodnitsky and Taqqu, 1994). Meyer et al. (1999) showed that fractional Brownian motion admits an almost sure expansion in a wavelet basis

$$\sum_{j,k=-\infty}^{\infty} 2^{-jH} (\Psi_H(2^j t - k) - \Psi_H(-k))\varepsilon_{j,k}, \tag{1.3}$$

where Ψ_H is some deterministic function and $\{\varepsilon_{j,k}\}$ are independent standard normal random variables, and hence that it has a representation (1.1). In the stable case $\alpha \in (0,2)$, for fixed H, there are infinitely many different H-self-similar processes with stationary increments (for examples, see Samorodnitsky and Taqqu, 1994; Pipiras and Taqqu, 2002b). Corollary 1.1 states that, under weak assumptions, none of these processes have a discrete linear representation (1.1). In particular, none of these processes will have a wavelet like expansion (1.3) where $\{\varepsilon_{j,k}\}$ are independent stable random variables.

Theorem 1.1 and Corollary 1.1 may seem discouraging, in particular as compared to the Gaussian case, because they rule out representations (1.1) which have simple structure and are easy to work with and often useful in practice. For example, wavelet expansion (1.3) provides a way to simulate a fractional Brownian motion (see Abry and Sellan, 1996). However, these results in the case $\alpha < 2$ suggest that one ought to look at alternatives. One can expect to represent, for example, a stationary or self-similar stable process as a series (1.1) but with *dependent* stable innovations, or one could still take these innovations independent and obtain approximations to the processes. Both of these alternative approaches can be found in the probabilistic literature. For example, a linear fractional stable motion which is one of the simplest stable self-similar processes with stationary increments (see Samorodnitsky and Taqqu, 1994), can be approximated as in Section 7.11 of Samorodnitsky and Taqqu (1994) by using (1.1)-type approximating sums of its integral representation. The same process is represented as the wavelet expansion series (1.3) with dependent innovations in Benassi and Roux (2000). The results of this note show that both of these approaches are natural because they are the best that one can do under the circumstances.

The rest of the note is organized as follows. In Section 2, we prove Theorem 1.1 and Corollary 1.1. The proofs of these results employ minimal integral representations of stable processes and their connections to non-singular flows. Since we work with linear processes of form (1.1), these sophisticated notions take more elementary and intuitive forms which may be of independent interest. In Section 3, we consider an alternative approach based on spectral measures which can be used when dealing with specific processes.

2. Proofs of Theorem 1.1 and Corollary 1.1

If the process $\{X(t)\}_{t\in\mathbb{R}}$ has representation (1.1), then we can represent it as

$$\{X(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_E f_t(x)M(dx)\right\}_{t\in\mathbb{R}}, \tag{2.1}$$

where $E=\mathbb{Z}$, $f_t(x)=f_t(n)$ for $x=n\in\mathbb{Z}$, M is a α-stable random measure (see Samorodnitsky and Taqqu, 1994) on E with a control measure

$$m(dx)=\sum_{n\in\mathbb{Z}}\sigma_n\delta_{\{n\}}(dx)$$

and a skewness intensity $\beta: E\mapsto[-1,1]$ defined by $\beta(x)=\sum_{n\in\mathbb{Z}}\beta_n 1_{\{x=n\}}$. The coefficients $\sigma_n>0$ and $\beta_n\in[-1,1]$ are the scale and the skewness coefficients, respectively, of independent strictly stable random variables ε_n ($\beta_n=0$ when $\alpha=1$). More generally, strictly stable processes might be defined by (2.1) on any measure space $(E,\mathscr{E},m)$ with a stable random measure M having a control measure m and a skewness intensity $\beta: E\mapsto[-1,1]$, and $\{f_t\}_{t\in\mathbb{R}}\subset L^\alpha(E,\mathscr{E},m)$. The collection $\{f_t\}_{t\in\mathbb{R}}$ is then said to be a spectral representation for the process $\{X(t)\}_{t\in\mathbb{R}}$.

In the proof of Theorem 1.1, we use the notion of a *minimal spectral representation* of a stable process. It is defined as follows (see Hardin (1982), Rosiński (1995) in the symmetric case and Rosiński (1994, 1998) in the more general strictly stable case).

Suppose that E is a subset of a Polish (that is, metric, complete and separable) space, $\mathscr{E}$ is the σ-algebra of the Borel subsets of E and m is a σ-finite measure on E. We write $A=B$ m-a.e.

if $m(A\Delta B)=0$ and say that two sub-σ-algebras of $\mathscr{E}$ are equal modulo m if their sets are equal m-a.e. Let $F=\{f_t, t\in\mathbb{R}\}$ and define supp(F), that is, the support of f_t, $t\in\mathbb{R}$, as a minimal (m-a.e.) set $A\in\mathscr{E}$ such that $m\{x\colon\ f_t(x)\neq 0,\ x\neq A\}=0$ for every $t\in\mathbb{R}$.

Definition 2.1. The spectral representation $\{f_t\}_{t\in\mathbb{R}}\subset L^\alpha(E,\mathscr{E},m)$ is called *minimal* for the process $\{X(t)\}_{t\in\mathbb{R}}$ if the following two conditions are satisfied:

$$\text{(M1)}\quad \mathrm{supp}(F)=E(m\text{-a.e.}),$$

$$\text{(M2)}\quad \rho(F)=\mathscr{E}(\text{modulo}\, m),$$

where $\rho(F)$ is the smallest σ-algebra, called ratio σ-algebra, generated by the extended-valued functions f/g with $f,g\in F$.

A more practical condition (M2′), equivalent to (M2), can be found in Theorem 3.8 of Rosiński (1998). It is also the one that we shall use. One says that a map $\phi: E\mapsto E$ is *non-singular* [1] when $m(\phi^{-1}(A))=0$ if $m(A)=0$ for $A\in\mathscr{E}$.

Definition 2.2. Condition (M2′) is said to hold if for every non-singular map ϕ: $E\mapsto E$ and a map $a: E\mapsto\mathbb{R}\backslash\{0\}$ such that, for every $t\in\mathbb{R}$,

$$f_t(\phi(x))=a(x)f_t(x)\quad m\text{-a.e.},\tag{2.2}$$

one has

$$\phi(x)=x\quad m\text{-a.e.}$$

The usefulness of minimal spectral representations will become apparent later. We now focus on processes of type (1.1) and show that one may suppose without loss of generality that the representation (1.1) is minimal. The proof below, in fact, shows how representation (1.1) can be modified to make it minimal. This modification also provides an idea of the type of redundancy that one seeks to eliminate in order to obtain a minimal representation.

Lemma 2.1. *Suppose that the process $\{X(t)\}_{t\in\mathbb{R}}$ has a representation (1.1). Then it has also a minimal representation*

$$\sum_{k\in K} g_t(k)\xi_k=\int_K g_t(k)M(\mathrm{d}k),$$

[1] In the ergodic literature, one finds two different definitions of a non-singular map $\phi: E\mapsto E$, namely,
(a) $m(A)=0$ implies $m(\phi^{-1}(A))=0$, for every $A\in\mathscr{E}$,
(b) $m(A)=0$ if and only if $m(\phi^{-1}(A))=0$, for every $A\in\mathscr{E}$.
(See, for example, P. Walters, *An Introduction to Ergodic Theory*, pp. 236–237 or K. Petersen, *Ergodic Theory*, p. 2 for (a), and U. Krengel, *Ergodic Theorems*, p. 3 for (b)). Rosiński (1998), on which property (2.2) is based, does not define non-singularity explicitly. The proofs of that paper, however, indicate that non-singularity is used in the sense (a). This was later confirmed by Jan Rosiński in a personal communication. We have, therefore, adopted definition (a) as well.
We also consider below "non-singular flows" $\{\phi_t\}_{t\in\mathbb{R}}$ (see (2.4)), but since these are invertible ($\phi_t^{-1}=\phi_{-t}$), non-singularity for the flow $\{\phi_t\}_{t\in\mathbb{R}}$, can be defined either as
(a′) $m(A)=0$ implies $m(\phi_t^{-1}(A))=0$, for every $t\in\mathbb{R}$ and $A\in\mathscr{E}$ or, equivalently, as
(b′) $m(A)=0$ if and only if $m(\phi^{-1}(A))=0$, for every $t\in\mathbb{R}$ and $A\in\mathscr{E}$.

where K is a countable set, $\{g_t(k), k \in K\}_{t\in\mathbb{R}}$ is a collection of deterministic functions, $\{\xi_k\}_{k\in K}$ is a sequence of independent strictly α-stable random variables and M is a α-stable random measure satisfying $M(\{k\}) = \xi_k$.

Proof. We assume

$$X(t) = \sum_{n=-\infty}^{\infty} f_t(n)\varepsilon_n, \quad t \in \mathbb{R}.$$

Consider the following relation on $\mathbb{Z}$: $n \sim l$ if and only if $\exists a(n,l) \neq 0$ such that $f_t(n) = a(n,l)f_t(l)$ for all $t \in \mathbb{R}$. It is clear that $\sim$ is an equivalence relation. Let K denote the set of equivalence classes and let $\mathbb{Z} = \sum_{k\in K} C_k$ be the decomposition of $\mathbb{Z}$ into the equivalence classes with respect to the relation $\sim$. Observe that the set K of equivalence classes is countable. Let us also take a representative $n_k, k \in K$, from each equivalence class C_k. It follows from the definition of C_k that

$$\left\{\sum_{n=-\infty}^{\infty} f_t(n)\varepsilon_n\right\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \left\{\sum_{k\in K} f_t(n_k)\left(\sum_{n\in C_k} a(n,n_k)\varepsilon_n\right)\right\}_{t\in\mathbb{R}}$$

$$=: \left\{\sum_{k\in K} g_t(k)\xi_k\right\}_{t\in\mathbb{R}}, \tag{2.3}$$

where we set

$$g_t(k) = f_t(n_k)$$

and where the random variables $\xi_k = \sum_{n\in C_k} a(n,n_k)\varepsilon_n$ are strictly α-stable and independent. Let us show that representation (2.3) is minimal for the process $\{X(t)\}_{t\in\mathbb{R}}$. Since one of the equivalence classes C_k may contain all those n such that $f_t(n) = 0$ for all $t \in \mathbb{R}$, we implicitly exclude this class from the representation (2.3). Then,

$$\mathrm{supp}\{g_t, t \in \mathbb{R}\} = K,$$

so that the first condition (M1) of minimality is satisfied. To show that the second condition (M2$'$) holds as well, let $\phi: K \mapsto K$ be a non-singular map between equivalent classes (since K is countable, "non-singular" map here means "any" map because $m(A) = 0$ implies $A = \emptyset$ and hence $m(\phi^{-1}(A)) = m(\phi^{-1}(\emptyset)) = 0$.) Let $a: K \mapsto \mathbb{R}\backslash\{0\}$ be another map such that

$$g_t(\phi(k)) = a(k)g_t(k) \quad \text{for all } t \in \mathbb{R}.$$

Then

$$f_t(n_{\phi(k)}) = a(k)f_t(n_k) \quad \text{for all } t \in \mathbb{R}.$$

The last relation implies that $n_{\phi(k)}$ and n_k belong to the same equivalence class. Since $n_{\phi(k)}$ and n_k are representatives of the classes $\phi(k)$ and k, respectively, and since $n_{\phi(k)}$ and n_k belong to the same class, this implies $\phi(k) = k$ and proves (M2$'$). $\square$

We shall now prove Theorem 1.1. The proof, given below, uses the notions of a flow and a related cocycle. A *flow* on a space $(E, \mathscr{E})$ is a collection $\{\phi_t\}_{t\in\mathbb{R}}$ of measurable maps $\phi_t: E \mapsto E$, $t \in \mathbb{R}$, which satisfy $\phi_0(x)=x$ and the *translation equation*

$$\phi_{s+t}(x)=\phi_s(\phi_t(x)), \tag{2.4}$$

for all $s,t\in\mathbb{R}$ and $x\in E$. A flow $\{\phi_t\}_{t\in\mathbb{R}}$ is said to be *non-singular* if the maps $\phi_t:E\mapsto E$, $t\in\mathbb{R}$, are non-singular, and it is said to be *measurable* if the map $\phi_t(x):\mathbb{R}\times E\mapsto E$ is measurable. A *cocycle* $\{a_t\}_{t\in\mathbb{R}}$ for the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is a measurable map $a_t(x):\mathbb{R}\times E\mapsto\{-1,1\}$ such that

$$a_{s+t}(x)=a_s(\phi_t(x))a_t(x) \tag{2.5}$$

for all $s,t\in\mathbb{R}$ and $x\in E$. Flows and cocycles have proved useful in the study of stable processes with an invariance property, such as stationarity or self-similarity (see Rosiński, 1995; Pipiras and Taqqu, 2002a).

Proof of Theorem 1.1. Suppose that $\{X(t)\}_{t\in\mathbb{R}}$ is a stationary stable process with a representation (1.1). We want to show that the process $\{X(t)\}_{t\in\mathbb{R}}$ is then trivial. By Lemma 2.1, we have that

$$\{X(t)\}_{t\in\mathbb{R}}\stackrel{\mathrm{d}}{=}\left\{\int_K g_t(k)M(\mathrm{d}k)\right\}_{t\in\mathbb{R}},$$

where the set K is countable, M is a strictly α-stable random measure on K with a control measure $m(\mathrm{d}k)=\sum_{l\in K}\sigma_l\delta_{\{l\}}(\mathrm{d}k)$, $\sigma_l>0$, and a skewness intensity $\beta(k)=\sum_{l\in K}\beta_l 1_{\{l=k\}}$, $\beta_1\in[-1,1]$, and $\{g_t\}_{t\in\mathbb{R}}$ is a minimal spectral representation for the process $\{X(t)\}_{t\in\mathbb{R}}$. Then, by Theorem 3.1 in Rosiński (1995) and the proof of Theorem 3.2 in Rosiński (1994), there is a measurable non-singular flow $\{\phi_t\}_{t\in\mathbb{R}}$ on K and a related cocycle $\{a_t\}_{t\in\mathbb{R}}$ such that, for each $t\in\mathbb{R}$ and $k\in K$,

$$g_t(k)=a_t(k)\left\{\frac{\mathrm{d}(m\circ\phi_t)}{\mathrm{d}m}(k)\right\}^{1/\alpha}g_0(\phi_t(k)), \tag{2.6}$$

the set $K_0=\{k\in K\colon\beta(k)\neq 0\}$ is invariant under the flow (that is, $K_0=\phi_t^{-1}(K_0)$ for all $t\in\mathbb{R}$) and the relations $|\beta\circ\phi_t|=|\beta|$ and $a_t=\beta\circ\phi_t/\beta$ hold on the set K_0. By Lemma 2.2 below, $\{\phi_t\}_{t\in\mathbb{R}}$ is an identity flow, that is, $\phi_t(k)=k$ for all $t\in\mathbb{R}$ and $k\in K$. Relation (2.6) reduces to $g_t(k)=a_t(k)g_0(k)$. Since relation (2.5) becomes $a_{s+t}(k)=a_s(k)a_t(k)$ for an identity flow, by taking $s=t$ in this relation, we obtain that $a_{2t}(k)=(a_t(k))^2=1$ for all $t\in\mathbb{R}$ and $k\in K$. Then, we have that

$$g_t(k)=g_0(k)$$

for all $t\in\mathbb{R}$ and $k\in K$ which implies that

$$\{X(t)\}_{t\in\mathbb{R}}\stackrel{\mathrm{d}}{=}\left\{\int_K g_0(k)M(\mathrm{d}k)\right\}_{t\in\mathbb{R}}=:\{Z_0\}_{t\in\mathbb{R}}.$$

By using $(X(t),X(0))=_d(Z_0,Z_0)$, we obtain that $P(X(t)=X(0))=P(Z_0=Z_0)=1$ and hence $X(t)=X(0)=:Z$ a.s. for all $t\in\mathbb{R}$. □

Remark. The assumption of the measurability of the function $f_t(n)$ in (1.1) is technical. One can reformulate it by requiring the discrete linear process (1.1) be measurable (see Samorodnitsky and Taqqu, 1994, Chapter 13).

The following lemma was used in the above proof.

Lemma 2.2. *Let K be a countable set and $\{\phi_t\}_{t\in\mathbb{R}}$ be a non-singular measurable flow on K. Then, $\{\phi_t\}_{t\in\mathbb{R}}$ is an identity flow, that is, $\phi_t(k)=k$ for all $k\in K$ and $t\in\mathbb{R}$.*

Proof. Fix $k\in K$. The flow $\{\phi_t\}_{t\in\mathbb{R}}$ takes this point k and moves it to $\phi_t(k)\in K$ at time t. Since the function $t\mapsto\phi_t(k)$ takes at most a countable number of values, there is $k_0\in K$ such that the set

$$\{t\in\mathbb{R}:\ \phi_t(k)=k_0\}$$

has a positive Lebesgue measure, and hence $\phi_{t_0}(k)=k_0$ for some t_0. Since $\phi_{t-t_0}(k_0)=\phi_{t-t_0}(\phi_{t_0}(k))=\phi_t(k)$, the set

$$\tau=\{t\in\mathbb{R}:\ \phi_t(k_0)=k_0\}$$

has a positive Lebesgue measure as well. Since τ is an additive group (if $t_1,t_2\in\tau$, then $t_1+t_2\in\tau$ because $\phi_{t_1+t_2}(k_0)=\phi_{t_1}(\phi_{t_2}(k_0))=\phi_{t_1}(k_0)=k_0$) and has a positive measure, Corollary 1.1.4 in Bingham et al. 1987 implies that $\tau=\mathbb{R}$. Hence,

$$\phi_t(k_0)=k_0\quad\text{for all } t\in\mathbb{R} \tag{2.7}$$

and, since $\phi_{t_0}(k)=k_0$, by applying the map ϕ_{-t_0} to both sides of this relation, we get

$$k=\phi_0(k)=\phi_{-t_0}(\phi_{t_0}(k))=\phi_{-t_0}(k_0)=k_0.$$

We conclude from (2.7) that

$$\phi_t(k)=k\quad\text{for all } t\in\mathbb{R} \text{ and } k\in K.\quad\square$$

Heuristically, Lemma 2.2 is based on the following idea. Consider a particle moving under the flow $\{\phi_t\}_{t\in\mathbb{R}}$. If K is countable, then the particle must stay at some point $k\in K$ for a finite amount of time, say $\varepsilon>0$. But by taking $s,t\in[0,\varepsilon]$ in the flow equation $\phi_{s+t}(k)=\phi_s(\phi_t(k))$, one gets $\phi_{s+t}(k)=k$ and hence the particle must stay at k forever. The flow, therefore, must be the identity flow.

We now prove Corollary 1.1.

Proof of Corollary 1.1. Consider first part (i). Suppose that the process $\{X(t)\}_{t\in\mathbb{R}}$ is H-self-similar and has a representation (1.1). Then, by applying the Lamperti's transformation (see Samorodnitsky and Taqqu, 1994, Section 7.1), the process

$$Y_1(t)=\mathrm{e}^{-tH}X(\mathrm{e}^t),\quad t\in\mathbb{R}$$

is stationary and

$$\{Y_1(t)\}_{t\in\mathbb{R}}\stackrel{\mathrm{d}}{=}\left\{\sum_{n=-\infty}^{\infty}\mathrm{e}^{-tH}f_{\mathrm{e}^t}(n)\varepsilon_n\right\}_{t\in\mathbb{R}}.$$

By Theorem 1.1, there is a strictly α-stable random variable Z_1 such that for all $t\in\mathbb{R}$, $Y_1(t)=Z_1$ a.s. The inverse transformation which leads Y_1 back to X is

$$X(t)=t^HY_1(\ln t),\quad t>0.$$

Hence we get that, for all $t > 0$, $X(t) = t^H Z_1$ a.s. To obtain an analogous relation for $t < 0$, consider the stationary process $Y_2(t) = e^{-tH} X(-e^t)$, $t \in \mathbb{R}$, which yields for $s > 0$, $X(-s) = s^H Y_2(\ln s) = s^H Z_2$ by a similar argument.

Suppose now that $\{X(t)\}_{t\in\mathbb{R}}$ has stationary increments. Under the assumptions of part (ii), the well-known transformation (see Cambanis and Maejima, 1989)

$$Y(t) = X(t) - \int_{-\infty}^{t} e^{-(t-s)} X(s)\,ds, \quad t \in \mathbb{R}, \tag{2.8}$$

defines a stationary process. It is well-defined because we suppose $\int_{-\infty}^{0} e^s |X(s)|\,ds < \infty$ a.s. Moreover, relation (2.8) is invertible in the sense that

$$X(t) - X(0) = Y(t) - Y(0) + \int_0^t Y(s)\,ds, \quad t \in \mathbb{R}. \tag{2.9}$$

Since transformation (2.8) preserves structure (1.1) of a discrete linear process, Theorem 1.1 implies that $Y(t) = Z$ a.s. for some strictly α-stable random variable Z. Then after substituting this Z into (2.9), we get $X(t) - X(0) = tZ$ a.s.

Part (iii) follows from the parts (i) and (ii) by using the fact that $X(0) = 0$ a.s. for a self-similar process $\{X(t)\}_{t\in\mathbb{R}}$ (see, for example, Samorodnitsky and Taqqu, 1994, p. 312). □

3. A different perspective

Theorem 1.1 and Corollary 1.1 provide general results valid for stationary, stationary increments or self-similar processes. Their proof was based on minimal representations and flows. One can sometimes use an alternative approach, based on "spectral measures", when dealing with a *specific* process. For each $n \geqslant 1$, there is a measure on the unit sphere S_n, called a "spectral measure", which characterizes the n-dimensional distributions of the stable process. It is known that processes of form (1.1) constitute a small subclass of stable processes because their spectral measures are discrete, that is, they are concentrated only on a countable number of points of the corresponding unit spheres. If one is given a (stationary) stable process, it is sometimes possible to show that it cannot be represented as a series (1.1) because the spectral measure of some of its finite-dimensional distributions is not discrete. We illustrate this method on the "linear fractional stable motion" process.

Example 3.1. Consider the so-called linear fractional stable motion $\{L(t)\}_{t\in\mathbb{R}}$ which is a symmetric α-stable, self-similar process with stationary increments having the integral representation

$$\{L(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} \left((t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right) M(du) \right\}_{t\in\mathbb{R}}, \tag{3.1}$$

where $\alpha \in (0,2)$, $H \in (0,1)$, $H \neq 1/\alpha$ and M is the so-called symmetric α-stable random measure with the Lebesgue control measure on $\mathbb{R}$. This means that the characteristic function of a vector

$(L(t_1),\ldots,L(t_n)), t_1,\ldots,t_n\in\mathbb{R}$, is given by

$$E\exp\left\{\mathrm{i}\sum_{k=1}^{n}\theta_k L(t_k)\right\}=\exp\left\{-\int_{\mathbb{R}}\left|\sum_{k=1}^{n}\theta_k\left((t_k-u)_+^{H-1/\alpha}-(-u)_+^{H-1/\alpha}\right)\right|^{\alpha}\,\mathrm{d}u\right\}.$$

For more information on linear fractional stable motion, see Samorodnitsky and Taqqu (1994). (One can also consider strictly α-stable random measures M in (3.1) which leads to strictly α-stable linear fractional stable motions. We assume here that M is symmetric α-stable for simplicity.)

Since linear fractional stable motion is self-similar and is not of the form $t^H Z$, Corollary 1.1 implies that it cannot be represented as a series (1.1). This result can also be obtained by using the approach based on "spectral measures" as follows. Let $L_0=(L(s),L(t))$ and $X_0=(X(s),X(t))$, $s\neq t$, s, $t\neq 0$, be two-dimensional distributions of the linear fractional stable motion $\{L(t)\}_{t\in\mathbb{R}}$ and a process $\{X(t)\}_{t\in\mathbb{R}}$ of form (1.1), respectively. Since we focus now on symmetric α-stable distributions, assume that the independent α-stable random variables ε_n in (1.1) are symmetric. By Theorem 2.4.3 in Samorodnitsky and Taqqu (1994), the vectors L_0 and X_0 are uniquely characterized by a symmetric finite measure on the unit sphere S_2, called a spectral measure. The spectral measure Γ_{X_0}, corresponding to the vector X_0, can be seen to be concentrated on a countable number of points

$$s_n=\left(\frac{f_s(n)}{(f_t^2(n)+f_s^2(n))^{1/2}},\frac{f_t(n)}{(f_t^2(n)+f_s^2(n))^{1/2}}\right)$$

and $-s_n$, $n\in\mathbb{Z}$, where f_t is given by (1.1). On the other hand, as in Samorodnitsky and Taqqu (1994, p. 116), the spectral measure Γ_{L_0} of the vector L_0 is given by

$$\Gamma_{L_0}(A)=\frac{1}{2}\int_{g^{-1}(A)}\mathrm{d}u+\frac{1}{2}\int_{g^{-1}(-A)}\mathrm{d}u, \tag{3.2}$$

where $g(u)=(g_1(u),g_2(u))$ with $g_1(u)=h_s(u)/(h_t^2(u)+h_s^2(u))^{1/2}$ and $g_2(u)=h_t(u)/(h_t^2(u)+h_s^2(u))^{1/2}$, and

$$h_t(u)=(t-u)_+^{H-1/\alpha}-(-u)_+^{H-1/\alpha},\quad u,t\in\mathbb{R}$$

is the kernel function in (3.1). Observe that the relation $g_1(u)=a$ for some $|a|\leqslant 1$ is equivalent to $h_s^2(u)(1-a^2)=a^2h_t^2(u)$. This last relation cannot hold for more than a countable number of points u and hence, by (3.2), $\Gamma_{L_0}(\{s\})=0$ for all $s\in S_2$. Since Γ_{X_0} is concentrated on a finite number of points, we have $\Gamma_{L_0}\neq\Gamma_{X_0}$ and hence that the vectors L_0 and X_0 have different distributions. This shows that linear fractional stable motion (3.1) cannot be represented as a series (1.1).

The method of proof in the previous example does not always work.

Example 3.2. Consider the strictly α-stable Lévy motion $\{L(t)\}_{t\in\mathbb{R}}$ which has stationary independent and strictly α-stable increments (and is self-similar with $H=1/\alpha$). Since the increments of stable Lévy motion are independent, any finite-dimensional distribution $L_0=(L(t_1),\ldots,L(t_n))$ is a linear combination of *independent* strictly α-stable random variables and hence, by Proposition 2.3.7 in Samorodnitsky and Taqqu (1994), the spectral measure Γ_{L_0} of the vector L_0 is concentrated on a finite number of points on the unit sphere S_n. Since the spectral measure Γ_{X_0} of a finite-dimensional distribution $X_0 = (X(t_1), \ldots, X(t_n))$ of a process $\{X(t)\}_{t\in\mathbb{R}}$ of the form (1.1) is also concentrated on a countable number of points on the sphere, we cannot immediately conclude by analyzing the spectral measures that stable Lévy motion cannot be represented as a series (1.1).

The strength of Theorem 1.1 resides in its generality, namely, that one does not have to work as in Example 3.1 with the particular stable process at hand, and show, by analyzing its spectral measures, that it does not have the representation (1.1). Theorem 1.1 and its Corollary 1.1 state that there are no continuous-time stationary or stationary increments or self-similar processes with the representation (1.1) except in trivial cases. These trivial cases are of the forms

$$X(t) = Z, \quad X(t) - X(0) = tZ, \quad X(t) = t^H Z.$$

The proof of Theorem 1.1 also illustrates how one can derive these results by using minimal representations and flows.

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