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Anisotropic Analysis of Some Gaussian Models

Aline Bonami and Anne Estrade

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ABSTRACT. Although the classical Fractional Brownian Motion is often used to describe porosity, it is not adapted to anisotropic situations. In the present work, we study a class of Gaussian fields with stationary increments and "spectral density." They present asymptotic self-similarity properties and are good candidates to model a homogeneous anisotropic material, or its radiographic images. Unfortunately, the paths of all Gaussian fields with stationary increments have the same apparent regularity in all directions (except at most one). Hence we propose here a procedure to recover anisotropy from one realization: computing averages over all the hyperplanes which are orthogonal to a fixed direction, we get a process whose Hölder regularity depends explicitly on the asymptotic behavior of the spectral density in this direction.

Motivation and Introduction

Thirty years ago, Mandelbrot and Van-Ness [17] have initiated the description of 1-dimensional data through Fractional Brownian Motion (FBM). Since then, Fractal Analysis is often used for the description of roughness or porosity of some d-dimensional material. The fundamental parameter of the d-dimensional FBM—the Hurst index H—is the index of regularity, while the fractal dimension of the graph is given by $D = d + 1 - H$. This model is well adapted when the material is homogeneous and isotropic. To take into account non-homogeneity, a generalization of FBM with a "Hurst index depending on the point" has been introduced simultaneously in [3] and [15]. Here we deal with homogeneous material and focus on anisotropy.

The present work has found its origin in pluridisciplinary discussions on the diagnosis of osteoporosis from X-ray pictures of bones (the use of radiographs being the simplest way—financially and technically speaking—to get information). For isotropic bones, it has been shown in [7] that, when modeling the level of grey along lines of the radiographs by FBM, the Hurst parameter appears as a good indicator of the alterations of the micro-architecture that are provoked by osteoporosis. But, in general, the assumption of isotropy

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is not valid. On the other hand, one may consider a bone as a homogeneous material, that is, the mean variations of the bone density around one point do not depend of this one. As a consequence, it is natural to model the bone density by a 3D-Gaussian field with stationary increments. We show that a projection in one direction preserves the properties of such models, so that the same assumptions can be made on bone radiographs: the level of grey at each point is modeled by a 2D-Gaussian field with stationary increments. Typical examples for those anisotropic Gaussian fields are "$1/f$-noises" in the terminology of signal theory, with spectral density $|\xi|^{-\alpha(\xi)}$ where the power $\alpha(\xi)$ depends on the direction of ξ.

For the material itself, as well as for the X-ray picture, it seems natural to consider the regularity in different directions in order to measure the anisotropy (and it is done in practice as in [16]). Indeed, it is easier to estimate the parameter of a one-dimensional process. Unfortunately, a characteristic of random fields with stationary increments is the following: they have the same regularity in all directions, except in at most one (this phenomenon is described in [5] for stationary random fields). Hence, in some way, they certainly look isotropic! The main idea of this work lies in getting another method to recover anisotropy.

Let us describe precisely our method (Directional Average Method) when applied to an X-ray picture. We assume that this one is modeled by a 2D-Gaussian field with spectral representation $\int_{\mathbb{R}^2}(e^{it.\xi}-1)|\xi|^{-h(\theta)-1}\,dW(\xi)$, where dW is a 2D-Brownian measure and $\theta = \mathrm{Arg}(\xi)$ is the angle between the x-axis and the direction of ξ. Moreover, h is a π-periodic continuous map with values between 0 and 1 which depends on the anisotropy of the X-ray picture. The function h is then recovered as follows. For any fixed direction θ, the picture is averaged over all the lines orthogonal to θ. A new 1D-process is obtained and its Hölder regularity is proved to be equal to $h(\theta)+\frac{1}{2}$. Hence an anisotropic analysis of the picture can be elaborated.

Although the starting point of this article was the analysis of 2D-pictures, we deal with a more general setting. We study d-parameter real-valued Gaussian fields with spectral representation

$$\left\{\int_{\mathbb{R}^d}\left(e^{it.\xi}-1\right)f^{\frac{1}{2}}(\xi)\,dW(\xi)\;;\;t\in\mathbb{R}^d\right\}$$

where $\{W(\xi);\,\xi\in\mathbb{R}^d\}$ is the complex Lévy Brownian field, with adapted real and imaginary parts such that the Wiener integral is real-valued. The function f—the spectral density—is any positive even function with adapted integrability assumptions. When f is not radial, the Gaussian field is not isotropic. We refer to [11] and [14] for other anisotropic models which have been proposed in the literature, but do not have stationary increments.

Then natural questions appear. Which properties of the FBM are preserved (self-similarity, Hölder regularity,...)? Is this class of Gaussian fields stable under projection? How to mesh the anisotropy? How to recover information on the anisotropy from one realization? Our answers give, in each case, sufficient conditions on the spectral density. We should emphasize that, in most cases, we adapt well-known properties to our context. The main originality of this article relies in the choice of the model itself, its invariance through projections, and the fact that this invariance may be used to recover the parameters.

The article is organized as follows. In the first section, we introduce the model, that is the class of Gaussian fields with spectral density, and give some examples. Such fields are known to have a continuous version under some decay assumption for the spectral density. The second section is devoted to self-similarity properties. We prove that the Gaussian field associated with an asymptotically homogeneous spectral density is locally asymptotically self-similar. Moreover, the tangent field belongs to the same class of Gaussian fields. In the third section we study Hölder regularity. We come back to the result of "apparent isotropy" which we already described, and observe the behavior of the Gaussian field in each direction. The last section deals with projections and averages. The class of Gaussian fields with spectral density is proved to be stable under projections. We then present our method to recover anisotropy by analyzing one-parameter Gaussian processes. Our main theorem (Theorem 1) describes the dependency between the Hölder regularity of the process obtained by averaging over all the hyperplanes orthogonal to a fixed direction on one hand, and the asymptotic behavior of the spectral density in this direction on the other hand. A particular attention is given to examples: four of them are followed all over the article and confronted with the general results.

1. Gaussian Field with Stationary Increments

All over the article, we consider *d-parameter random fields* (see [1] or [22] for a general presentation). By this terminology, we mean a map X from $\Omega \times \mathbb{R}^d$ into $\mathbb{R}$ such that, for all t in $\mathbb{R}^d$, $X(.,t) := X(t)$ is a random variable defined on a probability space Ω (equipped with a σ-field and a probability measure). When $d = 1$, we speak of a *process*. At this stage, no measurability with respect to the "space" variable t is required. Let us recall basic definitions involving the finite dimensional distributions of X. For two random fields X and Y, we write $X \stackrel{fdd}{=} Y$ when, for all n in $\mathbb{N}^*$ and all $t_1, \ldots, t_n$ in $\mathbb{R}^d$, the n-dimensional random variables $(X(t_1), \ldots, X(t_n))$ and $(Y(t_1), \ldots, Y(t_n))$ have the same distribution.

A d-parameter random field X is said:
- to have *stationary increments* if, for all t_0 in $\mathbb{R}^d$,

$$X(.+t_0) - X(t_0) \stackrel{fdd}{=} X(.) - X(0)\,;$$

- to be *self-similar of order α* if, for all λ in $\mathbb{R}^*$, $X(\lambda .) \stackrel{fdd}{=} \lambda^\alpha X(.)$;
- to be *isotropic* if, for all rotation R in $\mathbb{R}^d$, $X \circ R \stackrel{fdd}{=} X$.

The finite dimensional distributions of a centered Gaussian random field X are totally determined by the covariance function $(s,t) \mapsto \mathrm{Cov}(X(s), X(t))$.

1.1 Fractional Brownian Field

Let H belong to $(0,1)$. We begin this study with the (unique) real-valued centered random field, vanishing at the origin, which is simultaneously Gaussian, with stationary increments, H-self similar and isotropic: the celebrated *fractional Brownian field with Hurst index H*, $B^H = \{B^H(t); t \in \mathbb{R}^d\}$.

Its covariance function is given by

$$\mathrm{Cov}\left(B^H(s), B^H(t)\right) = c_{H,d}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right)\,,\ \forall s,t \in \mathbb{R}^d$$

with $|s|$ the Euclidean norm of s and $c_{H,d}$ a nonnegative constant depending on H and d.

A spectral (or harmonizable) representation is available (see [22] Chapter 4 or [21] Section 7), given by

$$\left\{\int_{\mathbb{R}^d} \frac{e^{it.\xi} - 1}{|\xi|^{\frac{1}{2}(2H+d)}}\, dW(\xi)\ ;\ t \in \mathbb{R}^d\right\} \tag{1.1}$$

where dW is a complex Brownian measure.

Any process obtained by restriction along a straight line Δ of $\mathbb{R}^d$ going through 0, $\{B^H(u); u \in \Delta\}$, is a one-parameter fractional Brownian motion with Hurst index H and therefore the Hölder critical exponent of its sample paths is H a.s. whatever the direction of the line. Indeed the fractional Brownian field B^H is isotropic. In the next section we force anisotropy by changing the map $\xi \mapsto |\xi|^{-\frac{1}{2}(2H+d)}$ in the spectral representation (1.1) into a general (non-radial) map $\xi \mapsto f^{\frac{1}{2}}(\xi)$.

1.2 Gaussian Fields with Prescribed Spectral Density

Let $X = \{X(t); t \in \mathbb{R}^d\}$ be a Gaussian field with mean zero and stationary increments. The finite dimensional distributions of $\{X(t) - X(0); t \in \mathbb{R}^d\}$ are completely given by *the variogram v of X*

$$v(t) = \frac{1}{4}\mathbf{E}\left((X(t) - X(0))^2\right) , \ \forall t \in \mathbb{R}^d$$

since the covariance function satisfies

$$\mathrm{Cov}(X(t) - X(0), X(s) - X(0)) = 2(v(t) + v(s) - v(t-s)) , \ \forall s, t \in \mathbb{R}^d .$$

Note that this identity characterizes the stationarity of the increments for centered Gaussian fields.

We will now investigate the real Gaussian fields with mean zero and stationary increments whose variogram is continuous and may be written

$$v(t) = \int_{\mathbb{R}^d} \sin^2(t.\xi/2) f(\xi)\, d\xi , \ \forall t \in \mathbb{R}^d \tag{1.2}$$

with f a positive function such that $\int_{\mathbb{R}^d} (1 \wedge |\xi|^2) f(\xi)\, d\xi < \infty$.

It is clear that the function f can be replaced in (1.2) by an even function g: just take $g(\xi) = \frac{1}{2}(f(\xi) + f(-\xi))$. Actually it is the only allowed transformation, as proved in the next lemma. In the sequel, μ will denote the measure on $\mathbb{R}^d$ given by

$$d\mu(\xi) = \left(1 \wedge |\xi|^2\right) d\xi .$$

Lemma 1.

If f and g are even positive functions in $\mathbf{L^1}(\mathbb{R}^\mathbf{d}, \mathbf{d}\mu; \mathbb{R})$ inducing the same variogram, i. e.,

$$\int_{\mathbb{R}^d} \sin^2(t.\xi/2) f(\xi)\, d\xi = \int_{\mathbb{R}^d} \sin^2(t.\xi/2) g(\xi)\, d\xi , \ \forall t \in \mathbb{R}^d ,$$

then $f = g$ a.e..

Proof. Indeed, let us show that an even $d\mu$-integrable function g, such that $\int_{\mathbb{R}^d} \sin^2 (t.\xi/2) g(\xi)\, d\xi$ vanishes for all t, is identically 0. It is sufficient to prove that, for φ an even function in the Schwartz class, then

$$\int_{\mathbb{R}^d} |\xi|^2 \varphi(\xi) g(\xi)\, d\xi = 0 .$$

But, using Fourier inversion formula, we may write

$$|\xi|^2 \varphi(\xi) = \int_{\mathbb{R}^d} \cos(t.\xi)\psi(t)\, dt = \int_{\mathbb{R}^d} (\cos(t.\xi) - 1)\psi(t)\, dt$$

for some other Schwartz function ψ. The use of Fubini's theorem allows to conclude. $\square$

Definition 1. Let X be a Gaussian field with mean zero, stationary increments and variogram given by (1.2), with f an even positive function in $\mathbf{L^1}(\mathbb{R}^\mathbf{d}, \mathbf{d}\mu; \mathbb{R})$. Then we call f the spectral density of X and say that X is a Gaussian field with spectral density f.

Let us now list some properties of such Gaussian fields. Let f be an even positive map in $\mathbf{L^1}(\mathbb{R}^\mathbf{d}, \mathbf{d}\mu; \mathbb{R})$ and X^f be a Gaussian field with spectral density f.

First, $X^f - X^f(0)$ has the same finite dimensional distributions as

$$\left\{ \int_{\mathbb{R}^d} \left(e^{it.\xi} - 1 \right) f^{\frac{1}{2}}(\xi)\, dW(\xi)\ ;\ t \in \mathbb{R}^d \right\}$$

since they are both Gaussian fields with mean zero and stationary increments and have the same variogram. A consequence of this representation is that any even positive function in $\mathbf{L^1}(\mathbb{R}^\mathbf{d}, \mathbf{d}\mu; \mathbb{R})$ is the spectral density of at least one centered Gaussian field with stationary increments. From now on, a *standard spectral density* is any even positive map in $\mathbf{L^1}(\mathbb{R}^\mathbf{d}, \mathbf{d}\mu; \mathbb{R})$.

Next, assuming that $X^f(0) = 0\ a.s.$, Lemma 1 shows that X^f *is self-similar if and only if* f *is a homogeneous map*, and in the same vein, X^f *is isotropic if and only if* f *is a radial map*. A consequence is that there is only "one" Gaussian field X^f which is both self-similar and isotropic: it is the fractional Brownian motion described in Section 1.1, with spectral density $\xi \mapsto |\xi|^{-\alpha}$ for some adapted α.

In most cases, we will content ourselves to consider properties of finite dimensional distributions, which are given by the variogram, since our main object is to propose adequate models as well as an analysis that allows to estimate their parameters. We will nevertheless consider the possibility of having a continuous modification of X^f. In this case, equality of finite dimensional distributions can be replaced by equality in law on the space of continuous paths on $\mathbb{R}^d$. Without any surprise (see [2] Corollary 2.2 for instance), a continuous modification of X^f exists when f decreases rapidly enough at infinity. More precisely, for $m \in (0, +\infty)$, we consider the following assumption on the standard spectral density f: there exists constants A, B in $(0, +\infty)$ such that

$$\mathbf{D}(m): \qquad f(\xi) \leq B|\xi|^{-(2m+d)}\ , \quad \text{for almost all } |\xi| > A\ .$$

The assumption $\mathbf{D}(m)$ is used in the next proposition.

Proposition 1.

Let f be a standard spectral density and X be a Gaussian field with spectral density f. If f satisfies assumption $\mathbf{D}(m)$ for some positive m, then there exists a field $\tilde{X}$, defined on the same probability space as X, whose paths are a.s. continuous, and such that for all $t \in \mathbb{R}^d$, $X(t) = \tilde{X}(t)\ a.s.$.

Whether there exists a continuous modification of a given random process, and which Hölder regularity can be expected for a modification, are quite old questions. In the 60's several criteria for the existence of a continuous modification have been proposed in two different contexts. One, due to Kolmogorov and Centsov, is concerned with Hölder regular modifications of random fields defined on $\mathbb{R}^d$ (see [12] Section 2.2 for instance and references therein). Another one, due to Dudley, Marcus and Shepp, Fernique, gives necessary and sufficient conditions for the existence of continuous modifications of Gaussian processes defined on a metric set (see [10] Chapter 15 or [1] Chapter 3 and references therein). In both cases, the criteria rely on a control of the k-th moment of the increments of the process (k big enough in the first case, $k = 2$ in the second case). In our context of Gaussian fields defined on $\mathbb{R}^d$, they can both be applied and both consist in a control of the variance of the increments. For the next proof, we chose the first one.

Proof. We use the Kolmogorov–Centsov criterion: for all positive T,

$$\forall s, t \in [-T, T]^d, \ \ \mathbf{E}\left(|X(t) - X(s)|^\alpha\right) \le C|t - s|^{d+\beta}$$

for some positive constants α, β and C.

Since X is Gaussian, then for all $k \in \mathbb{N}^*$, there exists a constant c_k such that

$$\forall s, t \in \mathbb{R}^d, \ \ \mathbf{E}\left(|X(t) - X(s)|^{2k}\right) = c_k \left(\mathbf{E}\left(|X(t) - X(s)|^2\right)\right)^k .$$

With k large enough, the next lemma allows to conclude.

Lemma 2.

Under the assumptions of Proposition 1, for all $T > 0$, there exists a positive constant C such that

$$\forall s, t \in [-T, T]^d, \ \mathbf{E}\left(|X(t) - X(s)|^2\right) \le C|t - s|^{2(1\wedge m)} .$$

Let us prove this inequality. Recall that $\mathbf{E}(|X(t) - X(s)|^2) = 4v(t - s)$ and use Assumption $\mathbf{D}(m)$ to get, for all t in $\mathbb{R}^d$,

$$\begin{aligned} v(t) &\le \frac{1}{4}|t|^2 \int_{|\xi|\le A} |\xi|^2 f(\xi)\, d\xi + B \int_{|\xi|>A} \sin^2(t.\xi/2)|\xi|^{-(2m+d)}\, d\xi \\ &\le |t|^2 \left(\frac{1}{4}\int_{|\xi|\le A} |\xi|^2 f(\xi)\, d\xi\right) + |t|^{2m}\left(B\int_{\mathbb{R}^d} \sin^2(\xi/2)|\xi|^{-(2m+d)}\, d\xi\right). \ \square \end{aligned}$$

Note that the inequality in Lemma 2 implies more smoothness than a.s. continuity. Actually (see the above references), if f satisfies $\mathbf{D}(m)$ then the paths of the modification $\tilde{X}$ are a.s. Hölder regular of order $(1 \wedge m) - \varepsilon$, for all $\varepsilon > 0$. This will in fact be fundamental below.

Before ending this section, let us quote that the assumptions on the spectral representation of the variogram that we consider seem relevant to describe a large class of Gaussian fields with stationary increments. Indeed, as a consequence of Bochner theorem (see [6] Proposition 3.1 or [22] Chapter 4), the variogram of any real centered process with stationary increments, if continuous, is given by

$$t \mapsto \int_{\mathbb{R}^d} (1 - \cos t\xi)\, dG(\xi) ,$$

where dG is a positive measure on $\mathbb{R}^d$ such that $\int_{|\xi|<1} |\xi|^2\, dG(\xi) + \int_{|\xi|>1} dG(\xi)$ is finite. So the only restriction imposed by the representation (1.2) is that dG is absolutely continuous, and later, with condition $\mathbf{D}(m)$, some behavior at infinity of the density.

1.3 Examples

We give hereafter some examples of spectral densities. They all behave like a power of $\xi \mapsto 1/|\xi|$ at infinity, and satisfy the assumption $\mathbf{D}(m)$ for some positive m.

Example 1. In the spectral representation of the fractional Brownian motion (1.1), we replace the constant H by a function of the direction of ξ. In other words, we consider an even positive function which is homogeneous of degree zero,

$$h(\lambda\xi) = h(\xi) \;\; \forall \xi \neq 0 \in \mathbb{R}^d \;, \;\; \forall \lambda \neq 0 \in \mathbb{R} \,,$$

which may be identified with an even function on the unit sphere S^{d-1} of $\mathbb{R}^d$ that we note h as well. We assume moreover, that h takes its values inside the interval $[m, M] \subset (0, 1)$, with $m = \text{essinf} \;\; h$ and $M = \text{esssup} \;\; h$. The spectral density that we consider is given by

$$\xi \in \mathbb{R}^d \;\; \mapsto \;\; \frac{1}{|\xi|^{2h(\xi)+d}} \;.$$

It provides what can be called an "anisotropic fractional Brownian field with directional Hurst index h," denoted by $X^{(h)}$. A spectral representation for $X^{(h)}$ is given by

$$\left\{ \int_{\mathbb{R}^d} \frac{e^{it.\xi} - 1}{|\xi|^{\frac{1}{2}(2h(\xi)+d)}} \, dW(\xi) \; ; \;\; t \in \mathbb{R}^d \right\} \;.$$

Computing the variogram with "polar" coordinates gives

$$v(t) = \int_{\mathbb{R}^d} \frac{\sin^2(t.\xi/2)}{|\xi|^{2h(\xi)+d}} \, d\xi = \int_{S^{d-1}} C(h(u)) \, |t.u|^{2h(u)} \, du \tag{1.3}$$

with, for all H in $(0, 1)$, $C(H) = \displaystyle\int_{\mathbb{R}^+} \frac{\sin^2(x/2)}{x^{2H+1}} \, dx$. This constant arises naturally when studying fractional Brownian motion, and is equal to $C(H) = \frac{\pi/8}{H\Gamma(2H)\sin(H\pi)}$ (see [21] (formula 7.2.13) for instance).

Example 2. Let h be as above, and take as spectral density

$$\xi \in \mathbb{R}^d \;\; \mapsto \;\; \frac{1}{\left(1 + |\xi|^2\right)^{\frac{1}{2}(2h(\xi)+d)}} \;.$$

In this example, we remark that integrability at the origin allows to separate the two terms in the spectral representation, and write the associated Gaussian field as

$$\int_{\mathbb{R}^d} \frac{e^{it.\xi}}{\left(1 + |\xi|^2\right)^{\frac{1}{2}(h(\xi)+d/2)}} \, dW(\xi) - \int_{\mathbb{R}^d} \frac{1}{\left(1 + |\xi|^2\right)^{\frac{1}{2}(h(\xi)+d/2)}} \, dW(\xi) \;.$$

The first term itself gives rise to a stationary process. Moreover, we can now allow M to be arbitrarily large.

Example 3. For a 2-parameter Gaussian field, let us choose H_1 and H_2 in $(0, 1)$ and take as spectral density

$$\xi \in \mathbb{R}^2 \;\; \mapsto \;\; \left(\frac{1}{|\xi_1|^{H_1+1} + |\xi_2|^{H_2+1}} \right)^2 \;.$$

Example 4. We borrow it from [3] p. 24. A spectral density is defined by

$$\xi \in \mathbb{R}^d \;\; \mapsto \;\; \frac{s^2\left(\frac{\xi}{|\xi|}\right)}{|\xi|^{2m+d}}$$

where $m \in (0, 1)$ and s is an even square-integrable map defined on S^{d-1}.

2. Asymptotic Self-Similarity

Self-similarity for a Gaussian field with stationary increments is clearly equivalent to homogeneity of the variogram. In the case of a Gaussian field with spectral density, it is equivalent to homogeneity of the standard spectral density, as already said. More precisely, self-similarity of order m corresponds to homogeneity of order $2m$ for the variogram and of order $-(2m+d)$ for the spectral density. Note that all such Gaussian fields are given by Example 4.

We now consider asymptotic self-similarity of the Gaussian field. This one may be considered either locally, or at infinity. We will see that it depends on the behavior of the spectral density either at infinity, or at 0. Let us start with the local property.

The local asymptotic self-similarity (*l.a.s.s.* property) will first be related to the asymptotic homogeneity of the variogram of the Gaussian field, then to the asymptotic homogeneity of the spectral density. One can find links between these notions in a more general context in [3] (Theorem 1.4). Let us recall the l.a.s.s. definition.

Definition 2. Let $\alpha > 0$. A field $X = \{X(t);\, t \in \mathbb{R}^d\}$ is locally asymptotically self-similar of order α at the point $t_0 \in \mathbb{R}^d$ if the finite dimensional distributions of

$$\left\{ \frac{X(t_0+\lambda t) - X(t_0)}{\lambda^\alpha};\ t \in \mathbb{R}^d \right\}$$

converge to the finite dimensional distributions of a non trivial field as $\lambda \to 0^+$. The limit field is called the tangent field at the point t_0.

Let us now define the asymptotic homogeneity of a function. We say that g, a positive function on $\mathbb{R}^d$, is *asymptotically homogeneous* of order α at ∞ if there exists a non zero function g_∞ such that, for almost every ξ in $\mathbb{R}^d$, $g_\lambda(\xi) = \lambda^{-\alpha} g(\lambda\xi)$ has limit $g_\infty(\xi)$ when λ tends to $+\infty$. In this case, the function g_∞ is clearly homogeneous of degree α, which fixes uniquely the parameter α. We define as well asymptotic homogeneity at 0, and use the notation g_0 for the limit.

The next lemma provides a sufficient condition on the variogram ensuring local asymptotic self-similarity.

Lemma 3.
Let X be a Gaussian field with mean zero and stationary increments. Assume that its variogram is asymptotically homogeneous of order $2m$ at 0, with limit function v_0. Then X is, at any point t_0, locally asymptotically self-similar of order m and the tangent field is Gaussian has stationary increments, mean zero, and variogram v_0.

Proof. Since we deal with finite dimensional distributions of Gaussian fields with mean zero, we only have to prove that

$$\mathrm{Cov}\left(\frac{X(t_0+\lambda t) - X(t_0)}{\lambda^m}, \frac{X(t_0+\lambda s) - X(t_0)}{\lambda^m} \right) \to 2(v_0(t) + v_0(s) - v_0(t-s)) ,$$

when λ tends to 0, which is a consequence of the stationarity of the increments of X and the asymptotic homogeneity of the variogram of X. □

We consider now conditions on f at ∞. In fact, we propose sufficient conditions which imply also the convergence in law of the Gaussian fields.

We first ask f to be asymptotically homogeneous at ∞, which is a natural condition, but which is not enough as we will see on the examples. So we also ask f to satisfy the additional assumption $\mathbf{D}(m)$.

Proposition 2.

Let $m \in (0,1)$ and let f be a standard spectral density which satisfies $\mathbf{D}(m)$ and is asymptotically homogeneous of order $-(2m+d)$ with limit function f_∞. Then X^f is, at any point t_0, asymptotically self-similar of order m with tangent field X^{f_∞}.

Moreover, denoting by $\tilde{X}^f$ and $\tilde{X}^{f_\infty}$ continuous versions of X^f and X^{f_∞}, then for all $t_0 \in \mathbb{R}^d$,

$$\lim_{\lambda \to 0^+} \left\{ \frac{\tilde{X}^f(t_0 + \lambda t) - \tilde{X}^f(t_0)}{\lambda^m};\ t \in \mathbb{R}^d \right\} = \left\{ \tilde{X}^{f_\infty}(t);\ t \in \mathbb{R}^d \right\} ,$$

where the convergence here is the convergence in distribution on the space of continuous paths on $\mathbb{R}^d$.

Remark. Note that f_∞ satisfies also $\mathbf{D}(m)$. Since it is homogeneous of order $-(2m+d)$, then the limit f_∞ is a standard spectral density. The limit field X^{f_∞} is self-similar of order m, and so given by Example 4. Moreover, Proposition 1 applies to both spectral densities f and f_∞, and provides continuous modifications for X^f and X^{f_∞}.

Proof. The first point concerning the (usual) local asymptotic self-similarity is a consequence of Lemma 3. As before, we decompose $v_\lambda = \lambda^{-2m} v(\lambda .)$ into two parts,

$$v_\lambda(t) = \lambda^{-2m} \int_{|\xi|<A} \sin^2(\lambda t.\xi/2) f(\xi)\, d\xi + \lambda^{-2m-d} \int_{|\xi|>\lambda A} \sin^2(t.\xi/2) f(\xi/\lambda)\, d\xi \ .$$

The first term is bounded by $C\lambda^{-2m+2}|t|^2$, which tends to 0, while Lebesgue's Theorem may be applied to prove the convergence of the second one.

This proves the finite dimensional distributions convergence. Let us now prove the tightness for the family $(Z^{(\lambda)} = \frac{\tilde{X}^f(t_0+\lambda\cdot) - \tilde{X}^f(t_0)}{\lambda^m})_{0<\lambda<1}$, which allows to conclude for the convergence in law on the space of continuous paths. We use the following tightness criterion, valid for each family $(Z^{(\lambda)})_{\lambda>0}$ of fields on $\mathbb{R}^d$ vanishing at the origin, (Kolmogorov criterion, see [12] p. 64): for all $T > 0$,

$$\forall s, t \in [-T, T]^d, \sup_{\lambda>0} \mathbf{E}\left(\left| Z^{(\lambda)}(t) - Z^{(\lambda)}(s) \right|^\alpha \right) \le C|t-s|^{d+\beta}$$

for some positive constants C, α and β.

Since $\tilde{X}^f$ is Gaussian with variogram v, for every positive integer k,

$$\mathbf{E}\left(\left| Z^{(\lambda)}(t) - Z^{(\lambda)}(s) \right|^{2k} \right) = C_k\, \lambda^{-2mk}\, v(\lambda(t-s))^k ,$$

and the above inequality is satisfied thanks to Lemma 2, for which we already used the assumption $\mathbf{D}(m)$. □

Let us remark that the situation is particularly simple in dimension one, where the asymptotic self-similarity follows from the behavior at ∞ of the spectral density,

$$f(\xi) \simeq c|\xi|^{-2m-1}, \qquad \xi \to \pm\infty \ .$$

Let us now test the examples of Section 1.3 with respect to asymptotic self-similarity. Recall that we already dealt with Example 4.

Example 1. Taking $m = \text{essinf}(h)$, we get for limit function

$$f_\infty(\xi) = \frac{\mathbf{1}_{\{h=m\}}(\xi)}{|\xi|^{2m+d}} ,$$

which is non zero if only if the set $\{h = m\}$ has positive measure (which is the case, for instance, if the map h takes only a finite number of values). By this, we mean that the intersection of this set with the unit sphere S^{d-1} has a positive measure for the Lebesgue measure on S^{d-1}.

The assumption $\mathbf{D}(m)$ is clearly satisfied. So the Gaussian field $X^{(h)}$ is, at any point, locally asymptotically self-similar of order m if the set $\{h = m\}$ has positive measure. The tangent field is anisotropic if h is not constant.

Conversely, if $\{\xi \in S^{d-1};\ h(\xi) = m\}$ has measure 0, let us prove that $X^{(h)}$ is not asymptotically self-similar, for any α. It is clearly not the case for $\alpha \le m$ since for any point t_0 the variance of the Gaussian process $\{\frac{X^{(h)}(t_0+\lambda t)-X^{(h)}(t_0)}{\lambda^\alpha};\ t \in \mathbb{R}^d\}$ tends to 0. On the other hand, for $\alpha = m + \varepsilon$ with $\varepsilon > 0$, for $\lambda \in (0, 1)$,

$$\lambda^{-2\alpha} v(\lambda t) \ge \lambda^{-\varepsilon} \int_{\frac{\xi}{|\xi|} \in E} \sin^2(t.\xi/2)\, d\xi ,$$

where $E = \{\xi \in S^{d-1};\ h(\xi) < m + \varepsilon/2\}$ has non-zero Lebesgue measure. So the above quantity tends to infinity when λ tends to 0^+.

We have just proved the following: *$X^{(h)}$ is locally asymptotically self-similar if and only if $\{\xi \in S^{d-1};\ h(\xi) = m\}$ has positive measure.*

Note that $X^{(h)}$ is also locally asymptotically self-similar at infinity if and only if $\{\xi \in S^{d-1};\ h(\xi) = M = \text{essup}(h)\}$ has positive measure.

Example 2. If h is taken as in Example 1 above, then the assumptions of Proposition 2 are fulfilled with $m = \text{essinf}(h)$ and the same limit function as in the last example; the associated field is then locally asymptotically self-similar of order m when the set $\{h = m\}$ has non zero measure. No asymptotic self-similarity is observed at infinity.

Example 3. In the case $H_1 < H_2$, the spectral density f is asymptotically homogeneous of degree $-2(H_2 + 1)$, with limit function $f_\infty(\xi) = |\xi_2|^{-2(H_2+1)}$. This function is not integrable at ∞ and hence is not a standard spectral density. Actually the spectral density f does not fulfill the assumption $\mathbf{D}(H_2)$, but only $\mathbf{D}(H_1)$. On the other hand, it is easy to see that X^f is not asymptotically self-similar.

This computation proves that the assumption $\mathbf{D}(m)$, with the right m, cannot be omitted in Proposition 2.

The first two examples lead to the less restrictive notion of *lass critical index*, which we define now.

Definition 3. The lass critical index of the field $X = \{X(t); t \in \mathbb{R}^d\}$ is the upper bound of the set of positive numbers α such that $\lambda^{-2\alpha} v_X(\lambda t)$ tends to 0 for a.e. value of t when λ tends to 0.

In the first two examples, the lass critical index is equal to m, without any additional assumption. More generally, we have the following proposition.

Proposition 3.

Let $m \in (0,1)$ and let f be a standard spectral density. Assume that

(i) *f satisfies $\mathbf{D}(\alpha)$ for all $\alpha < m$;*

(ii) *for all $\alpha > m$, there exists a constant $A > 0$ and a set $E \subset S^{d-1}$ of positive measure such that for almost all ξ in $\mathbb{R}^d$,*

$$|\xi| > A \text{ and } \frac{\xi}{|\xi|} \in E \Rightarrow f(\xi) > |\xi|^{-(2\alpha+d)}\,;$$

then X^f has m as lass critical index.

The proof is analogous to the previous ones, and we shall see a generalization later. One can prove with a direct proof that the Example 3 has also a lass critical index, equal to $H_1 + \frac{H_2-H_1}{2(H_2+1)}$. It is not a consequence of the previous proposition, which is adapted to the Examples 1 and 2.

3. Hölder Regularity

The present section is dedicated to the smoothness of sample paths, which can also be deduced from the variogram. From now on, we have the directional properties of a Gaussian field X with stationary increments in mind. We will consider the Hölder regularity of the sample paths of each Gaussian process obtained by restriction, $\{X(t)\,;\ t \in \Delta\}$, where Δ is a straight line in $\mathbb{R}^d$. We will see that it does not depend, in general, on the direction of Δ. Let us first describe how to measure the regularity of one-parameter Gaussian processes.

3.1 Hölder Critical Exponent for Gaussian Processes

Let $X = \{X(t);\, t \in \mathbb{R}\}$ be a Gaussian process with stationary increments. A well-known result relates the Hölder regularity of the sample paths with the behavior of the variogram function at the origin (see [4, 1, 8] or [2]). We first introduce the required assumption on the variogram.

Definition 4. Let $\beta = n + s$ with $n \in \mathbb{N}$, $s \in (0,1]$. The variogram function v satisfies the condition $\mathbf{H}(\beta)$ if v is $2n$-continuously differentiable, and

$$\begin{aligned} s &= \sup\left\{\alpha > 0;\ \left|v^{(2n)}(t) - v^{(2n)}(0)\right| = \circ\left(|t|^{2\alpha}\right), t \to 0\right\} \\ &= \inf\left\{\alpha > 0;\ |t|^{2\alpha} = \circ\left(\left|v^{(2n)}(t) - v^{(2n)}(0)\right|\right), t \to 0\right\}\ . \end{aligned}$$

Remark that if v satisfies $\mathbf{H}(\beta)$ with $\beta \in (0,1]$, then any Gaussian process X with variogram v admits a continuous modification (see Lemma 2). Note also that in this case, β is the lass critical index of X (see Section 2). Such processes are called *index-β Gaussian fields* in the terminology of [1].

We now give the definition of the Hölder critical exponent of a random process.

Definition 5. Let $\beta \in (0,1)$. A process $X = \{X(t)\,;\ t \in \mathbb{R}\}$ is said to have Hölder critical exponent β whenever it satisfies the two following properties

- for any $\alpha \in (0,\beta)$, the sample paths of X satisfy a.s. a uniform Hölder condition of order

α on any compact set, i. e., for any compact set K of $\mathbb{R}$, there exists a positive random variable A such that $a.s.$

$$|X(t) - X(s)| \leq A|s-t|^{\alpha} \ , \ \forall s, t \in K \ ;$$

- for any $\alpha \in (\beta, 1)$, a.s. the sample paths of X fail to satisfy any uniform Hölder condition of order α.

Let us now write how these two notions are related. The next proposition comes precisely from [1] Theorem 8.3.2 and Theorem 2.2.2.

Proposition 4.

Let $X = \{X(t); t \in \mathbb{R}\}$ be a Gaussian process with mean zero, and stationary increments and assume that its variogram v satisfies the condition $\mathbf{H}(\beta)$ for some positive non-integer β.

(i) *If $\beta \in (0,1)$, then any continuous version of X has Hölder critical exponent β;*

(ii) *If $\beta \in (n, n+1)$ with $n \in \mathbb{N}^*$ then X is n-times mean-square differentiable. Moreover, the n-th mean-square derivative $X^{(n)}$ of X is a Gaussian process with stationary increments and variogram function $t \mapsto (-1)^n(v^{(2n)}(t) - v^{(2n)}(0))$ and any continuous version of $X^{(n)}$ has Hölder critical exponent $\beta - n$.*

Remark that the last proposition does not allow to get fine estimates such as iterated logarithmic laws for the modulus of continuity of the sample paths as in [18] or [3]. We are only interested in critical Hölder exponents, and do not describe the behavior for the critical value. The proposition does not say anything about integer values of β, and is in fact less precise when $\beta \geq 1$.

3.2 Directional Regularity

We are now interested in the Hölder regularity in each direction of a d-parameter Gaussian field with stationary increments $X = \{X(t); t \in \mathbb{R}^d\}$. More precisely, we consider its restriction along a straight line, that is to say the process $\{X(t_0+tu); t \in \mathbb{R}\}$ for $u \in S^{d-1}$ and $t_0 \in \mathbb{R}^d$.

Definition 6. Let X be a d-parameter random field with stationary increments and let u be any direction in S^{d-1}. If the process $\{X(tu); t \in \mathbb{R}\}$ has Hölder critical exponent $\beta(u)$, we say that X admits $\beta(u)$ as directional regularity in direction u.

The stationarity of the increments of X implies that $\beta(u)$ is also the Hölder critical exponent of all processes obtained by restricting X to any straight line of direction u.

In [5] the directional regularity of any 2-parameter stationary random field is proved to be constant except in at most one direction where it can be larger (see also [19] for more general results). We prove hereafter, in our context of Gaussian fields with stationary increments, a similar result based on the "directional variogram."

Proposition 5.

Let X be a d-parameter Gaussian field with mean zero, stationary increments and variogram v. Suppose that for all u in S^{d-1}, the map: $t \mapsto v(tu)$ satisfies the assumption $\mathbf{H}(\beta(u))$ for some $\beta(u)$ in $(0,1)$. Then the map $\beta : u \mapsto \beta(u)$ takes at most d different values. Moreover, it is constant except, perhaps, on the intersection of the sphere with a subspace of dimension at most $d-1$ where it may take larger values.

Proof. We denote also by β its extension as a homogeneous function of degree 0 on $\mathbb{R}^d \setminus \{0\}$.

We choose $u_1, \dots, u_k$ in $\mathbb{R}^d \setminus \{0\}$ and u in the vector space generated by $u_1, \dots, u_k$. Then $u = \sum_{j=1}^k a_j u_j$ with some $a_1, \dots, a_k$ in $\mathbb{R}$ and

$$X(tu) - X(0) = \sum_{j=1}^k \left(X\left(\sum_{i=1}^j t a_i u_i\right) - X\left(\sum_{i=1}^{j-1} t a_i u_i\right)\right) .$$

By stationarity of the increments,

$$v(tu) = \mathbf{E}\left((X(tu) - X(0))^2\right) \;\le\; C \sum_{j=1}^k v(t a_j u_j) .$$

Then, for all $\alpha < \min(\beta(u_1), \dots, \beta(u_k))$, the quantity $|t|^{-2\alpha} v(tu)$ tends to 0 with t. Hence the map β is such that $\beta(u) \ge \min(\beta(u_1), \dots, \beta(u_k))$ for all u in the vector space generated by $u_1, \dots, u_k$.

Let us suppose now the existence of $d+1$ vectors with $d+1$ different values of β. Then one of them—say u—must be in the vector space generated by the d others—say $u_1, \dots, u_d$-, and then $\beta(u) > \min(\beta(u_1), \dots, \beta(u_k))$. One can always assume that $\beta(u_1) < \dots < \beta(u_d)$. If $\beta(u) > \beta(u_1)$ then u must belong to the vector space generated by $u_2, \dots, u_d$, because, otherwise, exchanging u and u_1 gives $\beta(u_1) \ge \beta(u)$. Then $\beta(u) > \min(\beta(u_2), \dots, \beta(u_d))$. Iterating the procedure yields a contradiction. This proves the first conclusion of the proposition.

Moreover, let us denote by β_0 the smallest value of β on $\mathbb{R}^d \setminus \{0\}$, realized at u_0. The set of values u for which $\beta(u) > \beta_0$ clearly generates a proper subspace of $\mathbb{R}^d$. □

Example. An example of a 2-parameter Gaussian field with stationary increments and non-constant directional regularity is given by Example 3 of Section 1.3 via the spectral density:

$$\xi \in \mathbb{R}^2 \;\mapsto\; \left(|\xi_1|^{H_1+1} + |\xi_2|^{H_2+1}\right)^{-2} ,$$

for any $0 < H_1 < H_2 < 1$. Using Proposition 4, the directional regularity is proved to be equal to $\frac{H_1+H_2+2H_1H_2}{2(H_2+1)}$ in all directions except in the direction of the vector $(0,1)$, where it is equal to $\frac{H_1+H_2+2H_1H_2}{2(H_1+1)}$.

This example can be easily adapted to any dimension, to find examples where the function β takes any number of values between 1 and d.

Actually, among all the examples presented in Section 1.3, the above example is the only one with non-constant directional Hölder exponent. A general result is established in the next subsection.

3.3 Anisotropic Gaussian Fields with Constant Directional Regularity

We prove now that, whenever the spectral density is bounded at ∞ by $C|\xi|^{-(2\beta+d)}$ and equivalent to it inside some cone, then the directional regularity of the associated d-parameter Gaussian field is constant and equal to β.

Proposition 6.

Let f be a standard spectral density and v^f its associated variogram defined by (1.2). *We assume that β is a positive number such that*

(i) *f satisfies $\mathbf{D}(\alpha)$ for all $\alpha < \beta$;*

(ii) *for all $\alpha > \beta$, there exists a constant $A > 0$ and a set $E \subset S^{d-1}$ of positive measure such that for almost all ξ in $\mathbb{R}^d$,*

$$|\xi| > A \ \ and \ \ \frac{\xi}{|\xi|} \in E \Rightarrow f(\xi) > |\xi|^{-(2\alpha+d)} ;$$

then for all $u \in S^{d-1}$, the map: $t \in \mathbb{R} \ \mapsto \ v^f(tu)$ satisfies condition $\mathbf{H}(\beta)$.

Remark. With Proposition 4, assumptions (i) and (ii) for a non-integer value of β imply that X^f admits β as directional regularity in any direction. Furthermore, when β is less than one, Proposition 3 states that β is also the lass critical index of X^f.

Examples. Examples 1 and 2 of Section 1.3 satisfy the required assumption with $\beta = \text{essinf}(h)$, so does Example 4 with $\beta = m$.

Proof. Since $v^f \circ R = v^{f\circ R}$ for any rotation R in $\mathbb{R}^d$ such that $R(0) = 0$ and since the assumptions are also satisfied by $f \circ R$, we only prove that

$$v \ : \ t \in \mathbb{R} \ \mapsto \ v(t) = v^f(t\mathbf{1})$$

satisfies condition $\mathbf{H}(\beta)$ with $\mathbf{1} = (0, \ldots, 0, 1) \in S^{d-1}$. For all $t \in \mathbb{R}$,

$$v(t) = \int_{\mathbb{R}^d} f_t(\xi)\, d\xi \ ,$$

with $f_t(\xi) = \sin^2(t\xi_d/2) f(\xi) = \frac{1}{2}(1 - \cos(t\xi_d)) f(\xi)$ and ξ_d the last coordinate of ξ. Let us denote by n the integer such that β lies in $(n, n+1]$.

First step: v is $(2n)$-differentiable. Indeed, from the identity

$$2\frac{\partial^{2n}}{\partial t^{2n}}\left(\sin^2(tx/2)\right) = (-1)^n x^{2n}\cos(tx) \ ,$$

we get the inequality

$$\left|\frac{\partial^{2n}}{\partial t^{2n}} f_t(\xi)\right| \le \frac{1}{2}|\xi_d|^{2n} f(\xi) \ .$$

The right hand side is integrable by the assumption (i). So v is $2n$-differentiable, and

$$(-1)^n \left(v^{(2n)}(t) - v^{(2n)}(0)\right) = \int_{\mathbb{R}^d} |\xi_d|^{2n} \sin^2(t\xi_d/2) f(\xi)\, d\xi \ .$$

Second step: for all $\alpha \in (n, \beta)$, the quantity $(-1)^n|t|^{2(n-\alpha)}(v^{(2n)}(t) - v^{(2n)}(0))$ tends to 0 when t tends to 0. Clearly, we might as well show that, for all such α, the same quantity is uniformly bounded for $0 < t < 1$. We write

$$|t|^{2(n-\alpha)} \int_{\mathbb{R}^d} |\xi_d|^{2n} \sin^2(t\xi_d/2) f(\xi)\, d\xi = I_1 + I_2$$

with I_1, I_2 the integrals on the sets $\{|\xi| < A\}$ and $\{|\xi| > A\}$, respectively. In the first one, $|\xi_d|^{2n}\sin^2(t\xi_d/2)$ can be bounded by $t^2|\xi|^{2n+2}$. The fact that I_1 is bounded follows at once from the fact that f is in $\mathbf{L^1}(\mathbb{R}^{\mathbf{d}}, \mathbf{d}\mu; \mathbb{R})$. In the second one, we change variables and use the condition $\mathbf{D}(\alpha)$ (assumption (i)) to obtain the bound

$$I_2 \le C \int_{|\xi|>A|t|} \sin^2(\xi_d/2)|\xi|^{2n-d-2\alpha}\, d\xi \ ,$$

which is clearly finite.

Third step: for all $\alpha > \beta$**, the quantity** $(-1)^n|t|^{2(n-\alpha)}(v^{(2n)}(t) - v^{(2n)}(0))$ **tends to** ∞ **when** t **tends to** 0**.** Clearly, as before, we might as well show that, for all such α, the same quantity is uniformly bounded below for $|t|$ small enough. By assumption (ii), take $A > 0$ and $E \subset S^{d-1}$ of positive measure such that

$$|\xi| > A \text{ and } \frac{\xi}{|\xi|} \in E \Rightarrow f(\xi) > |\xi|^{-(2\alpha+d)} ,$$

and take $|t| < A^{-1}$. Then

$$\begin{aligned} \int_{|\xi|>A} |\xi_d|^{2n}\sin^2(t\xi_d/2) f(\xi)\, d\xi \quad &\ge \quad \int_{|\xi|>A\,;\ \frac{\xi}{|\xi|}\in E} |\xi_d|^{2n} \frac{\sin^2(t\xi_d/2)}{|\xi|^{2\alpha+d}}\, d\xi \\ &\ge \quad |t|^{2(\alpha-n)} \int_{|\xi|>1;\ \frac{\xi}{|\xi|}\in E} |\xi_d|^{2n} \frac{\sin^2(\xi_d/2)}{|\xi|^{2\alpha+d}}\, d\xi \ , \end{aligned}$$

which allows to conclude. □

It is interesting to note that, although the Hölder critical exponent is the same in all directions, the Lipschitz constant can strongly depend on the direction. For instance in Example 1, if the set $\{h = m\}$ has positive measure, then, for each direction u, the variogram function deduced from (1.3) will be

$$v(tu) = \left(C(m) \int_{h^{-1}(\{m\})} |u.\alpha|^{2m}\, d\alpha \right) |t|^{2m} + o\left(|t|^{2m}\right) ,$$

and the constant before $|t|^{2m}$ clearly depends on u. The same phenomenon can be observed in Example 4. This type of analysis of anisotropy is studied in [5] where the directional Lipschitz constant is called "the topothesy function."

4. Projections and Averages

4.1 Projection of a Gaussian Field with Spectral Density

We investigate here the effect of a weighted projection on the Gaussian fields with stationary increments under consideration.

Let X be a d-parameter random field with continuous covariance, with $d \ge 2$. Let k be an integer between 1 and $d-1$. Let us choose a function φ (the weight function) in the set $\mathcal{W}_{d-k}$ of continuous maps from $\mathbb{R}^{d-k}$ to $\mathbb{R}$ with compact support and integral equal to 1. When X is continuous, we want to define *the projection of X with weight φ* as the average, with weight φ, in the first $d-k$ coordinates,

$$p(X,\varphi)(t) = \int_{\mathbb{R}^{d-k}} X(s,t)\varphi(s)\,ds\ ,\ \forall t \in \mathbb{R}^k\ .$$

We will see that the choice of the weight function φ does not play any role in the results, as soon as it is sufficiently smooth. For $k = d-1$, $p(X,\varphi)$ is the $d-1$-parameter random field obtained by projection of X onto an hyperplane. When $d = 3$, $p(X,\varphi)$ can be interpreted as the X-ray picture of a piece (determined by the window φ) of a material modeled by a 3D random field X. In all cases, we recognize a weighted Radon transform or X-ray transform. One may consult [13] for more general models which arise in tomography. For $d = 2$, the random field X can model the grey-level of an X-ray picture, and $p(X,\varphi)$ can be interpreted as the average, with weight φ, of X over all "horizontal" straight lines.

The definition of the projection as an integral is not convenient in our context and asks for strong assumptions, while we are mainly interested in second order statistics. Moreover, we keep in mind that integrals will be replaced by finite sums in order to be able to compute them on real data. This leads us to define the weighted projection of X by the following formula, which is inspired by [22] Chapter 1.4, and which coincides with the previous one on continuous trajectories,

$$p(X,\varphi)(t) = \lim_{n\to\infty} 2^{-n(d-k)} \sum_{j\in\mathbb{Z}^{d-k}} \varphi\left(2^{-n}j\right) X\left(2^{-n}j,t\right)\ ,\ \forall t \in \mathbb{R}^k\ . \qquad (4.1)$$

The existence of the limit, and the fact that it defines a random field of the same class, is given in the following proposition.

Proposition 7.

Let $\varphi \in \mathcal{W}_{d-k}$ and X be a d-parameter Gaussian field which vanishes at 0, has mean zero, stationary increments, with continuous variogram v_X. For $n \in \mathbb{N}$, let

$$Y_n(t) = 2^{-n(d-k)} \sum_{\gamma\in 2^{-n}\mathbb{Z}^{d-k}} \varphi(\gamma)X(\gamma,t)\ ,\ \forall t \in \mathbb{R}^k\ .$$

(i) *Then, for all $t_1, t_2, \ldots, t_l$, the sequence $(Y_n(t_1), Y_n(t_2), \ldots, Y_n(t_l))$ has a limit in $L^2(\Omega,\mathbb{R}^l)$. Moreover, the limit defines a Gaussian field $Y = p(X,\varphi)$ which has mean zero and stationary increments.*
(ii) *If X is a Gaussian field with spectral density f, then $p(X,\varphi)$ is a Gaussian field with spectral density $T(f)$ given by*

$$T(f)\ :\ \xi \in \mathbb{R}^k\ \mapsto\ \int_{\mathbb{R}^{d-k}} \left|\hat{\varphi}(x)\right|^2 f(x,\xi)\,dx\ . \qquad (4.2)$$

Moreover, if f satisfies the assumption $\mathbf{D}(m)$ for some m in $(0,1)$, then $T(f)$ does also (with the same m).

Recall that condition $\mathbf{D}(m)$ is sufficient to ensure the existence of a continuous modification (see Proposition 1).

Note that the second point proves that the class of Gaussian fields with spectral density is stable under projections. Therefore, when a material is modeled by a (3-parameter) Gaussian field with spectral density, then X-ray pictures of this material can be modeled in the same way.

Proof. The proof of the convergence is elementary but tedious. We will sketch it when $l = 1$. Its generalization does not present any difficulty. To prove that the sequence $Y_n(t)$ is a Cauchy sequence in $L^2(\Omega)$, it is sufficient to prove that the covariance $\mathrm{Cov}(Y_m(t), Y_n(t))$ has a limit when m and n tend to ∞. Using the variogram of X, we can write $\mathrm{Cov}(Y_m(t), Y_n(t))$ as

$$2^{-(m+n)(d-k)} \sum_{\gamma\in 2^{-m}\mathbb{Z}^{d-k}} \sum_{\gamma'\in 2^{-n}\mathbb{Z}^{d-k}} \varphi(\gamma)\varphi\left(\gamma'\right) \mathrm{Cov}(X(\gamma,t), X\left(\gamma',t\right) \ .$$

This last covariance may be written using the variogram,

$$\mathrm{Cov}\left(X(\gamma,t), X\left(\gamma',t\right)\right) = 2\left(v_X(\gamma,t) + v_X\left(\gamma',t\right) - v_X\left(\gamma-\gamma',0\right)\right. .$$

We recognize Riemann sums of the integral of a continuous function, which converge to the corresponding integral.

Once we have proved the convergence in $L^2(\Omega)$, it follows immediately that the limit Y is also Gaussian with mean zero. Let us prove that it has stationary increments, and compute its variogram. As before, for all t and t' in $\mathbb{R}^k$, $\mathbf{E}(|Y(t)-Y(t')|^2)$ may be written as

$$\lim_{n\to\infty} 2^{-2n(d-k)}\mathbf{E}\left(\left|\sum_{j\in\mathbb{Z}^{d-k}} \left(X\left(2^{-n}j,t\right) - X\left(2^{-n}j,t'\right)\right)\varphi\left(2^{-n}j\right)\right|^2\right) .$$

Since v_X is continuous and φ is continuous, the limit is again an integral:

$$2\int_{\mathbb{R}^{2(d-k)}} \left(v_X\left(s-s',t'-t\right) + v_X\left(s-s',t-t'\right) - 2v_X\left(s-s',0\right)\right)\varphi(s)\varphi\left(s'\right)\,ds\,ds'.$$

This shows that $\mathbf{E}((Y(t)-Y(t'))^2)$ depends only on $t-t'$, which proves that Y has stationary increments, and gives an explicit formula for the variogram v_Y.

(ii) Suppose now that X has a spectral density f. The previous computation of the variance of the increments of Y gives, for all t in $\mathbb{R}^k$,

$$v_Y(t) = \frac{1}{2}\int_{\mathbb{R}^{2(d-k)}} \left(v_X\left(s-s',-t\right) + v_X\left(s-s',t\right) - 2v_X\left(s-s',0\right)\right)\varphi(s)\varphi\left(s'\right)\,ds\,ds'.$$

Let us use Fubini's theorem and denote by $\check{\varphi} : s \ \mapsto \ \varphi(-s)$ to get

$$\begin{aligned} v_Y(t) &= \frac{1}{2}\int_{\mathbb{R}^{d-k}} (v(s,t) + v(s,-t) - 2v(s,0))\,\varphi * \check{\varphi}(s)\,ds \\ &= \frac{1}{2}\int_{(\xi_1,\xi_2)\in\mathbb{R}^{d-k}\times\mathbb{R}^k} (1-\cos(t.\xi_2))\,f(\xi_1,\xi_2) \\ &\quad \times\left(\int_{\mathbb{R}^{d-k}} \cos(s.\xi_1)\,\varphi * \check{\varphi}(s)\,ds\right) d\xi_1\,d\xi_2 \\ &= \frac{1}{2}\int_{(\xi_1,\xi_2)\in\mathbb{R}^{d-k}\times\mathbb{R}^k} (1-\cos(t.\xi_2))\,f(\xi_1,\xi_2)\left|\hat{\varphi}(\xi_1)\right|^2\,d\xi_1\,d\xi_2 \\ &= \int_{\mathbb{R}^k} \sin^2(t.\xi_2/2)T(f)(\xi_2)\,d\xi_2 \ . \end{aligned}$$

Then $T(f)$, given by (4.2), is the required spectral density for Y.

In order to prove that $T(f)$ inherits from f the assumption $\mathbf{D}(m)$, just note that, for almost all $\xi\in\mathbb{R}^k$ such that $|\xi| > A$, we have

$$T(f)(\xi) \le \left(||\varphi||_\infty^2 \int_{\mathbb{R}^{d-k}} \left(1+|y|^2\right)^{-\frac{1}{2}(2m+d)}\,dy\right)\times|\xi|^{-(2m+k)} \ . \qquad \square$$

4.2 Regularity of the Averages

We have in mind possible applications to the analysis of some material (or X-ray pictures) through the analysis of one-parameter processes. So, from now on, we will only deal with averages over hyperplanes of d-parameter Gaussian fields, that is to say with projections where the weight function φ belongs to $\mathcal{W}_{d-1}$ (with the notations of previous subsection). One moment of reflection allows to see the possibility of generalizations to other situations.

We prove now that if the spectral density f of X behaves like $\xi \mapsto |\xi|^{-(2\beta+d)}$ asymptotically in the "vertical" direction $\mathbf{1} = (0, \ldots, 0, 1)$, then the process of "horizontal" averages $p(X, \varphi)$ has Hölder critical exponent $\beta + \frac{1}{2}(d-1)$. Using Proposition 4, this will be the case if the variogram of $p(X, \varphi)$ satisfies $\mathbf{H}(\beta + \frac{1}{2}(d-1))$.

Proposition 8.

Let f be a standard spectral density and assume the existence of positive constants m and β such that
(i) *f satisfies $\mathbf{D}(m)$;*
(ii) *for all $\alpha < \beta$, there exists a constant $A > 0$ and a neighborhood $E \subset S^{d-1}$ of $(0, \ldots, 0, 1)$ such that for almost all ξ in $\mathbb{R}^d$,*

$$|\xi| > A \text{ and } \frac{\xi}{|\xi|} \in E \Rightarrow f(\xi) \leq |\xi|^{-(2\alpha+d)} .$$

(iii) *for all $\alpha > \beta$, there exists a constant $A > 0$ and a neighborhood $E \subset S^{d-1}$ of $(0, \ldots, 0, 1)$ such that for almost all ξ in $\mathbb{R}^d$,*

$$|\xi| > A \text{ and } \frac{\xi}{|\xi|} \in E \Rightarrow f(\xi) > |\xi|^{-(2\alpha+d)} .$$

Let φ belongs to $\mathcal{W}_{d-1}$ such that $|\hat{\varphi}|^2$ satisfies $\mathbf{D}(\beta)$, then, the variogram of the projection $p(X, \varphi)$ satisfies condition $\mathbf{H}(\beta + \frac{1}{2}(d-1))$.

Proof. Recall that the projection $p(X, \varphi)$ is defined by (4.1) with $k = 1$ and has a spectral density $T(f)$ given by (4.2). Using Proposition 6 in one dimension, it is sufficient to prove the next lemma.

Lemma 4.

Under the assumptions of Proposition 8, the spectral density $T(f)$ is such that

- *for all $\alpha < \beta + \frac{1}{2}(d-1)$, $T(f)$ satisfies $\mathbf{D}(\alpha)$;*
- *for all $\alpha > \beta + \frac{1}{2}(d-1)$, there exists a constant $A > 0$ such that*

$$|\xi| > A \Rightarrow T(f)(\xi) > |\xi|^{-(2\alpha+1)} .$$

Proof of the lemma. Let $\alpha < \beta + \frac{1}{2}(d-1)$. By assumptions (i) and (ii) there exists a constant $A > 0$ and a neighborhood $E \subset S^{d-1}$ of $(0, \ldots, 0, 1)$ such that for almost all ξ in $\mathbb{R}^d$,

$$\begin{aligned} |\xi| > A \quad &\Rightarrow \quad f(\xi) \leq |\xi|^{-(2m+d)} , \\ |\xi| > A \text{ and } \frac{\xi}{|\xi|} \in E \quad &\Rightarrow \quad f(\xi) \leq |\xi|^{-[2(\alpha - \frac{1}{2}(d-1))+d]} = |\xi|^{-(2\alpha+1)} . \end{aligned}$$

We can assume that E is of the form $\{(x,\xi) \in \mathbb{R}^{d-1} \times \mathbb{R}\ ;\ |x| < \eta|\xi|\}$ for some positive constant η. For ξ in $\mathbb{R}$ such that $|\xi| > A$, we write $T(f)(\xi) = T_1(f)(\xi) + T_2(f)(\xi)$, where

$$\begin{aligned} T_1(f)(\xi) &= \int_{x\in\mathbb{R}^{d-1};\ (x,\xi)/|(x,\xi)|\in E} \left|\hat{\varphi}(x)\right|^2 f(x,\xi)\, dx \\ &\le |\xi|^{-(2\alpha+1)} \times \int_{\mathbb{R}^{d-1}} \left|\hat{\varphi}(x)\right|^2\, dx \ = \ C|\xi|^{-(2\alpha+1)}\,, \end{aligned}$$

and

$$\begin{aligned} T_2(f)(\xi) &= \int_{x\in\mathbb{R}^{d-1};\ (x,\xi)/|(x,\xi)|\notin E} \left|\hat{\varphi}(x)\right|^2 f(x,\xi)\, dx \\ &\le C \int_{x\in\mathbb{R}^{d-1};|x|\ge\eta|\xi|} |x|^{-(2\beta+d-1)}\, |x|^{-(2m+d)}\, dx \\ &\le C|\xi|^{-(2\beta+2m+2d-2)} \le C|\xi|^{-(2\alpha+1)}\,. \end{aligned}$$

The constant C may change from one line to another. The assumption on φ has been used for the second inequality. Both inequalities for $T_1(f)$ and $T_2(f)$ prove the first part of the lemma.

For $\alpha > \beta + \frac{1}{2}(d-1)$, by assumption (iii),

$$|\xi| > A \text{ and } \frac{\xi}{|\xi|} \in E \Rightarrow f(\xi) > |\xi|^{-(2\alpha+1)}\,.$$

As before, we may assume that $E = \{(x,\xi) \in \mathbb{R}^{d-1} \times \mathbb{R}\ ;\ |x| < \eta|\xi|\}$ for some positive constant η. Then, for ξ in $\mathbb{R}$ such that $|\xi| > A$,

$$\begin{aligned} T(f)(\xi) &\ge \int_{x\in\mathbb{R}^{d-1};\ |x|<\eta|\xi|} \left|\hat{\varphi}(x)\right|^2 f(x,\xi)\, dx \\ &\ge \left(\int_{|x|<\eta A} \left|\hat{\varphi}(x)\right|^2\, dx\right) |\xi|^{-(2\alpha+1)}\,. \end{aligned}$$

This finishes the proof of the lemma, as well as the proof of the proposition. □

Remark that we can similarly describe the asymptotic self-similarity for the average-process $p(X,\varphi)$. Sufficient conditions to get a lass critical index (see Proposition 3) are provided by Lemma 4, at least when $\beta+\frac{1}{2}(d-1)$ belongs to $(0,1)$. In the case $\beta+\frac{1}{2}(d-1) > 1$, a lass property could certainly be observed, not for the process $p(X,\varphi)$ itself, but for the process of nth-order increments (see [20]) with n the integer part of $\beta + \frac{1}{2}(d-1)$.

4.3 The Directional Average Method

We now describe the *directional average method*, which allows to recover the asymptotic properties of the function f in each direction.

Let f be a standard spectral density and X^f be d-parameter Gaussian field with spectral density f. For any direction u in S^{d-1}, we average the field X^f over all the hyperplanes orthogonal to u.

More precisely, we first choose a weight function in the set $\mathcal{W}_{d-1}$. For any direction u in S^{d-1}, we obtain the *average-process in the direction u* by the following prescription:

- first compose X^f with a rotation R_u of center 0 which maps the direction $\mathbf{1} = (0, \dots, 0, 1)$ onto u,
- then compute the projection with weight φ, that is, consider

$$Y^{f,\varphi,u}(t) = p(X \circ R_u, \varphi)(t) \,, \tag{4.3}$$

which is defined a.s. for all t in $\mathbb{R}$.

We state now our main theorem.

Theorem 1.

Let f be a standard spectral density and assume the existence of positive constants m and M such that
(i) *f satisfies $\mathbf{D}(m)$;*
(ii) *for all u in S^{d-1}, there exists $\beta(u) \in (0, M]$ such that*

- *for all $\alpha < \beta(u)$, there exists a constant $A > 0$ and a neighborhood $E \subset S^{d-1}$ of u such that for almost all ξ in $\mathbb{R}^d$,*

$$|\xi| > A \text{ and } \frac{\xi}{|\xi|} \in E \Rightarrow f(\xi) \leq |\xi|^{-(2\alpha+d)} .$$

- *for all $\alpha > \beta(u)$, there exists a constant $A > 0$ and a neighborhood $E \subset S^{d-1}$ of u such that for almost all ξ in $\mathbb{R}^d$,*

$$|\xi| > A \text{ and } \frac{\xi}{|\xi|} \in E \Rightarrow f(\xi) > |\xi|^{-(2\alpha+d)} .$$

Let $\varphi \in \mathcal{W}_{d-1}$ be such that $|\hat{\varphi}|^2$ satisfies $\mathbf{D}(M)$. Then, for all u in S^{d-1}, the variogram of the average-process in the direction u, $Y^{f,\varphi,u}$ defined by (4.3)*, satisfies the assumption* $\mathbf{H}(\beta(u) + \frac{1}{2}(d-1))$.

Proof. Since all the conditions are invariant by rotations of center 0, it is sufficient to prove that the variogram of $Y^{f,\varphi,\mathbf{1}}$ satisfies the assumption $\mathbf{H}(\beta + \frac{1}{2}(d-1))$ with $\beta = \beta(\mathbf{1})$ and $\mathbf{1} = (0, \dots, 0, 1)$.

But this is claimed in Proposition 8. Just note that if $|\hat{\varphi}|^2$ satisfies the assumption $\mathbf{D}(M)$ with $\beta \leq M$ then it also satisfies $\mathbf{D}(\beta)$. □

Let us come back to the examples of the Section 1.3 and see how the previous theorem does apply to obtain non trivial results.

Examples 1 and 2. If h is an even continuous map from S^{d-1} into $[m, M] \subset (0, 1)$, the spectral density

$$\xi \in \mathbb{R}^d \;\mapsto\; f(\xi) = \frac{1}{|\xi|^{2h(\xi)+d}} \quad \left(\text{or } \frac{1}{\left(1 + |\xi|^2\right)^{\frac{1}{2}(2h(\xi)+d)}} \right)$$

satisfies the assumption (ii) of the theorem with $\beta(u) = h(u)$ for all $u \in S^{d-1}$. Then any weight φ in $\mathcal{W}_{d-1}$ such that $|\hat{\varphi}(\xi)| = O(|\xi|^{-\frac{1}{2}(1+d)})$ at infinity, provides directional averages which have Hölder critical exponent $h(u) + \frac{1}{2}(d-1)$ for all directions u.

With a slight refinement, one can prove that the result continues to hold when h has only right and left limits at each point; in this case the Hölder exponent is given by $\beta(u) = \min(h(u^+), h(u^-))$.

In the Example 2, there is no reason to limit ourselves to values of M below 1.

Example 3. In the case $H_1 < H_2$, assumptions (i) and (ii) are satisfied with $m = H_1$, $M = H_2$ and

$$\beta(\theta) = H_1 \ \text{if} \ \theta \neq \frac{\pi}{2}; \quad \beta\left(\frac{\pi}{2}\right) = H_2 \, .$$

So, again, we get a directional analysis which is constant except for one value.

Example 4. Just take $m = M = \beta(u)$ for all $u \in S^{d-1}$ and the theorem applies, but gives again apparent isotropy.

Conclusion

Our starting point was the fact that, basically, there does not exist a stationary random field which presents different directional Hölder critical exponents in different directions. So, if we are dealing with a material or with an image which presents anisotropy, then one has to choose a model which has the same property, and the measurement of anisotropy has to be done with other tools. The present work provides an analysis method for anisotropic 2D- or 3D-data which is based on Hölder critical exponents of directional averages. We have proved that there does exist random fields for which this analysis allows to recover the information on the anisotropy of the model. Clearly Examples 1 and 2 are good candidates for Gaussian fields with anisotropic regularity.

It remains to see whether they are good candidates for real data. To test different tools, classical methods for fractional Brownian motion simulations may be adapted to produce simulations. Estimation of the directional regularity, using by now standard methods (see [8]), will require to replace the integral averages by discrete ones, which has to be done carefully if one does not want to loose the smoothing effect that we have exploited here.

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MAPMO-UMR 6628, Département de Mathématiques, Université d'Orléans
45067 Orléans Cedex 2, France
e-mail: aline.bonami@labomath.univ-orleans.fr

MAPMO-UMR 6628, Département de Mathématiques, Université d'Orléans
45067 Orléans Cedex 2, France
e-mail: anne.estrade@labomath.univ-orleans.fr