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Correlated Gaussian systems exhibiting additive power-law entropies

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Abstract

We show, on purely statistical grounds and without appeal to any physical model, that a power-law q-entropy S_q, with $0 < q < 1$, can be *extensive*. More specifically, if the components X_i of a vector $X \in \mathbb{R}^N$ are distributed according to a Gaussian probability distribution, the associated entropy $S_q(X)$ exhibits the extensivity property for special types of correlation among the X_i. We also characterize this kind of correlation.
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1. Introduction

The Boltzmann–Gibbs logarithmic expression

$$S_{BG}\{p(k)\} = -\sum_k p(k) \ln p(k), \tag{1}$$

where one sums over microstates labelled by k, constitutes a cornerstone of our present understanding of Nature. If one knows a priori the expectation value $U = \langle H \rangle$ of the pertinent Hamiltonian, S_{BG} is maximized by Gibbs' probability distribution (PD) for the canonical ensemble [1–4], usually referred to as the equilibrium Boltzmann–Gibbs (BG) distribution

$$p_G(k) = \frac{\exp(-\beta E_k)}{Z_{BG}}, \tag{2}$$

with E_k the energy of the microstate k, $\beta = 1/k_B T$ the inverse temperature, k_B Boltzmann's constant, and Z_{BG} the partition function. This PD can safely be regarded as the most notorious and renowned PD in the field of statistical mechanics.

In the last 15 years this PD has found a counterpart in the guise of power-law distributions with which an "alternative" thermostatistics (usually referred to by the acronym "NEXT" for "NonEXTensive") can be built that constitutes nowadays not only a very active field but also, for many people, a new paradigm for statistical mechanics, with applications to several scientific disciplines [5–8]. Power-law distributions are certainly ubiquitous in physics (critical phenomena are just a conspicuous example [9]). It is argued [6] that, in the same manner that the BG-thermostatistics addresses thermal equilibrium states on the basis of Boltzmann's molecular chaos hypothesis (*Stosszahl-ansatz*), the alternative thermostatistics deals with phenomena in natural, artificial, and social systems that do not accommodate with such a simplifying hypothesis. However, NEXT uses the whole BG theoretical machinery, as for instance, the maximum entropy principle (MaxEnt) [4].

2. The extensivity question

The power law entropy S_q, with nonextensivity index $q \in \mathbb{R}$ [5]

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$$S_q = \frac{1}{q-1}\left(1 - \sum_k p_k^q\right), \qquad S_1 = S_{BG}, \tag{3}$$

is a nonextensive information measure: If A and B are independent systems then a priori $S_q(A + B) \neq S_q(A) + S_q(B)$. This assertion has come into question quite recently [6]. There exist composite systems for which the correlation among its components is of such a nature that S_{BG} becomes *nonextensive*, while S_q becomes *extensive* [6,10–12]. These references deal with exceedingly interesting examples of this new facet of q-thermostatistics. In particular, Refs. [11,12] need the special concept of *asymptotic scale invariance*. Ref. [10] considers both a classical and a quantum example in the thermodynamic limit. In the present communication we address a somewhat more general situation in the sense that we do not have any particular model in mind but a whole *class* of systems: Gaussian ones, of enormous importance in several areas of scientific endeavor.

3. Our main result

We will not need appeal neither to the *Stosszahl-ansatz* nor to *asymptotic scale invariance*. Our arguments are of a purely statistical nature, which can be regarded as giving them more generality. We will concern ourselves in what follows with the continuous instance in $\mathbb{R}^N$. Let the N components of vector $X \in \mathbb{R}^N$ be distributed according to a probability distribution f and write Tsallis' entropy $S_q(X)$ as

$$S_q(X) = \frac{1}{q-1}\left(1 - \int f^q\right). \tag{4}$$

Our main result is easily stated and reads as follows. Let f be of a Gaussian character. Then:

Theorem 1. *If $0 \leqslant q \leqslant 1$ and $N \in \mathbb{N}$ then there exists a positive definite matrix K and an N-variate Gaussian vector X with covariance matrix K such that X verifies the extensivity condition*

$$S_q(X) = \sum_{i=1}^{N} S_q(X_i). \tag{5}$$

3.1. Proof of the theorem

A centered Gaussian random variable with dimension N and covariance matrix K has a probability distribution function (PDF)

$$f_X(X) = \frac{1}{|2\pi K|^{1/2}} \exp\left(-\frac{1}{2}X^T K^{-1} X\right), \tag{6}$$

and thus its associated S_q entropy is easily seen to write

$$S_q(X) = \frac{1 - |2\pi K|^{\frac{1-q}{2}} q^{-N/2}}{q-1}. \tag{7}$$

One may assume without loss of generality that each component X_i of X has unit variance so that its associated q-sub-entropy reads

$$S_q(X_i) = \frac{1 - (2\pi)^{\frac{1-q}{2}} q^{-1/2}}{q-1}. \tag{8}$$

Now, the condition for extensivity adopts the appearance

$$S_q(X) = \sum_{i=1}^{N} S_q(X_i). \tag{9}$$

Alternatively one can write

$$1 - |2\pi K|^{\frac{1-q}{2}} q^{-N/2} = N\left(1 - (2\pi)^{\frac{1-q}{2}} q^{-1/2}\right), \tag{10}$$

or, equivalently,

$$|K|^{\frac{1-q}{2}} = N(2\pi)^{(1-N)\frac{1-q}{2}} q^{\frac{N-1}{2}} - (N-1)q^{N/2}(2\pi)^{N\frac{q-1}{2}}. \tag{11}$$

Notice that the right-hand side above is positive, since the function $f_N(q)$

$$f_N(q) = \frac{N}{N-1}\frac{(2\pi)^{\frac{1-q}{2}}}{\sqrt{q}}, \qquad 0 \leqslant q \leqslant 1, \tag{12}$$

decreases with q in this interval, verifies $f_N(1) = \frac{N}{N-1} > 1$ and thus $f_N(q) > 1$ on $[0, 1]$. Thus, Eq. (11) can be solved as

$$|K| = \left(N(2\pi)^{(1-N)\frac{1-q}{2}} q^{\frac{N-1}{2}} - (N-1)q^{N/2}(2\pi)^{N\frac{q-1}{2}}\right)^{\frac{2}{1-q}}. \tag{13}$$

It only remains now to find a matrix K with unit diagonal entries that satisfies the condition (13). This entails that one has to choose $\frac{N(N-1)}{2}$ correlation coefficients

$$K_{i,j} = E[X_i X_j] = \langle X_i X_j \rangle, \qquad 1 \leqslant i < j \leqslant N. \tag{14}$$

In the two following subsections, we provide two examples of such matrix.

3.2. Example: All uncorrelated components except for two of them

An obvious choice for the off-diagonal elements of K is

$$K_{i,j} = \begin{cases} \sigma, & i = 2, \ j = 1 \text{ or } i = 1, \ j = 2, \\ 0, & \text{else } (i \neq j), \end{cases}$$

so that $|K| = 1 - \sigma^2$ where σ should verify

$$\begin{cases} \sigma^2 = 1 - \left(N(2\pi)^{(1-N)\frac{1-q}{2}} q^{\frac{N-1}{2}} \right. \\ \qquad \left. - (N-1)q^{N/2}(2\pi)^{N\frac{q-1}{2}}\right)^{\frac{2}{1-q}}, \\ 0 < \sigma < 1, \end{cases}$$

which has a unique solution since function

$$g_N : [0, 1] \rightarrow [0, 1],$$
$$g_N(q) = 1 - \left(N(2\pi)^{(1-N)\frac{1-q}{2}} q^{\frac{N-1}{2}} \right.$$
$$\qquad \left. - (N-1)q^{N/2}(2\pi)^{N\frac{q-1}{2}}\right)^{\frac{2}{1-q}}, \tag{15}$$

is decreasing with q, verifies $g_N(0) = 1$ and $g_N(1) = 0$ and thus is one to one. In Fig. 1 below, the function $g_N(q)$ is plotted versus q for values $N = 2, 5, 10$ and 20 from bottom to top.
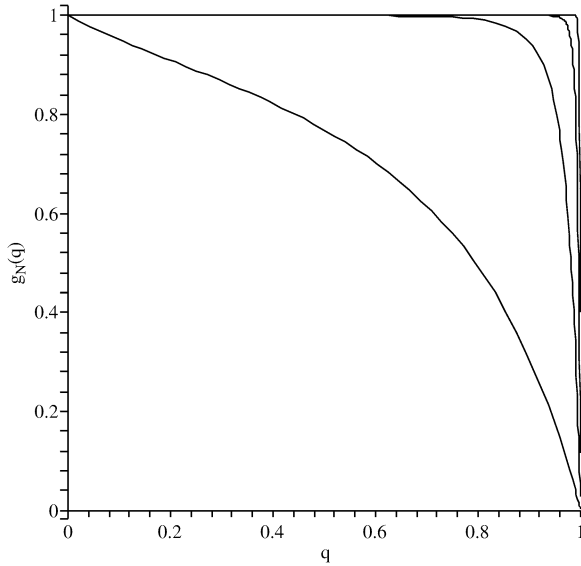
Fig. 1. Function $g_N(q)$ versus q for values $N = 2, 5, 10$ and 20 (bottom to top).

Fig. 2. Unique value of σ as a function of q for $N = 2, 3$ and 5 (bottom to top).

It is apparent that, for large values of N, this curve approaches a constant (equal to unity) in the interval $[0, 1[$, while it vanishes for $q = 1$. This means that, *for a fixed value of q, a more and more intense correlation degree becomes necessary between X_1 and X_2 as N grows so as to ensure extensivity for the system at hand*. The same conclusion is reached in Ref. [10] for their models.

3.3. Example: All equally correlated components

Another obvious choice is to select a correlation matrix of the form

$$K_{i,j} = \begin{cases} \sigma, & i \neq j, \\ 1, & \text{else}, \end{cases}$$

so that each pair of distinct components has the same degree of correlation. In such scenario we have

$$|K| = (1 - \sigma)^{N-1}\big(1 + (N-1)\sigma\big),$$

which is, for a given N, a decreasing function $g(\sigma)$ of σ such that $g(0) = 1$ and $g(1) = 0$. Thus, for given values of N and q, there exists a unique value of σ such that condition (13) holds. In Fig. 2 these values of σ are plotted as a function of q for $N = 2, 3$ and 5, respectively.

Identical conclusions to those of the preceding subsection are arrived at here, except that a smaller correlation degree is now needed to reach extensivity, because the correlation is in this instance spread over a larger number of components than previously.

4. Conclusion

We have shown that, for *any* Gaussian distributed vector in $\mathbb{R}^N$, Tsallis' entropy becomes *extensive* for $q \in [0, 1]$ if an adequate type of correlation among its components exists. This correlation imposes special conditions on the covariance matrix that apply only to its determinant and can always be
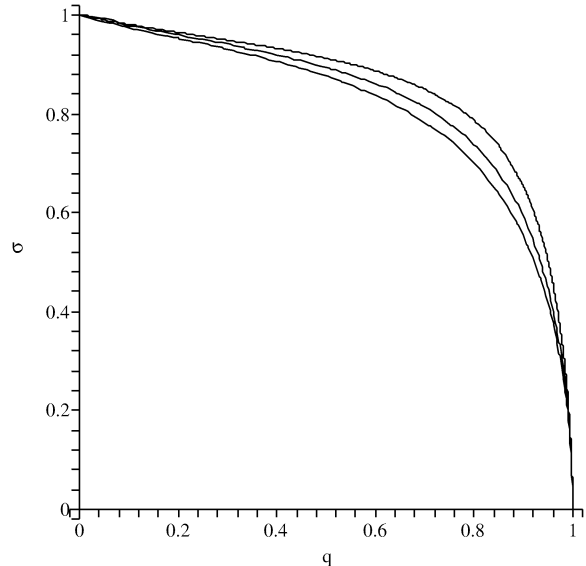
determined. Moreover, we provided a constructive proof that explicitly yields the covariance matrix that fulfills the desired purpose. We did not need appealing neither to the *Stosszahlansatz* nor to *asymptotic scale invariance*. Our arguments being of a purely statistical nature, they can be fairly said to considerably generalize the results described in [6].

It would be of interest to undertake further work on the themes here addressed involving (i) ascertaining whether our approach can be applied to Q-Gaussians and, if so, what would be the relation between Q and q for fixed N and (ii) concocting new examples for investigating the thermodynamic limit. This is of relevance because, in the present communication (like in [12], and in a couple of examples in the recent PNAS [6] by Tsallis, Gell-Mann and Sato), it is NOT possible to maintain q strictly below unity up to the thermodynamic limit $N \to \infty$. It would then be of interest to produce examples in which we could indeed have $q < 1$ up to the thermodynamic limit.[1]

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