Scaling and Multifractal: From Theory to Applications.

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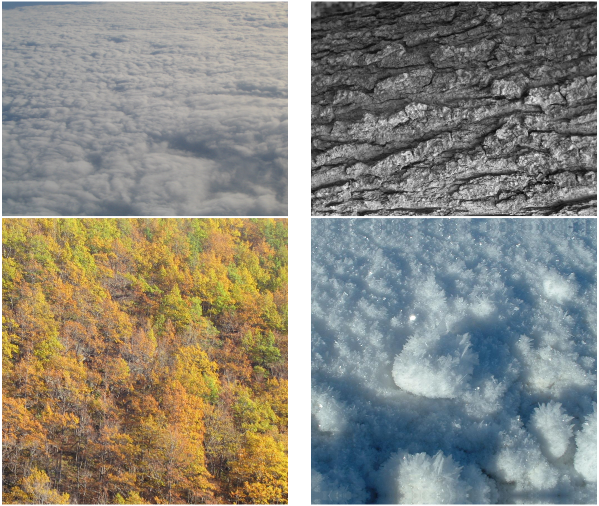
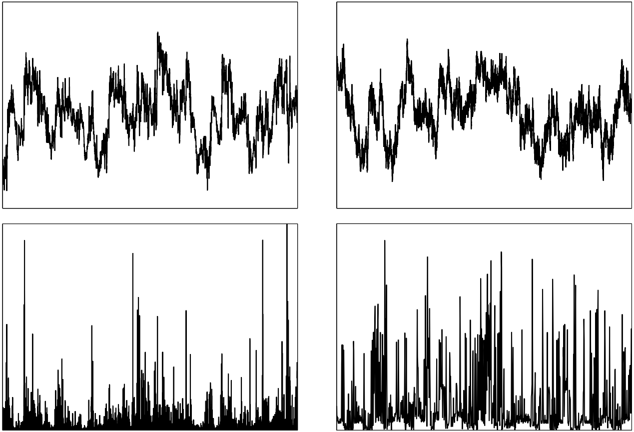
Empirical data : Images



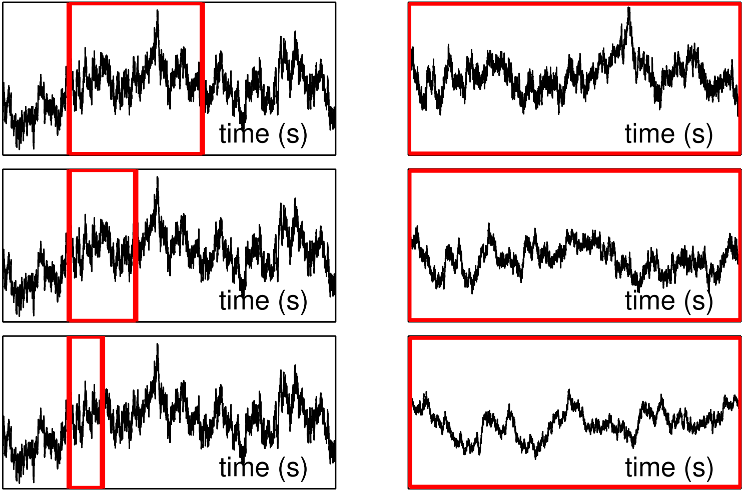
Empirical data : Signals



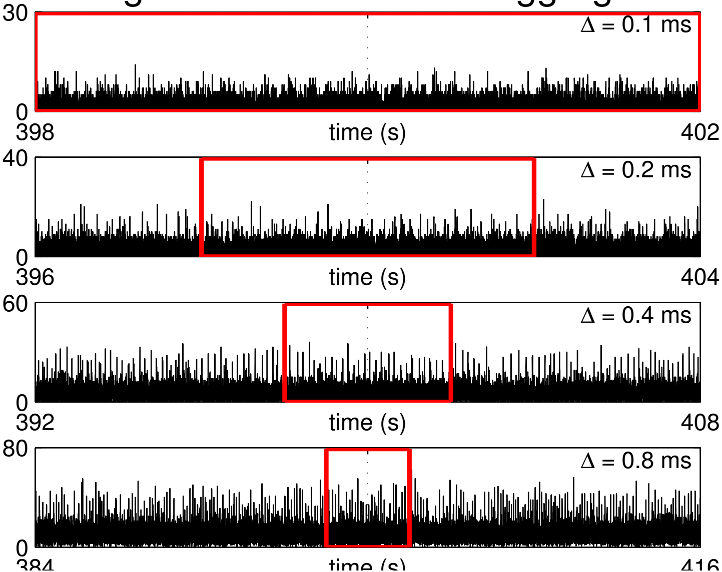
$\Longrightarrow$ describing/analyzing/modeling ?

$\Longrightarrow$ irregularity ? variability ?

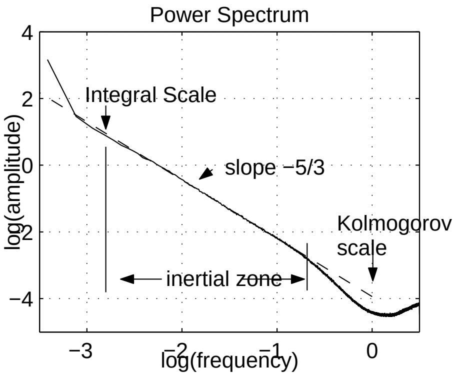
Scaling ? Covariance under dilatation

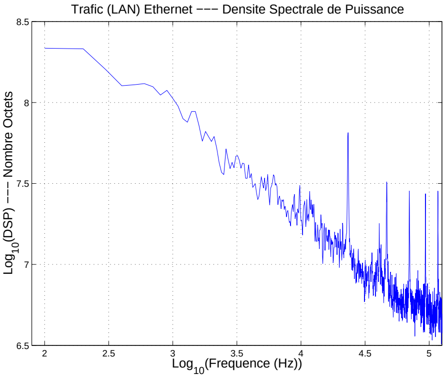


scaling ? Covariance under aggregation



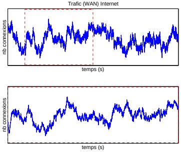
Scaling ? Power-law spectrum.

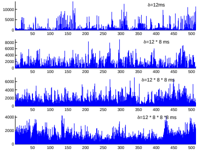


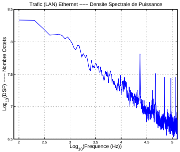


Scaling : Intuitive definition

- Definition : no characteristic scale !
- Equivalently : all scales are equally characteristic !
- Manifestation : cannot distinguish the whole from the part.

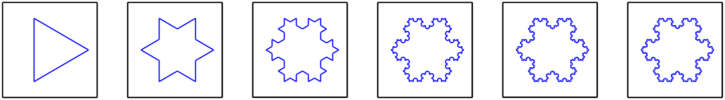






- Modeling :

rather than characteristic scale indentification,
seek for mechanisms relating scales.



Scaling and power laws

$$\begin{aligned} X(t/a) &= g(1/a)X(t), \\ X(at) &= g(a)X(t) \\ X(at) &= X(a)X(t) \end{aligned}$$

$$\begin{aligned} X(t_1 \cdot t_2) &= X(t_1) \cdot X(t_2) \\ 1D \Rightarrow |t_1 \cdot t_2|^{-\gamma} &= |t_1|^{-\gamma} \cdot |t_2|^{-\gamma} \end{aligned}$$

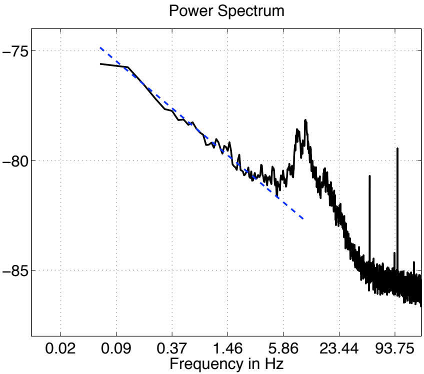
$$1D \Rightarrow \boxed{X(t) = |t|^{-\gamma}}$$

$$\boxed{\text{Scaling} \Rightarrow \text{Power Law}}$$

$1/f$-process (Model 1)

- 2nd order stationary $1/f$-process

$$\Gamma_Y(\nu) = C|\nu|^{-\gamma}, \ \gamma > 0, \ \ \nu_m \le |\nu| \le \nu_M, \ \frac{\nu_M}{\nu_m} \gg 1$$



Data MEG, Courtesy, Ph. Ciuciu, Neurospin, France

Scale Invariance Analysis Tool 1 : Spectrum Analysis

- Spectrum Estimation

  (time windowed) Periodogram or Welch estimator
  $|\tilde{Y}_{k,T}(\nu)|^2$ spectral estimate in $t \in [t_k - T/2, t_k + T/2]$
  $\hat{\Gamma}_Y(\nu) = \sum_k |\tilde{Y}_{k,T}(\nu)|^2$

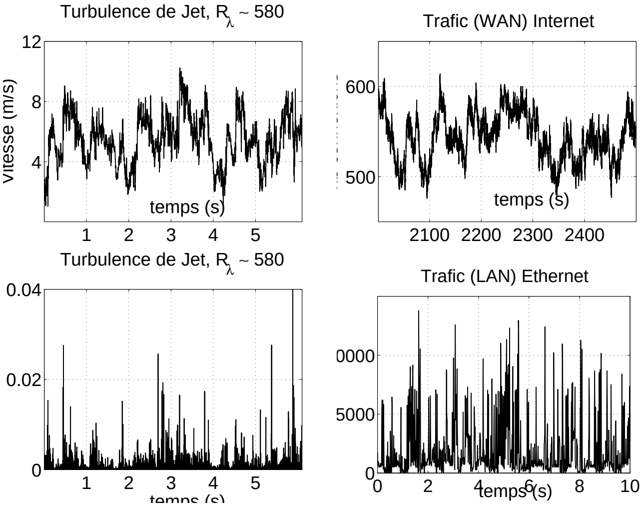
- $1/f$-spectrum :

  $\frac{1}{K}\sum_k |\tilde{Y}_{k,T}(\nu)|^2 = C|\nu|^{-\gamma}$

- Estimation :

  $\hat{\gamma} \rightarrow \log \frac{1}{K}\sum_k |\tilde{Y}_{k,T}(\nu)|^2$ versus $\log |\nu|$
  involve $\hat{\gamma}$ in analysis, detection, classification

Scaling in applications ?



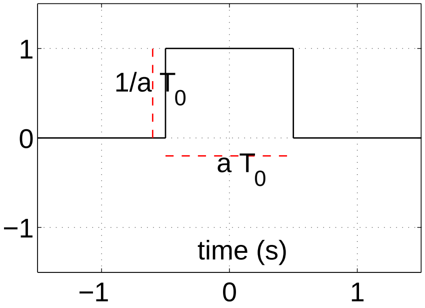
Scaling in applications ?

- hydrodynamic turbulence,
- statistical physics (long range interactions),
- astrophysics and cosmology (universe structure),
- geophysics (failures, earthquake),
- hydrology (nivology),
- physiology, biology, body rhythms (heart, gait),
- brain activity (fMRI),
- genomic,
- internet,
- geography (population repartition),
- artificial vision (intelligence),
- financial market,
- ...

Scaling analysis : Aggregation

Average within box of size a

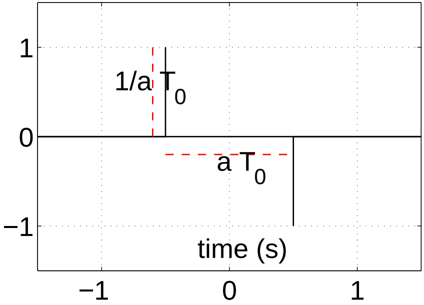
$$T_X(a,t) = \frac{1}{aT_0}\int_t^{t+aT_0} X(u)du$$



Scaling analysis : Increments

Difference over step lag of size a

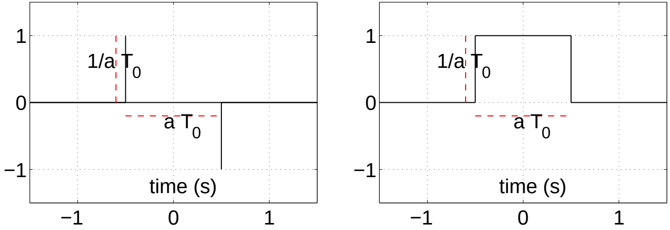
$$T_X(a,t) = X(t + aT_0) - X(t)$$



Scaling analysis : multiresolution analysis

- $X(t) \to T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0\left(\frac{u-t}{a}\right)$

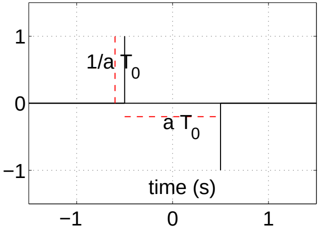
increment	Aggregation
DIFFERENCE	AVERAGE
$X(t+aT_0) - X(t)$	$\frac{1}{aT_0} \int_t^{t+aT_0} X(u)du$

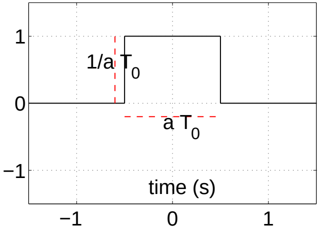


Scaling analysis : multiresolution analysis

- $X(t) \to T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0(\frac{u-t}{a})$

increment	Aggregation	?
DIFFERENCE	AVERAGE	?
$X(t + aT_0) - X(t)$	$\frac{1}{aT_0} \int_t^{t+aT_0} X(u)du$	?



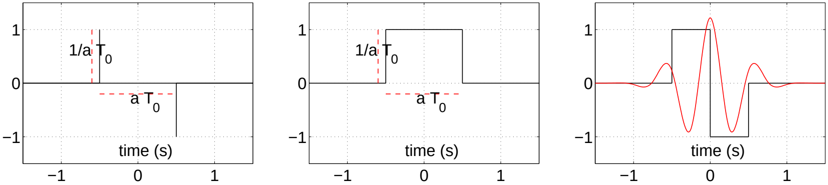


?

Multiresolution analysis

- $X(t) \rightarrow T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0(\frac{u-t}{a})$

increment difference	aggregation average	wavelet diff. of average



Continuous Wavelet Transform

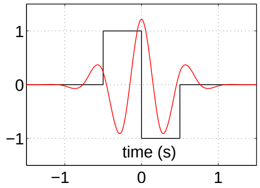
- Fourier Transform : $X(t) \Longrightarrow \tilde{X}(\nu) = \langle X, e_\nu \rangle$.

  Fourier Basis : $e_\nu(t) = \exp(\imath 2\pi\nu t)$

  Interpretation : ever lasting pure tone

- Continuous Wavelet Transform : $T_X(a, t) = \langle X, \psi_{a,t} \rangle$

  Mother-wavelet : $\int \psi_0(u) du = 0$,



Wavelet-basis : $\psi_{a,t}(u) = \frac{1}{|a|}\psi_0(\frac{u-t}{a})$

Interpretation : Joint time and frequency energy content

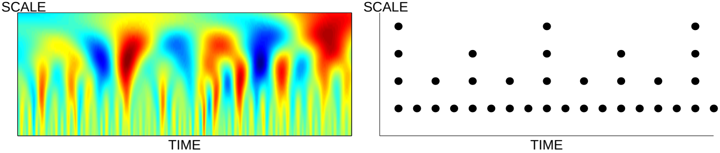
Continuous wavelet transform

$$X(t) \rightarrow T_X(a, t) = \langle \frac{1}{a}\psi\left(\frac{u-t}{a}\right) | X \rangle$$

Interpretation : Joint time and frequency energy content

Discrete Wavelet Transform

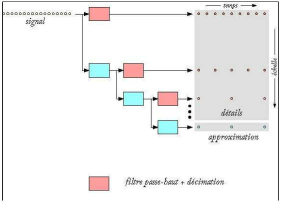
- Redundancy (reproducing kernel transform)



- Critical sampling : $(j, k) \Leftarrow (a = 2^j, t = k2^j)$

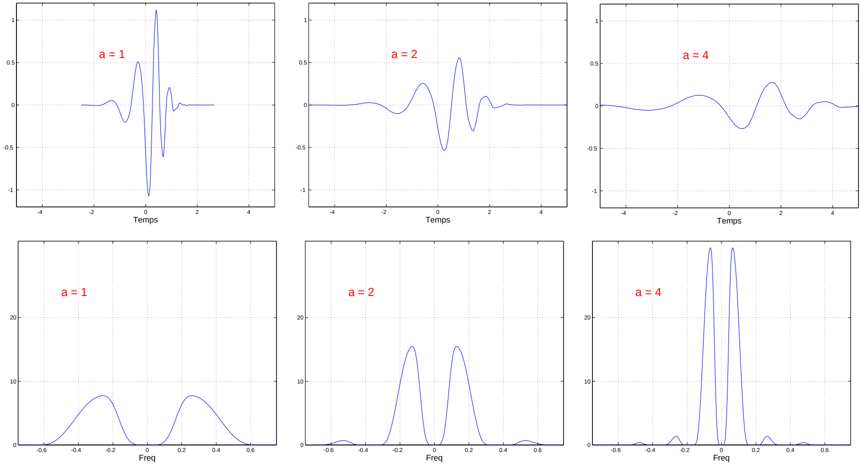
$$d_X(j, k) = T_X(a = 2^j, t = k2^j) \; ; \; \psi_{j,k}(u) = 2^{-j}\psi_0(2^{-j}u - k)$$

- Multiresolution analysis
  - Shift invariant nested spaces (Meyer, Mallat, Daubechies)
  - Critical sampling and frames
  - Orthonormal basis
  - Fast pyramidal recursive algorithm



Wavelet and scaling : Dilation operator

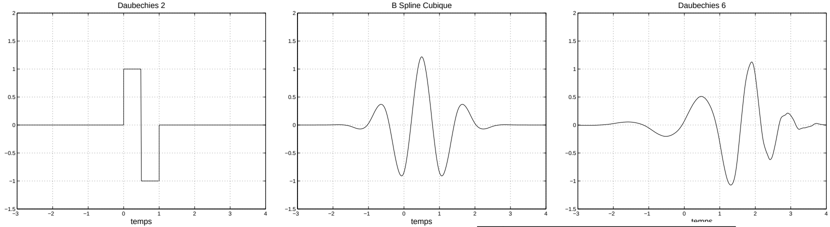
- Dilation (change of scale) operator : $\frac{1}{|a|}\psi_0(\frac{t}{|a|})$



Wavelet and scaling : Vanishing moments

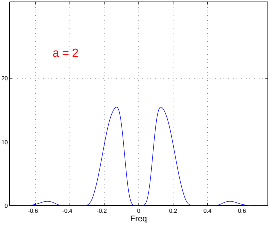
- Number of vanishing moments :

$$N_\psi \geq 1, \int t^k \psi_0(t)dt \equiv 0, \quad k = 0, 1, \ldots, N_\psi - 1.$$



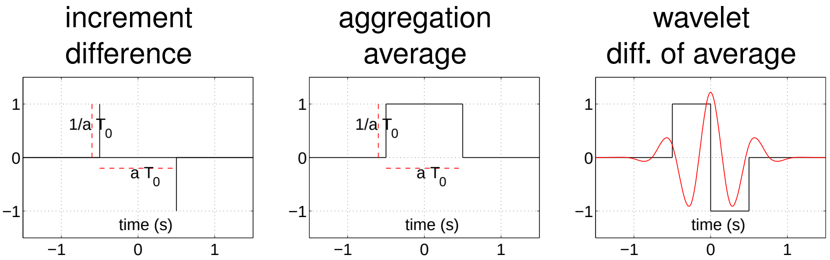
- Fourier transform :

$|\Psi(\nu)| \sim_{|\nu| \to 0} C|\nu|^{N_\psi}$



Multiresolution analysis

- $X(t) \to T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0\left(\frac{u-t}{a}\right)$

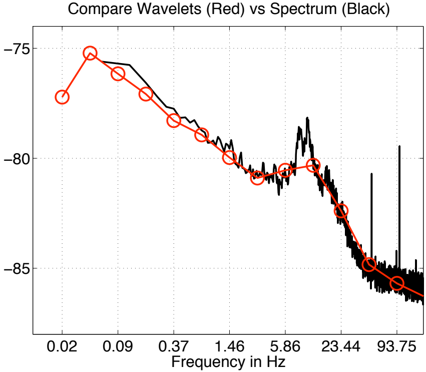


N_ψ

Number of vanishing moments

Spectrum Analysis versus Wavelet Analysis

- Wavelet Analysis : $\mathbf{E}|T_X(a,k)|^2 = \int \Gamma_X(\nu)|\tilde{\Psi}(a\nu)|^2 d\nu$
- $1/f-$process : $\frac{1}{n_a}\sum_{k=1}^{n_a}|T_X(a,k)|^2 \simeq C_q a^{\gamma-1}$
- $1/f-$process : $\hat{\Gamma}_Y(\nu) = \sum_k |\tilde{Y}_{k,T}(\nu|^2 \simeq C|\nu|^{-\gamma}$
- $\nu \simeq \nu_0/a$ and $q = 2$



Data FMRI, Courtesy, Ph. Ciuciu, Neurospin, France

Scale invariance, wavelet and multiresolution

- Signal, Image : $X(t) \to$ wavelet coefficients : $T_X(a, t)$,

- Scaling :

  $|T_X(a, k)|$ covariant w.r.t. a change of the analysis scale a

  $\Rightarrow$Power Laws : $\boxed{\frac{1}{n_a} \sum_{k=1}^{n_a} |T_X(a, k)|^q \simeq C_q a^{\zeta(q)}}$

- Range of scales : $a \in [a_m, a_M], \;\; a_M / a_m \gg 1$,
- Statistical orders : $q \in [q_m, q_M], \;\; q_m < 0 < q_M$,

- Scaling exponents : $\zeta(q)$: $\Rightarrow$ estimation,

$\Rightarrow$ detection, identification, classification.

From $1/f$ processes to ...

- Definition : 2nd order stationary $1/f$-process
$$\Gamma_Y(\nu) = C|\nu|^{-\gamma},\ \gamma > 0,\ \ \nu_m \le |\nu| \le \nu_M,\ \frac{\nu_M}{\nu_m} \gg 1$$

- $\nu_M \to +\infty$:$\Rightarrow$ fractal sample path

- $\nu_m \to 0$:$\Rightarrow$ Long Range dependence, self-similarity

Long Range Dependence

- Definitions : Y 2nd stationary process with

  Spectrum : $\Gamma_Y(\nu) = c_f|\nu|^{-\gamma}, 0 < \gamma < 1, \nu \to 0.$

  Covariance :

  $c_X(\tau) = \mathbb{E}Y(t)Y(t+\tau) = c_\tau|\tau|^{-\beta}, 0 < \beta < 1, \tau \to +\infty$

  with $\gamma = 1 - \beta$ and $c_f = 2(2\pi)\sin((1-\gamma)\pi/2)c_\tau$.

- Scaling : Power-law $\Rightarrow$ No Characteristic Scale,
- $\int_0^{+\infty} c_X(\tau)d\tau = +\infty$ Sum of Covariance Infinite,

- Consequences :

  Aggregation : $T_X(a,t) = \frac{1}{aT_0}\int_t^{t+aT_0} X(u)du,$

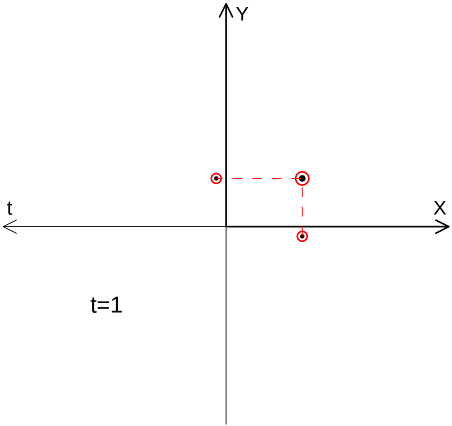
  $$\Rightarrow \text{Var}\, T_X(a,t) \sim Ca^{\gamma-1}, \quad a \to +\infty$$

$\Rightarrow$ Time Averages are poor estimates of Ensemble Averages

- Limitation : 2nd order only !

Random Walk : additive model

$$X(t+\tau) = X(t) + \underbrace{\delta_\tau X(t)}_{\text{Steps or Increments}}$$



Random Walk

- Random Walk : $X(t+\tau) = X(t) + \underbrace{\delta_\tau X(t)}_{\text{Steps or Increments}}$
- Statistical properties of the steps :

  A1 : Stationary,
  A2 : Independent,
  A3 : Gaussian,
  $\Rightarrow$ Ordinary Random Walk, Ordinary Brownian Motion,
  $\Rightarrow \mathbf{E}X(t)^2 = 2D|t|$, Einstein relation,
  $\Rightarrow \mathbf{E}X(t)^q = 2D|t|^{q/2}$, $q > -1$.
- Anomalies :

  $\Rightarrow \mathbf{E}X(t)^2 = 2D|t|^\gamma$,
  $\Rightarrow \mathbf{E}X(t)^2 = \infty$.
- Self Similar Random Walks ?

Self-Similar process

- Definition : $\{X(t)\}_{t\in\mathcal{R}} \overset{fdd}{=} \{a^H X(t/a)\}_{t\in\mathcal{R}}$

  Dilation Factor : $\forall a > 0$,

  Self-Similarity Exponent : $H > 0$.
- Scaling :

Self-Similar process and scaling

- $0 < H < 1$,
- Scaling :

$\Rightarrow$ $\mathbf{E}|X(t)|^q = \mathbf{E}|X(1)|^q \, |t|^{qH}$

$\Rightarrow$ non stationary processes

Self-Similar process and stationary increments

- Stationary increments :
  $\{X(t+\tau)-X(t)\}_{t\in\mathcal{R}} \overset{fdd}{=} \{X(0+\tau)-X(0)\}_{t\in\mathcal{R}}$

- Finite variance : $\mathbf{E}X(t)^2 < +\infty$

$\Rightarrow$ $0 < H < 1$,
$\Rightarrow$ $\mathbf{E}X(t)X(s) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$,
$\Rightarrow$ $\sigma^2 = \mathbf{E}X(1)^2$.

Self-Similar process and correlation

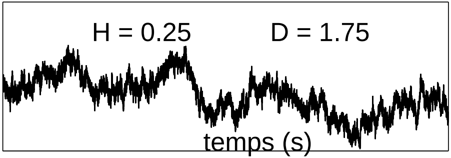
- Stationary increments and finite variance :

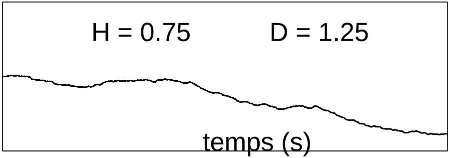
$$\Rightarrow \quad \mathbf{E}(X(t+\tau)-X(t))(X(s+\tau)-X(s)) \sim_{|t-s|\gg\tau} \frac{\sigma^2 H(2H-1)}{\tau^{2H}}|t-s|^{2H-2},$$

$\Rightarrow$ $H = 1/2$ no correlation,

$\Rightarrow$ $1 > H > 1/2$ positive correlation,

$\Rightarrow$ $1/2 > H > 0$ negative correlation,





Self-Similar, long range dependence and $1/f$

- Increment process :

$$\begin{array}{ll} & Y_\tau(t) = X(t+\tau) - X(\tau), \\ & \mathbf{E} Y_\tau(t) Y_\tau(s) \sim C_\tau |t-s|^{2H-2}, \\ & \Gamma_Y(\nu) = C_f |\nu|^{-(2H-1)} \\ \Rightarrow & 1 > H > 1/2 \Rightarrow 1 > 2-2H > 0 \\ \Rightarrow & \text{Long range dependence} \\ \Rightarrow & \text{Time averages poorly estimates ensemble averages} \end{array}$$

- Linear Filter :

$$\begin{array}{ll} & Y_\tau(t) = X(t+\tau) - X(\tau) = (X * \psi)(t), \\ & \psi(t) = \delta(t+\tau) - \delta(t), \\ & \Gamma_Y(\nu) = |1 - \exp(\imath 2\pi\nu\tau)|^2 \Gamma_X(\nu), \\ & \Gamma_Y(\nu) \sim |\nu|^2 \Gamma_X(\nu) \ \ |\nu| \to 0, \\ \Rightarrow & \text{"} \Gamma_X(\nu) \sim C_f |\nu|^{-(2H+1)} \text{"}, \quad |\nu| \to 0 \end{array}$$

Self-Similar process and scaling

- Stationary increments and finite variance :

$\Rightarrow \quad 0 < H < 1,$

$\Rightarrow$ Power Law :
$\mathbf{E}|X(t + a\tau_0) - X(t)|^q = C_q|a|^{qH},$
for all scales : $\forall a > 0$.

Self Similar Random Walk (Model 2)

- Random Walk : $X(t+\tau) = X(t) + \underbrace{\delta_\tau X(t)}_{\text{Steps or Increments}}$

- Ordinary :

  A1 : Stationary,
  A2 : Independence,
  A3 : Gaussianity,

- Self-similar :

  A1 : Stationary,
  A2 : Self-Similarity,

  Independence $\Rightarrow$ Correlation amongst increments :
  $0 < H < 1$.
  Gaussian $\Rightarrow$ Marginal distributions stable under addition (Gaussian, Stable, Hermite).

Self Similar Random Walk

- Random Walk : $X(t+\tau) = X(t) + \underbrace{\delta_\tau X(t)}_{\text{Steps or Increments}}$

- Self-similar :

  A1 : Stationary,
  A2 : Self-Similarity,

  Gaussian $\Rightarrow$ Fractional Brownian Motion $0 < H < 1$: $B_H(t)$
  $H = 1/2 \Rightarrow$ (Ordinary) Brownian Motion $B(t)$,

  Increments $\Rightarrow G_H(t) = B_H(t+1) - B_H(t)$ Frac. Gauss. Noise,
  $H = 1/2 \Rightarrow$ White Gaussian Noise.

Wavelets and self-similar processes

(Flandrin, 89, 92)

- 1. Wavelet and Self Similarity.
- 2. Wavelet and Non Stationarity.
- 3. Wavelet and Long Range Dependence.

Wavelets and Self-Similarity

- Self-Similarity :

$$\{X(t)\} \overset{d}{=} \{a^H X(t/a)\} \Rightarrow \{d_X(0,k)\} \overset{d}{=} \{2^{-jH} d_X(j,k)\}$$

- Marginal Distributions :

$$P_j(d) = \frac{1}{a} P_{j'}\left(\frac{d}{a}\right), \quad a = \left(\frac{2^{j'}}{2^j}\right)^H.$$

- Sketch of Proof :

$$\begin{aligned} d_X(j,k) &= \int X(u)\psi(2^{-j}u - k)2^{-j}du \\ &= \int X(2^j u)\psi(u-k)du \\ &\overset{d}{=} 2^{jH} \int X(u)\psi(u-k)du \\ &= 2^{jH} d_X(0,k). \end{aligned}$$

- Key-Point : Dilation Operator $\psi_{a,0}(u) = \frac{1}{a}\psi\left(\frac{u}{a}\right)$.

Wavelet and non-Stationarity

(with stationary increments)

- $\{d_X(j,k), k \in \mathcal{Z}\}$ stationary sequences for each scale $a = 2^j$.

- Sketch of Proof :

$$\begin{array}{rcl} d_X(0, k+k_0) & = & \int X(u)\psi(u-k-k_0)du \\ & = & \int X(u+k_0)\psi(u-k)du \\ & = & \int [X(u+k_0) - X(k_0)]\psi(u-k)du \\ & \stackrel{d}{=} & \int [X(u) - X(0)]\psi(u-k)du \\ & = & \int X(u)\psi(u-k)du \\ & = & d_X(0,k). \end{array}$$

- Key-Point : $N_\psi \geq 1, \quad \int \psi(t)dt = 0.$

Wavelet and long range dependence

- $\mathbf{E}d_X(j,k)d_X(j',k') = ?$

- Sketch of Proof :

$$\mathbf{E}X(t)X(s) = \sigma^2/2\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

$$\mathbf{E}d_X(j,k)d_X(j',k') = \int\int dtds\mathbf{E}X(t)X(s)\psi_{j,k}(t)\psi_{j',k'}(t),$$

$$= \underbrace{\int ds\psi_{j',k'}(s)}_{0}\int dt|t|^{2H}\psi_{j,k}(t) +$$

$$\underbrace{\int dt\psi_{j,k}(t)}_{0}\int ds|s|^{2H}\psi_{j',k'}(s) +$$

$$\int\int dtds|t-s|^{2H}\psi_{j,k}(t)\psi_{j',k'}(s).$$

Wavelets and Long Range Dependence - Con't

$$\mathbf{E}d_X(j,k)d_X(j',k') = \int du|u|^{2H} \underbrace{\int dv \psi_{j,k}(v+u/2)\psi_{j',k'}(v-u/2)}_{\Psi_{j,j',k,k'}(u)}.$$

$$\mathbf{E}d_X(j,k)d_X(j',k') = \int d\nu|\nu|^{-(2H+1)}2^{(j+j')/2}|\tilde{\psi}(2^j\nu)\tilde{\psi}(2^{j'}\nu)|\exp(-\imath 2\pi(2^jk-2^{j'}k')).$$

- But

$$\begin{aligned} |\nu|^{-(2H+1)}|\tilde{\psi}(2^j\nu)\tilde{\psi}(2^{j'}\nu)| &\simeq_{|\nu|\to 0} |\nu|^{-(2H+1)}|(2^j\nu)|^N|(2^{j'}\nu)|^N \\ &\simeq_{|\nu|\to 0} 2^{(j+j')N}|\nu|^{2N_\psi-2H-1}. \end{aligned}$$

- Hence, when $|2^jk-2^{j'}k'| \to +\infty$,

$$\mathbf{E}d_X(j,k)d_X(j',k') \sim K|2^jk-2^{j'}k'|^{2(H-N_\psi)}.$$

- Key-Points :
  Dilation Operator and Number of vanishing Moments.

Wavelets and Long Range Dependence - Con't

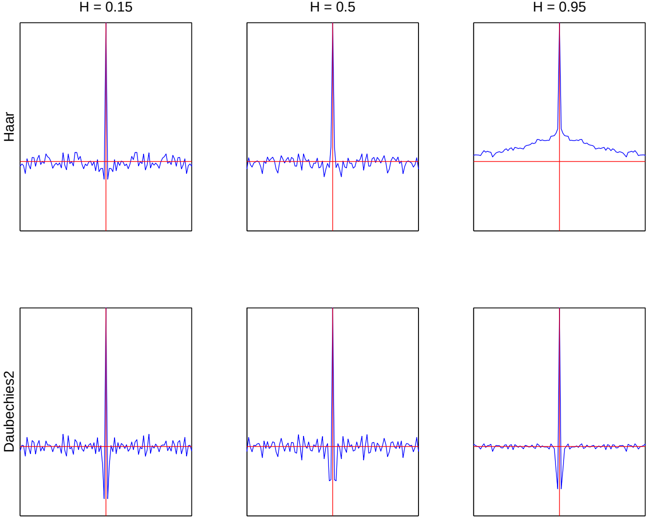
- Hence, when $|2^j k - 2^{j'} k'| \to +\infty$,

  $$\mathbf{E} d_X(j,k) d_X(j',k') \sim K |2^j k - 2^{j'} k'|^{2(H - N_\psi)} .$$

$\Rightarrow$ $\{d_X(j,k)\}$ are short range dependent
as soon as $N_\psi > H + 1/2$.

- Key-Points :
  Dilation Operator and Number of vanishing Moments.

Wavelets and Long Range Dependence - Con't



Wavelets and Self-Similar Processes - Summary

- Self-Similarity :

$$\{X(t)\} \stackrel{d}{=} \{a^H X(t/a)\} \Rightarrow \{d_X(0,k)\} \stackrel{d}{=} \{2^{-jH} d_X(j,k)\}$$

- Non-stationarity (with stationary increments) :

$\{d_X(j,k), k \in \mathcal{Z}\}$ stationary sequences for each scale 2^j

$$\Rightarrow \mathbf{E}|d_X(j,k)|^q = |d_X(0,0)|^q 2^{jqH}, \ \forall a = 2^j, \ q > -1.$$

- Long range dependence :

$\{d_X(j,k)\}$ Short Range Dependent if $N > H + 1/2$.

$$|2^j k - 2^{j'} k'| \to +\infty, \ |\mathrm{Cov}\, d_X(j,k) d_X(j',k')| \leq D|2^j k - 2^{j'} k'|^{2(H-N_\psi)},$$

Weak correlation amongst wavelet coefficients.

- Interpretations :

$$X(t) = \sum_k a_X(J,k)\varphi_{J,k}(t) + \sum_{j,k} d_X(j,k)\psi_{j,k}(t).$$

Scaling analysis : Logscale Diagrams

- Principle :

$$\mathbf{E}|d_X(j,k)|^q = |d_X(0,0)|^q 2^{jqH} \Rightarrow \text{log-log plots}$$

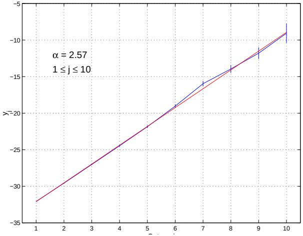
- Estimation : short-range dependence $\Rightarrow$

Ensemble averages $\rightarrow$ Time Averages

$$\mathbf{E}|d_X(j,k)|^q \Rightarrow 1/n_j \sum_k |d_X(j,k)|^q = S(2^j,q)$$

- Logscale Diagrams :

$$\log_2 S(2^j,q) \text{ versus } \log_2 2^j = j \Rightarrow qH$$



Wavelets and $1/f$-processes

- Spectral Analysis :
  Let Y be a 2nd Order stationary process,
  Let ψ have central frequency ν_0 and bandwith $\Delta\nu_0$.

$$\begin{aligned} \mathbf{E}|d_Y(j,k)|^2 &= \int \Gamma_Y(\nu)|\Psi(2^j\nu)|d\nu \\ &\simeq 2^{-j}\Gamma_Y(2^{-j}\nu_0) \text{ within bandwith } 2^{-j}\Delta\nu_0. \end{aligned}$$

- Let Y be Long Range Dependent :
  $\Gamma_Y(\nu) = C|\nu|^{-\gamma},\ 0 < \gamma < 1,\ |\nu| \to 0,$

$\Rightarrow$ Power Law : $\mathbf{E}|d_X(j,k)|^2 \sim C2^{j(\gamma-1)},\ j \to +\infty,$

- Decorrelation :
  $\mathbf{E}d_X(j,k)d_X(j,k')^2 \sim C|k-k'|^{\gamma-1-2N_\psi},\ |k-k'| \to +\infty,$

$\Rightarrow$ Short Range Dependence as soon as $N > \gamma/2$.

$\Rightarrow$ Logscale Diagram :

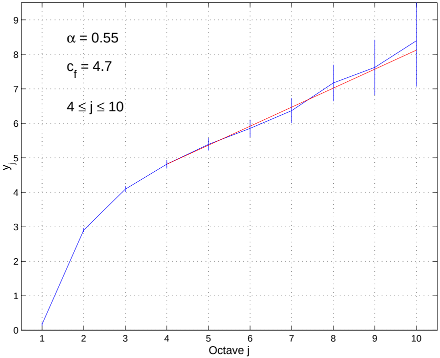
$$\log_2 S(2^j, 2) \text{ versus } \log_2 2^j = j \Rightarrow \gamma$$

Wavelets and $1/f$-processes

$\Rightarrow$ Power Law : $\mathbf{E}|d_X(j,k)|^2 \sim C2^{j(\gamma-1)},\ j \to +\infty,$

$\Rightarrow$ Logscale Diagram :

$$\log_2 S(2^j, 2) \text{ versus } \log_2 2^j = j \Rightarrow \gamma$$



Linear regression

- Weighted Linear Fits : $\hat{\gamma} = \sum_j w_j \log_2 S_2(2^j, 2)$

$$w_j = \frac{B_0 j - B_1}{a_j(B_0 B_2 - B_1^2)}, \quad B_k = \sum_j j^k / a_j \ a_j > 0 \text{ arbitrary numbers}$$

- Analytical Performances : $q = 2$
  - *i*) Assume : process is Gaussian,
  - *ii*) Idealisation : wavelet coefficients are exactly independent
- Bias :

$$\mathbf{E} \log_2 S(2^j, 2) = \log_2 \mathbf{E} S(2^j, 2) + \underbrace{\Gamma'(n_j/2) - \log_2(n_j/2)}_{g_j}.$$

$$\Rightarrow \mathbf{E}\hat{\gamma} = \gamma + \sum_j w_j g_j.$$

- Variance :

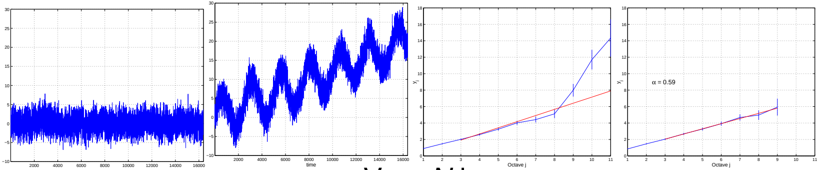
$$\mathrm{Var}\,\hat{\gamma} \simeq \left((2\log_2(e))^2(\textstyle\sum_j w_j^2 \sigma_j^2)\right)/n,$$

min. if $a_j = \sigma_j^2 = \mathrm{Var} \log_2 S(2^j, 2) \simeq 2^j$.

- Actual Performances : close to MLE.
- Conceptual and Practical Simplicity : DWT + Linear Fit

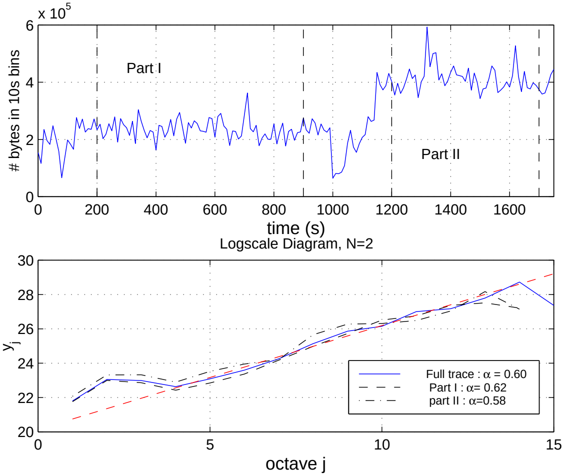
Scaling versus non stationarity : superimposed trends

- Linear transform :
  $Y(t) = X(t) + T(t) \Rightarrow d_Y(j,k) = d_X(j,k) + d_T(j,k)$
- If $T(t)$ Polynomial of degree P, then $d_T \equiv 0$ when $N_\psi > P$,
- If $T(t)$ smooth trend, then the d_T decrease as N increases.

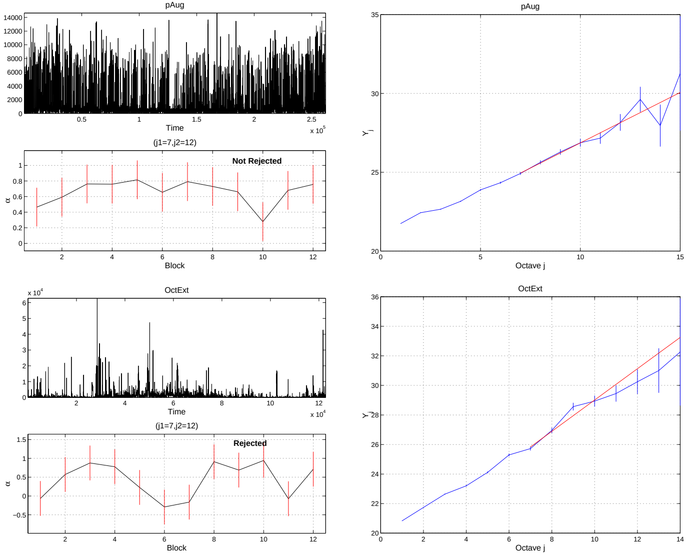


Vary N !

Superimposed Trends - Ethernet Data



Constancy of Scaling along time



Beyond self-similarity. . . ?

- Self-Similarity :

  Power Laws : $\mathbf{E}|d_X(j,k)|^q = C_q(2)^{jqH}$
  For all scales : $\forall a = 2^j$,
  For all orders : $q > -1$,
  A single parameter qH.

- Beyond :

  Power Laws : $\mathbf{E}|d_X(j,k)|^q = C_q(2)^{j\zeta(q)}$
  $\zeta(q)$ non linear concave function of q,
  For a limited range of scales : $a_m \leq a \leq a_M$,
  For a limited range of orders : $q_m \leq q \leq q_M$,
  A collection of scaling parameters $\zeta(q)$.

$\Rightarrow$ Multifractal

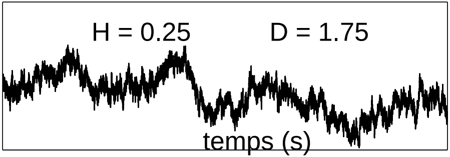
The other end of the spectrum - sample path regularity

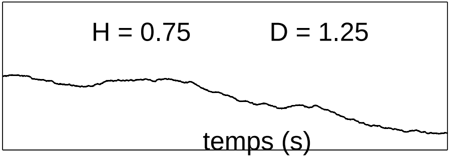
- $1/f$-processes : $\Gamma_Y(\nu) \simeq C|\nu|^{-\gamma},\ |\nu| \to +\infty$

$\Rightarrow$ $\mathbf{E}Y(t+\tau)Y(t) \sim \sigma^2(1 - C|\tau|^{\gamma})\,\tau \to 0,$

- Self-similar process :
  $\mathbf{E}|X(t+a\tau_0) - X(t)|^q = C_q|a|^{qH},\ a \to 0.$

$\Rightarrow$ Fractal Sample Path - Hölder Regularity





$\Rightarrow$ Multifractal analysis

$\Rightarrow$ Multifractal processes

$\Rightarrow$ Multifractal formalism

Multifractal Analysis

- **Local regularity** of $X(t)$ at $t_0 : 0 < \alpha < 1$

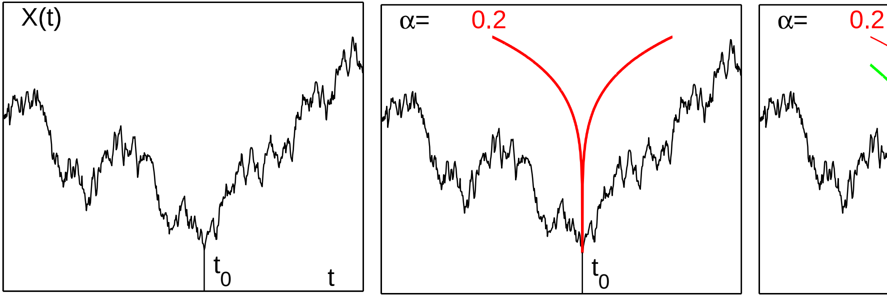
  Compare : $|X(t) - X(t_0)| < C|t - t_0|^\alpha$
- **Hölder Exponent** : $h(t_0) = \sup_\alpha\{\alpha : X \in C^\alpha(t_0)\}$

  Extend differentiability to non integer : $0 < h(t_0 < 1$

  $\lim_{|\mathbf{t}-\mathbf{t_0}|\to 0} \frac{|X(\mathbf{t})-X(\mathbf{t_0})|}{|\mathbf{t}-\mathbf{t_0}|^{h(t_0}} = C$

  $h(t_0) \to 1 \Rightarrow$, smooth, very regular,

  $h(t_0) \to 0 \Rightarrow$, rough, very irregular



Hölder exponent and Wavelets (Intuition)

- Local singularity :

$$X(t_0) \in C^{h(t_0)}, h(t_0) \text{ non (even) Integer }, P \leq h(t_0) < P+1,$$
$$X(t) \simeq_{t \to t_0} X(t_0) + \sum_{k=1}^{P} X^{(k)}(t_0) \frac{(t-t_0)^k}{k!} + C|t-t_0|^{h(t_0)}.$$

- Mother Wavelet with $N_\psi \leq P$:

$$T_X(a, t_0) \simeq_{a \to 0} a^N \int_{\mathcal{R}} u^N \psi_0(u) du.$$

- Mother Wavelet with $N_\psi \geq P+1$:

$$T_X(a, t_0) \simeq_{a \to 0} a^{h(t_0)} C \int_{\mathcal{R}} |u|^h \psi_0(u) du.$$

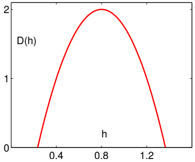
Hölder and Wavelets (theory)

Multifractal (or singularity) spectrum

- Data : a collection of singularities
  $|X(\mathbf{t}) - X(\mathbf{t_0})| \leq C|\mathbf{t} - \mathbf{t_0}|^{h(\mathbf{t_0})}$
- Fluctuations of local regularity : $h(\mathbf{t})$?
- not interested in h for each $(\mathbf{t})$!
- Instead, set $E(h)$ of points $\mathbf{t}$ with same h : $h(\mathbf{t}) = h$,
- Fractal dimension of $E(h)$,
- Actually Hausdorff dimension of $E(h)$, ▸ Hausdorff
- Multifractal spectrum : ▸ $D(h)$

$$D(h) = \dim_{\text{Haussdorf}}(E(h)).$$

$$0 \leq D(h) \leq d,$$
$$D(h) = -\infty \text{ if } E(h) = \{\emptyset\},$$



$\Rightarrow$ Global (geometrical) description of the fluctuations of the local regularity

- How to measure $D(h)$ from a single finite length observation ?

Multifractal processes

- Mono-Fractal :

  A single h : $\forall t$, $h(t) = h$, $D(h) = d\delta(h - H)$,
  Scaling exponents are linear in q : $\zeta(q) = qH$,
  Finite variance Self-similar processes,
  Fractional Brownian motion.

- Multi-Fractal :

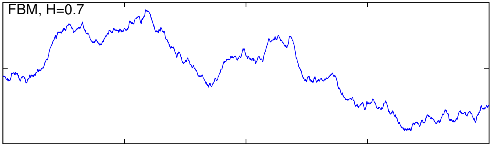
  Many different h,
  leaving on fractal subset of $\mathcal{R}^d$,
  Bell-shaped $D(h)$,
  Scaling exponents are concave q : $\zeta(q) \neq qH$,
  Multiplicative cascades (Mandelbrot)

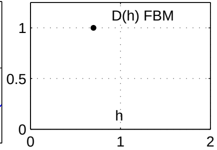
- Multi-Fractional : is something else !

  $h(t)$ is a deterministic function of time,
  $D(h) \equiv 0$,
  non stationary increments

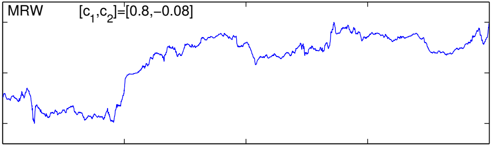
1D Examples

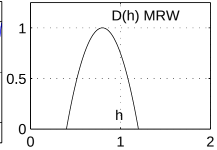
1. Fractional Brownian Motion (H-sssi)





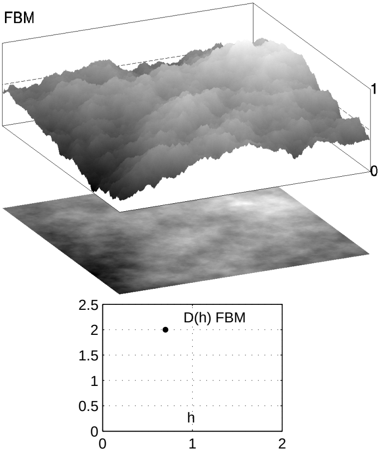
2. Multifractal Random Walk (MF)



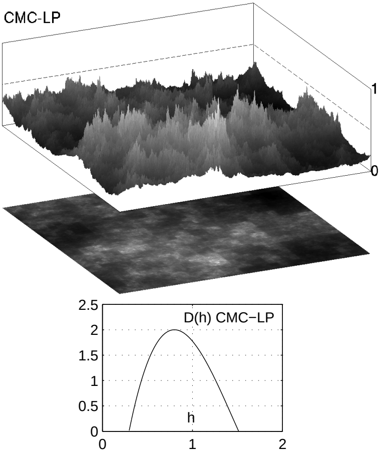


2D Examples

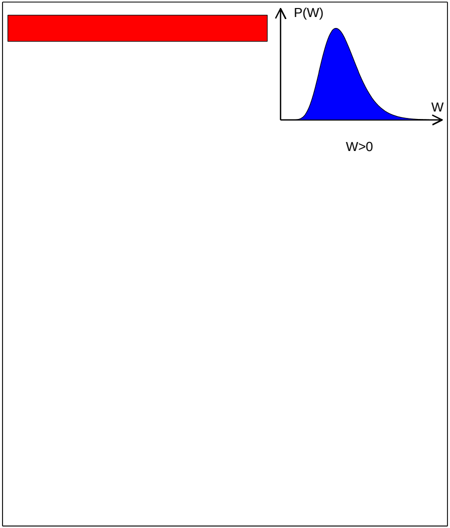
FBM (H-sssi)



Multiplicative casc. (MF)



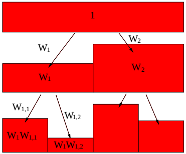
Mandelbrot Multiplicative Cascades (Model 3)

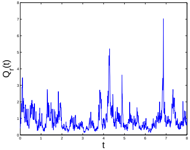


Mandelbrot Multiplicative Cascades (Model 3)

- Definition :

  Split Dyadic Intervals $I_{j,k}$ into two,
  I.I.D. positive (mean one) Multipliers $W_{j,k}$
  $Q_J(t) = \Pi_{\{(j,k):1\leq j\leq J, t\in I_{j,k}\}} W_{j,k}$,

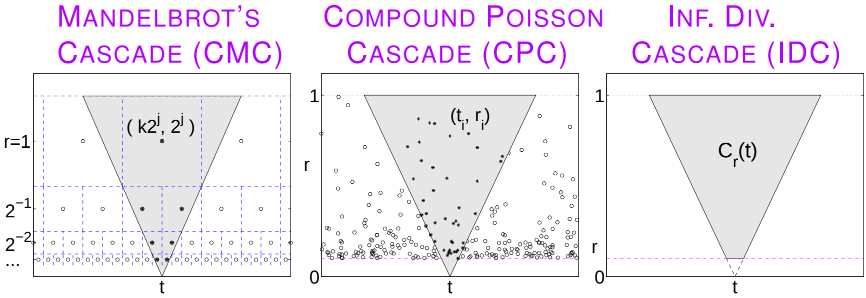




- Implications :

  Cascades, Multiplicative Structure,
  Power Laws,
  $\mathbf{E}\left(1/2^j \int_{k2^j}^{(k+1)2^j\tau_0} X(u)du\right)^q = C_q|2^j|^{\zeta_q}$,
  Multiple Exponents $\zeta_q = q - \log_2 \mathbf{E} W^q$, Non Linear in q,
  Fine Scales $a = 2^j \to 0$, $a \ll L$ Integral Scale,
  No Characteristic Scale of Time beyond an Integral Scale.
  Non Stationarity,
  Local Holder Exponent,
  MultiFractal Sample Paths, MultiFractal Spectrum $D(h)$.

Multiplicative Cascades (Model 3)



$$Q_r(t) = \Pi \; W_{j,k}, \qquad = \Pi \; W_{j,k}, \qquad = \exp \int dM(t', r'),$$

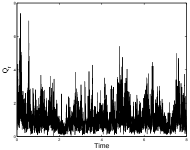
$$\varphi(q) = -\log_2 \mathbf{E} W^q, \quad = -q(1 - \mathbf{E} W) + 1 - \mathbf{E} W^q, \quad = \rho(q) - q\rho(1),$$

$$A(t) = \lim_{r \to 0} \int_0^t Q_r(u) du,$$

For a Range of qs, $\mathbf{E}|A(t + a\tau_0) - A(t)|^q = c_q |a|^{q+\varphi(q)}$,

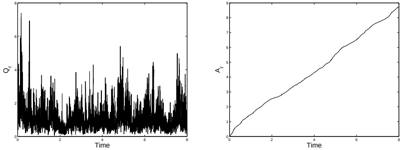
MultiFractal Processes (Model 3)

- Density $Q_r(t) = \Pi \; W_{j,k} \quad \mathbf{E}\left(\frac{1}{a}\int_t^{t+a\tau_0} Q_r(u)du\right)^q = c_q a^{\varphi(q)}$,



- Measure : $A(t) = \lim_{r\to 0} \int_0^t Q_r(u)du$,

$$\mathbf{E}|A(t+a\tau_0) - A(t)|^q = c_q|a|^{q+\varphi(q)},$$



- FBM in Multifractal Time : $V_H(t) = B_H(A(t))$,

How to measure the multifractal spectrum ?

- How to measure $D(h)$?
- from a single finite length observation ?

How to measure the multifractal spectrum ?

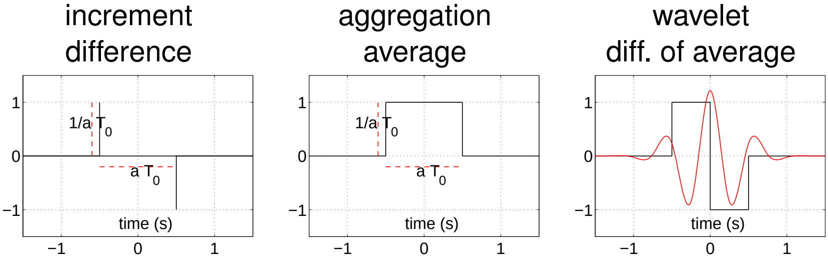
- How to measure $D(h)$?
- from a single finite length observation ?

$\Rightarrow$ Intuition : estimate h for each t !

- Not interested in measuring $h(t)$ for each t
- may vary from one sample path to another for stochastic processes,
- may vary widely between two very close t and t',
- Measuring $h(t)$ for each t makes no sense ! intrinsical obstruction !
- Actual measure are done on real data with finite resolution :
  - $\Rightarrow$ cannot do $|t - t_0| \to 0$,
  - $\Rightarrow$ on any subset, all h are present.
- Solution :
  - $\Rightarrow$ Multifractal Formalisms.

Multiresolution analysis

- $X(t) \to T_X(a,t) = \langle f_{a,t} | X \rangle, \quad f_{a,t}(u) = \frac{1}{a} f_0(\frac{u-t}{a})$



Multifractal formalism

- Multiresolution Quantities : $T_X(a,t)$,
- Structure functions : $S(a,q) = \frac{1}{n_a}\sum_{k=1}^{n_a} |T_X(a,t)|^q$,
- Power laws : $S(a,q) \simeq c_q |a|^{\zeta(q)}, \quad a \to 0$,

$$\begin{aligned} S_n(a,q) &\simeq && a^d \sum_h a^{-D(h)} a^{hq}, \\ &\simeq && \sum_h a^{d-D(h)+hq}, \\ &\sim_{a\to 0} && c_q a^{\zeta(q)} \end{aligned}$$

Saddle-point argument : $\Rightarrow$ Legendre transform

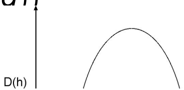
$$\zeta(q) = \min_{q\neq 0} (d + hq - D(h)).$$

- Scaling function : $\zeta(q) = \liminf_{a\to 0} \frac{\ln S(a,q)}{\ln a}$,
- Legendre transform : $\zeta(q) \to D(h)$.

$$D(h) = \min_{q\neq 0}(d + qh - \zeta(q))$$

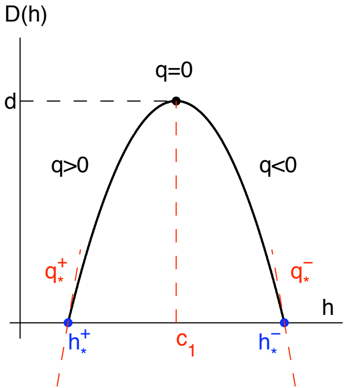
- Multifractal formalism $\to$ Scaling analysis :

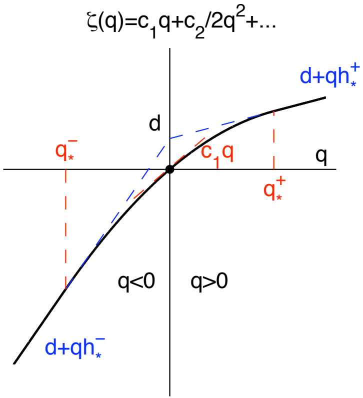
$S(a,q) \simeq C_q a^{\zeta(q)}$,



Legendre transform

Linearization effect





Linearization effect - Multiplicative cascades

- Multifractal spectrum for multiplicative cascades

  Moments : $\mathbf{E}|d_X(j,k)|^q = c_q 2^{j\lambda(q)}, q \in [q_c^-, q_c^+]$.
  Multipliers : $\lambda(q) =$ functional $(q, \mathbf{E}W^q)$),
  Legendre : $D_\lambda(h) = min_q(d + qh - \lambda(q))$,
  Multifractal Spectrum

  $D(h) = D_\lambda(h)$, if $D_\lambda(h) \geq 0$,,
  $D(h) = -\infty$, otherwise ;

- Scaling exponents (Structure functions) :

  $S(2^j, q) = (1/n_j)\sum_k |d_X(j,k)|^q = c_q 2^{j\zeta(q)}, q \in [q_*^-, q_*^+]$
  $\zeta(q) = min_h(d + qh - D(h))$,
  $\zeta(q) = \lambda(q), q_*^- \leq q \leq q_*^+$,
  $\zeta(q) = d + qh_M, q_*^+ \leq q$,
  $\zeta(q) = d + qh_m, q \leq q_*^-$.

- Disentangling a confusion : $\zeta(q) \neq \lambda(q)$.

$\Rightarrow$ Multifractal $\neq$ Multiplicative !

Integral scale

- Power Law : $S(a,q) = C_q a^{\zeta(q)}$,
- Hölder inequality $\Rightarrow$ $S(a,q)$ convex in q,
- $\ln S(a,q)$ convex in q,
- $\zeta(q)$ concave in q,

$\Rightarrow$ $\ln a \leq \frac{(\ln C_q)''}{(\zeta(q))''}$,

- Scaling with concave $\zeta(q)$, only at fine scales,
- Multifractal theory : $a \to 0$.

Log-cumulants

- Polynomial expansion :

$$\zeta(q) = \sum_{p \geq 1} c_p \frac{q^p}{p!} = c_1 q + \frac{c_2}{2!} q^2 + \frac{c_3}{3!} q^3 + \frac{c_4}{4!} q^4 + \cdots$$

- $C(j,p)$: **cumulants** of $\ln |d_X((j,\cdot)|$

$$C(j,p) = c_{0,p} + c_p \ln 2^j$$

- $D(h) = d + \frac{c_2}{2!}\left(\frac{h-c_1}{c_2}\right)^2 + \frac{-c_3}{3!}\left(\frac{h-c_1}{c_2}\right)^3 + \frac{-c_4+3{c_3}^2/c_2}{4!}\left(\frac{h-c_1}{c_2}\right)^4 + \ldots$

- $\zeta(q)$, $D(h) \rightarrow (c_1, c_2, c_3, c_4)$

- Discrimination :

  self-similar : $\zeta(q)$ linear , $\Rightarrow \forall p \geq 2 : \; c_p \equiv 0$
  multiplicative cascade : $\zeta(q)$ non linear, $\Rightarrow \exists p \geq 2 : c_p \neq 0$

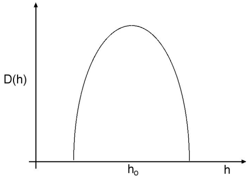
Multifractal formalism

- What Multiresolution Quantities : $T_X(a, t)$?
- Structure functions : $S(a, q) = \frac{1}{n_a} \sum_{k=1}^{n_a} |T_X(a, t)|^q$,
- Power laws : $S(a, q) \simeq c_q |a|^{\zeta(q)}, \quad a \to 0$,
- Scaling function : $\zeta(q) = \liminf_{a \to 0} \frac{\ln S(a,q)}{\ln a}$,
- Legendre transform : $\zeta(q) \to D(h)$.

$$D(h) = \min_{q \neq 0} (d + qh - \zeta(q))$$

- Multifractal formalism $\to$ Scaling analysis :

$S(a, q) \simeq C_q a^{\zeta(q)}$,
$a \to 0$,
$q \geq 0$ AND $q \leq 0$.



How well does it work ?

- Usual choices :

  increment, aggregation, wavelet
- Unsatisfactory :

  $q \leq 0 \Rightarrow S(a, q)$ instable

  $\zeta(q) \rightarrow D(h)$: not valid in general
- $T_X(a, t)$ need to be hierarchical (Jaffard, 2004).

  oscillations : $K_a = [t - a, t + a]$

  $$T_X(a, t) = \sup_{u \in K_a} X(u) - \inf_{u \in K_a} X(u),$$

  hierarchical : if $a_1 \leq a_2$, then $T_X(a_1, t) \leq T_X(a_2, t)$
- Solution (Jaffard et al., 2006) :

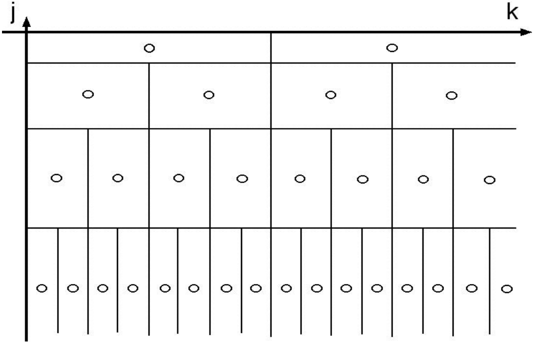
  increments $\rightarrow$ oscillations,

  wavelet coefficients $\rightarrow$ wavelet Leaders

Wavelet Leaders

- Discrete Wavelet Transform : $\lambda_{j,k} = [k2^j, (k+1)2^j)$

$$d_X(j,k) = \langle \frac{1}{2^j}\psi\left(\frac{t-2^jk}{2^j}\right)|X(t)\rangle \,,$$



- Wavelet Leaders : $3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$

$$\boxed{L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_{X,\lambda'}|}$$

2D Wavelet leaders

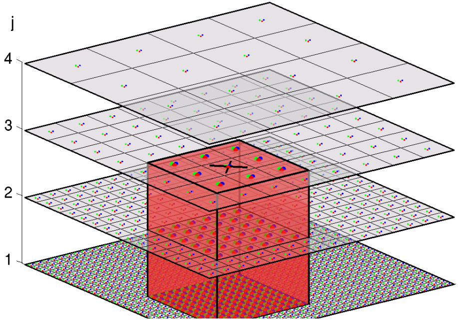
- 2D Discrete Wavelet Transform :

  $\lambda_{j,k_1,k_2} = \{[k_1 2^j, (k_1+1)2^j), [k_2 2^j, (k_2+1)2^j)\}$

  $d_X^{(m)}(j, k_1, k_2),\ m = 1, 2, 3$ (L_1 norm).

- Leaders : $3^2\lambda_{j,k_1,k_2} = \bigcup_{n_1,n_2=\{-1,0,1\}} \lambda_{j,k_1+n_1,k_2+n_2}$

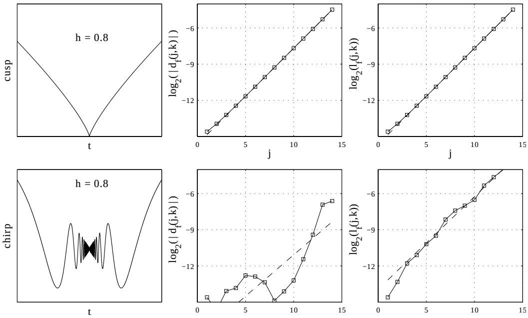
  $L_X(j,k) = \sup_{m=1,2,3,\ \lambda' \subset 3^2\lambda_{j,k_1,k_2}} |d_{X,\lambda'}^{(m)}|.$



Wavelet Leaders and Limitations

- $q < 0$: Obviously solved : Leaders are large
  $\Rightarrow$ MultiFractal Spectrum over its Entire Range,
- Oscillating (Chirp-type) Singularities :

  Cusp :
  $|X(t) - X(t_0)| \sim_{|t-t_0|\to 0} |t - t_0|^h \Rightarrow |L_X(j,k)| \sim_{|2^j|\to 0} 2^{jh}$
  Chirp : $|X(t) - X(t_0)| \sim_{|t-t_0|\to 0} |t - t_0|^h \sin(\frac{1}{|t-t_0|^\beta}) \Rightarrow$
  $|L_X(j,k)| \sim_{|2^j|\to 0} 2^{jh}$



Multifractal formalism

- Multiresolution Quantities : $L_X(a, t)$,
- Structure functions : $S(a, q) = \frac{1}{n_a} \sum_{k=1}^{n_a} |L_X(a, k)|^q$,
- Scaling function : $\zeta(q) = \liminf_{a \to 0} \frac{\ln S(a,q)}{\ln a}$,
- Legendre transform : $\zeta(q) \to D(h) = min_{q \neq 0}(d + qh - \zeta(q)$.

- Valid (Jaffard et al., 2006) :
  for $q \geq 0$ AND $q \leq 0$,
  for all classes of processes,
  on condition that positive Hölder exponents only,
  extends to higher dimension d.

Multifractal formalism

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- Valid (Jaffard et al., 2006) :
  for $q \geq 0$ AND $q \leq 0$,
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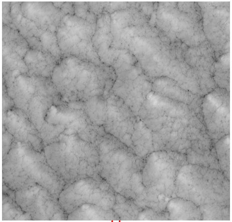
  extends to higher dimension d.

Multifractal Formalism

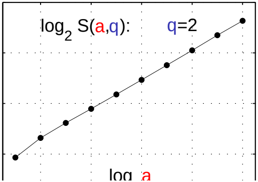
$$X(t) \rightarrow d_X(a,t) \rightarrow L_X(a,t)$$



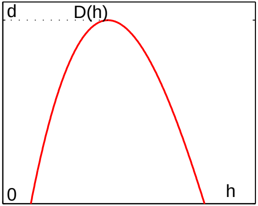
$\Downarrow$

$$S(a,q) = \frac{1}{n_a}\sum_{k=1}^{n_a} L_X(a,k)^q$$

$$S(a,q) = \frac{1}{n_a}\sum_{k=1}^{n_a} |L_X(a,k)|^q$$



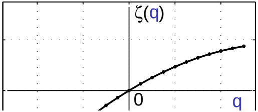
$$D(h) = \min_{q\neq 0}\left(d + qh - \zeta(q)\right)$$



$\Uparrow$

$$S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \rightarrow 0$$

$$\zeta(q) = \liminf_{a\rightarrow 0} \frac{\ln S(a,q)}{\ln a} S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \rightarrow 0$$



Estimation for multifractal attributes

1. $X(t) \rightarrow d_X(j,k) \rightarrow L_X(j,k)$

2. Structure functions and log-cumulants :

   $\log_2 S(j,q) \simeq G_q + \zeta(q) \log_2 2^j$
   $C(j,p) = c_p^0 + c_p \ln 2^j$

3. Weighted linear fits :

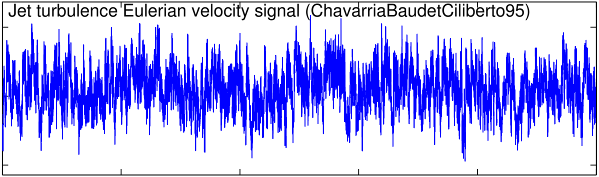
   $\hat{\zeta}(q) = \sum_{j=j_1}^{j_2} w_j \log_2 S(j,q)$
   $\hat{c}_p = (\log_2 e) \sum_{j=j_1}^{j_2} w_j \hat{C}(j,p)$

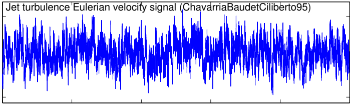
4. Multifractal Spectrum :

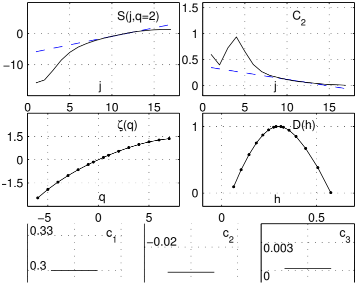
   $\hat{D}(h) = \min_{q \neq 0}(1 + qh - \hat{\zeta}(q))$
   Parametric formulation : $\hat{D}(q), \hat{h}(q)$.

Multifractal formalism at work : 1D

- Multifractal attributes estimation,
- from a single finite length observation

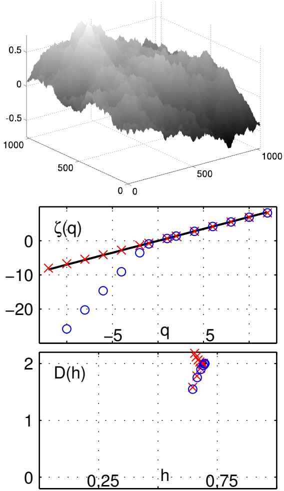




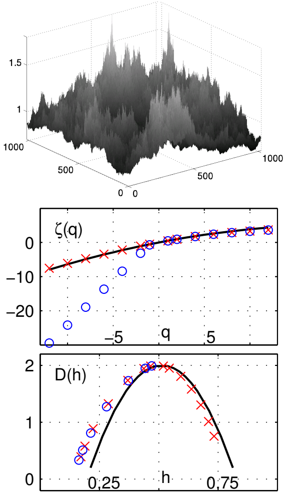


Multifractal formalism at work : 2D

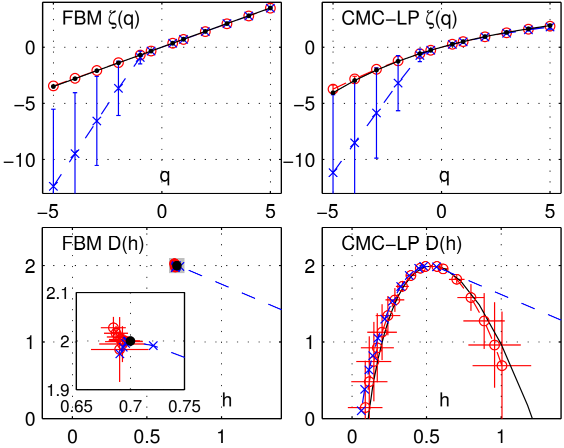
Fractional Brownian motion



Multiplicative cascade



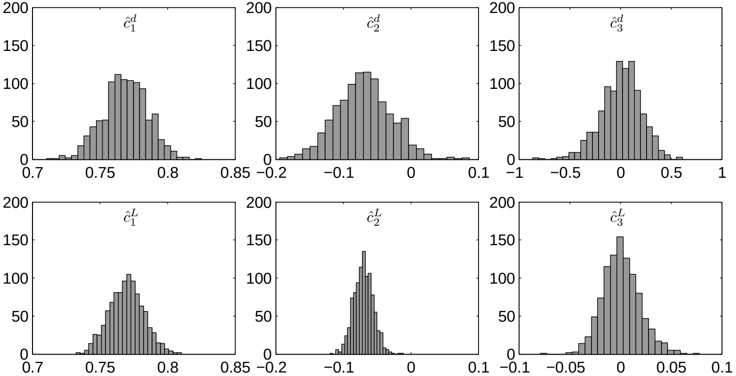
Estimation performance



Estimation performance

- Leader based estimates outperform wavelet based ones :
- mostly for $c_p, p \geq 2$ (multifractal properties),

$$\zeta(q) = c_1 q + c_2 q^2/2 + c_3 q^3/6 + \dots$$



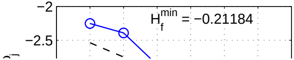
Leaders and Positive Hölder

- Leader MF formalism applies if X has positive uniform regularity
  - $\Rightarrow$ from finite data, Leaders can always be practically computed even if undefined
  - $\Rightarrow$ uniform regularity needs to be tested a priori
- X has uniform Hölder regularity ▸ Unif. Holder
  - $\Rightarrow$ X has positive Hölder exponent only !

- Minimal regularity computed a priori from wav. coefficients

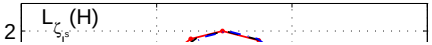
$$h_{min} = \liminf_{2^j \to 0} \frac{\ln \sup_{m,\mathbf{k}} |d_X^{(m)}(j,\mathbf{k})|}{\ln 2^j}$$

$h_{min} > 0 \Rightarrow$ uniform Hölder regularity.



Positive Hölder and (fractional) Integration

- What if $h_{min} < 0$?
  - No Multifractal formalism ?
  - Issue both for images and signals !
- Solution $\Rightarrow$ (fractional) Integration :
  - $(\widehat{I^\gamma X})(\xi) = (1 + |\xi|^2)^{\gamma/2} \hat{X}(\xi)$.
  - if $\gamma > h_{min}$, $I^\gamma X$ is uniformly Hölder function
  - integrate enough then apply wavelet Leader formalism.
  - avoid to actually compute fractional integration.
- Pseudo (fractional) Integration :
  - compute $d_X^{(m)}(j, k_1, k_2)$ directly from X,
  - $d_X^{(m),\gamma}(j, k_1, k_2) = 2^{\gamma j} d_X^{(m)}(j, k_1, k_2)$.
  - $L_X^\gamma(j, k_1, k_2) = \sup_{m,\, \lambda' \subset 3\lambda_{j,k_1,k_2}} |d_X^{(m),\gamma}(\lambda')|$.
  - MF spectrum from $L_X^\gamma(j, k_1, k_2)$ is identical to that of $(\widehat{I^\gamma X})(\xi)$
  - vary γ with $\gamma > h_{min}$, Deduce $D(h) = D^\gamma(h - \gamma)$

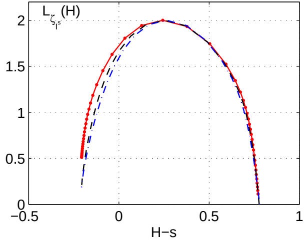


Open issue

- Pseudo (fractional) Integration...

$$d_X^{(m),\gamma}(j,k_1,k_2) = 2^{\gamma j} d_X^{(m)}(j,k_1,k_2).$$
$$L_X^{\gamma}(j,k_1,k_2) = \sup_{m,\,\lambda' \subset 3\lambda_{j,k_1,k_2}} |d_X^{(m),\gamma}(\lambda')|.$$

vary γ with $\gamma > h_{min}$, Deduce $D(h) = D^{\gamma}(h-\gamma)$



- ... is not always valid : theoretical issue

chirp (or oscillating) singularities

MF spectrum not well defined if both negative h and chirp !

Wavelet coefficients or Leaders ?

- Wavelet coefficients :
  - Estimate global attributes : self-similarity, LRD, global regularity
    - Compute h_{min}, Estimate c_1 and $\zeta(2)$,
    - but can hardly see deviation from $\zeta(q) = qH$.
- Wavelet Leaders :
  - Estimate multifractal attributes
    - Estimate c_p, $p \geq 2$
    - $\zeta(q)$ (for both positive and negative qs),
    - can perfectly estimate concave $\zeta(q)$,

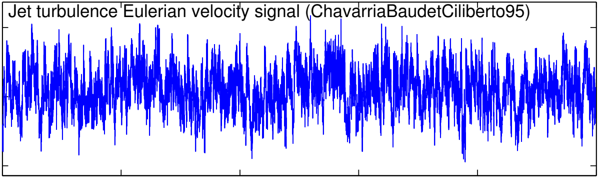
$\Rightarrow$ Coefficients and Leaders are to be used in a complementary way !

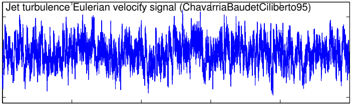
First, wavelet coef. ($h_{min} > 0$, c_1)
Second, Wavelet Leaders.

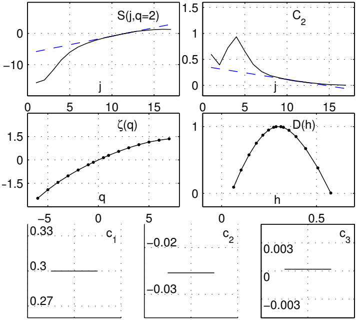
$\Rightarrow$ even if not truly interested in multifractal properties, local regularity and Hölder exponent, Leaders are needed to measure concave $\zeta(q)$

Estimation from a single observation

- Multifractal attributes estimation :







Confidence intervals for multifractal estimates ?

- Issues :
  - confidence intervals for multifractal attribute estimates ?
  - from a single observation,
  - from a finite length observation.
- State of the art ?
  - no theoretical results so far,
  - Gaussian asymptotics perform poorly,
  - almost totally overlooked issue.
- Why ?
  - multifractal processes : a large class
  - amongst which, multiplicative cascades,
  - often with heavy tail marginals,
  - and intricate (power law type) dependence structures,
- Our proposition : Non parametric bootstrap

  Drawing with replacement procedure

Bootstrap for multifractal analysis

- Multiplicative multifractal Processes :

  intricate, power law, dependencies
  non stationary,
  heavy tails.

  $\Rightarrow$ no bootstrap in the time domain

- Wavelet domain :

  decorrelation properties ? as for fBm ?
  time-scale residual dependency

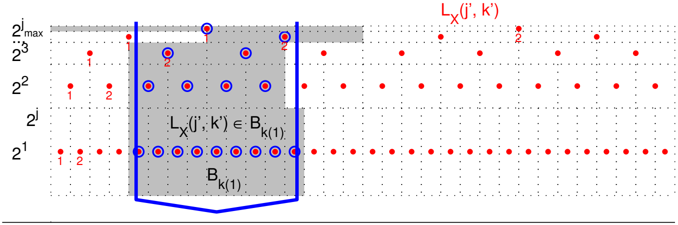
$\Rightarrow$ **Time-scale block bootstrap in the wavelet domain !**

Time-scale block bootstrap

- Time-scale block (1D) :

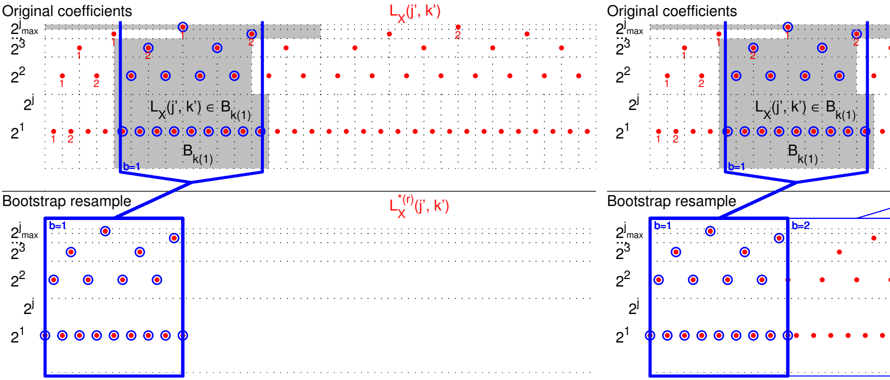
  timescale strip
  of time length $2l$
  through all scales

$$\mathcal{B}_k = \{L_X(j', k') : |k - k'2^{j'}| \leq l,\ 1 \leq j' \leq j_{max}\},\ 1 \leq k \leq n$$



Time-scale block bootstrap

- From the $\{L(j,k)\}$:
  draw with replacement, $B = \lceil \frac{N}{2l} \rceil$ blocks $\mathcal{B}_k$:
  chain them, $\rightarrow \{L^{*(r)}(j,k)\}$



Wavelet leaders time-scale blok bootstrap

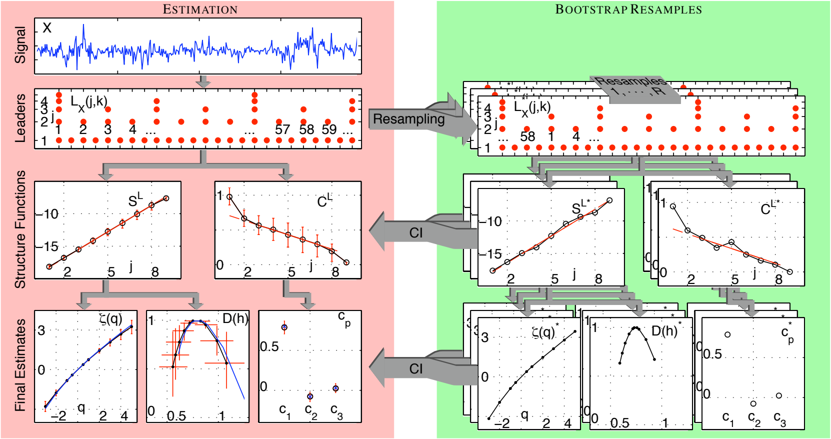
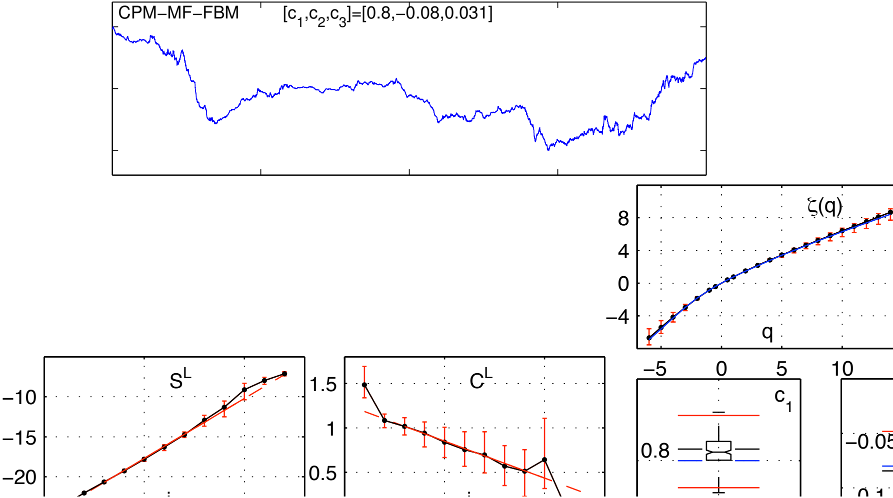


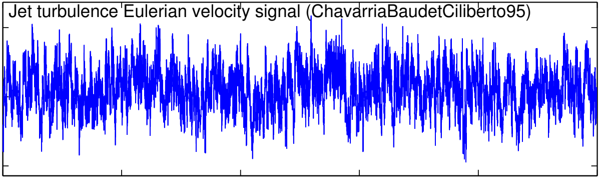
Illustration : 1D example

- fBm in multifractal time :



Turbulence

- Estimation and confidence intervals :



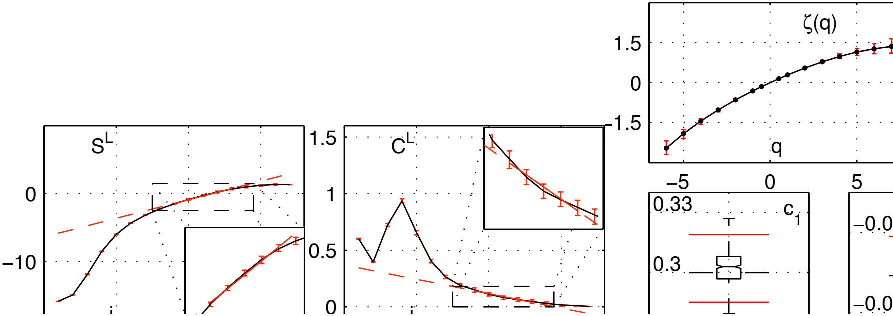


Illustration : 2D Example

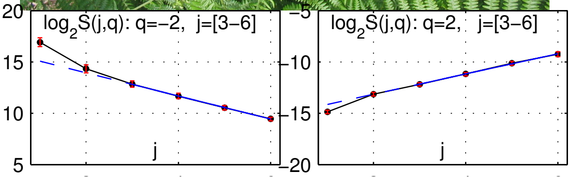
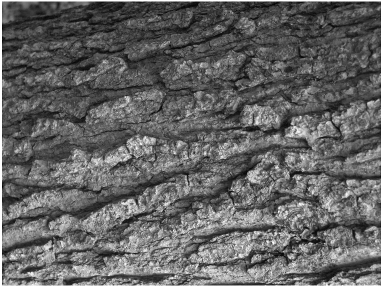


Illustration : 2D Example



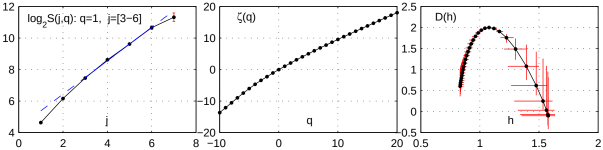
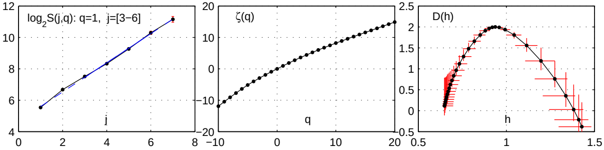


Illustration : 2D Example



Hypothesis tests

- Multifractality tests :

  test for a prescribed value of a given multifractal attribute
  example 1 : $c_1 = H > 0.5$? Long range dependence ?
  example 2 : $c_2 = 0$ or $c_2 \neq 0$? test for multifractal ?

$$\zeta(q) = c_1 q + c_2 q^2/2 + \ldots$$

  $\zeta(q)$ linear or not ? $\Rightarrow$ practical test for MF

- Stationarity-type test :

  are the estimated MF attributes constant along time ?

Open Issues

- Bootstrap procedures validated by numerical simulations.
- Why does Bootstrap theoretically work ?
- What are the actual dependence structure of multiresolution quantities for multifractal processes ?
- How does the number of vanishing moments help ?

$\Rightarrow$ Under current investigations !

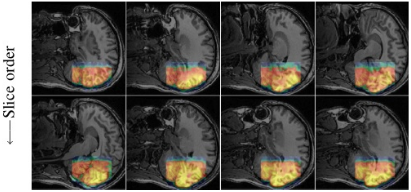
$\Rightarrow$ Preliminary results :

Dependence structure is long range (power-law) slowly decreasing type !

Vanishing moments help much less : decorrelation but no independence !

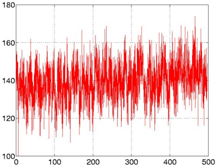
fMRI : Active versus Resting States

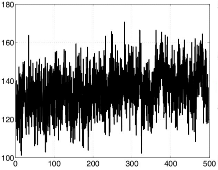
- fMRI data - task-related activation of brain regions

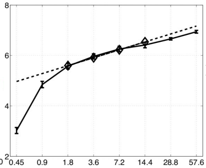


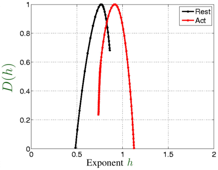
collaboration P. Ciuciu, CEA/NeuroSpin

time series [active] [resting] *scale invariance* *multifractal spectra*









Texture Classification

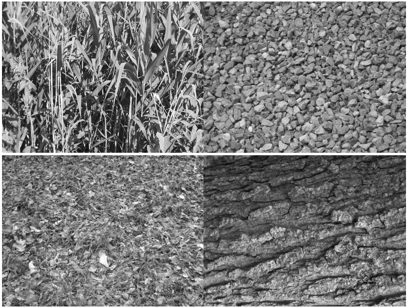
Collaboration J. Hui and S. Zuowei, NUS, Singapore

- texture image database :

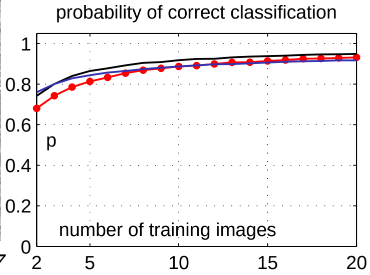
  40 classes $\times$ 50 images – size 1280 $\times$ 960

  rotation, illumination, angle, viewpoint invariance

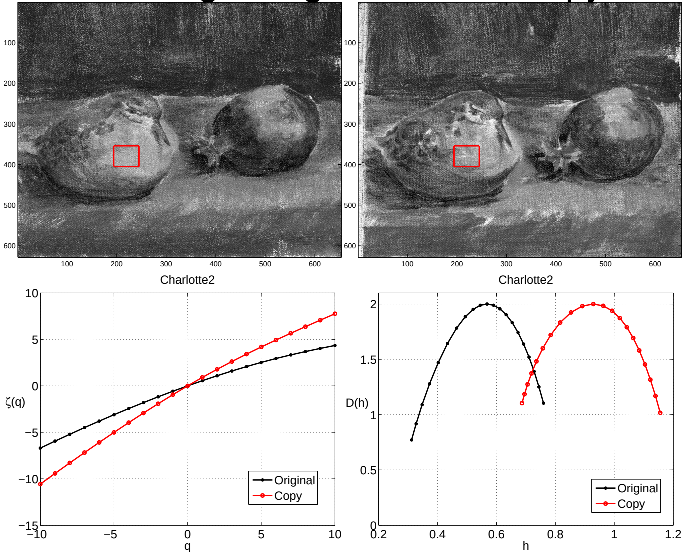
- accurate, robust, meaningful, efficient classification



U Maryland high-resolutions texture database, 2007



Painting : Original versus Copy



Conclusions

- Scaling :
  - no characteristic scales
  - all scales are characteristic
  - inter-scale relations
  - power laws
- Modeling :
  - 2nd order stationary processes,
  - Long range dependence,
  - self-similarity, random walks, addition
  - multifractality, cascades, multiplication
- Analysis :
  - aggregation, increments $\Rightarrow$ Multiresolution
  - wavelet coefficients ($N_\psi > h$, $N_\psi > H$)
  - wavelet Leaders
- Misc. :
  - Bootstrap,
  - 1D (signal), 2(images), nD

More information

- WEB Site :
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  - perso.ens-lyon.fr/herwig.wendt
- References (articles, talks)
- MATLAB Toolboxes (Wavelets, scaling and fractal)

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- L. Delbeke, Leuwen Univ., Belgique,
- M. Taqqu, V. Pipiras, Boston Univ., USA,
- M.-E. Torres, Univ. de Entre-Rios, Parana, Argentina,
- R. Riedi, Rice University, USA,
- P. Borgnat, S. Roux, SISYPHE, ENS Lyon, France,
- P. Chainais, B. Lashermes, H. Wendt, SISYPHE, ENS Lyon, France,
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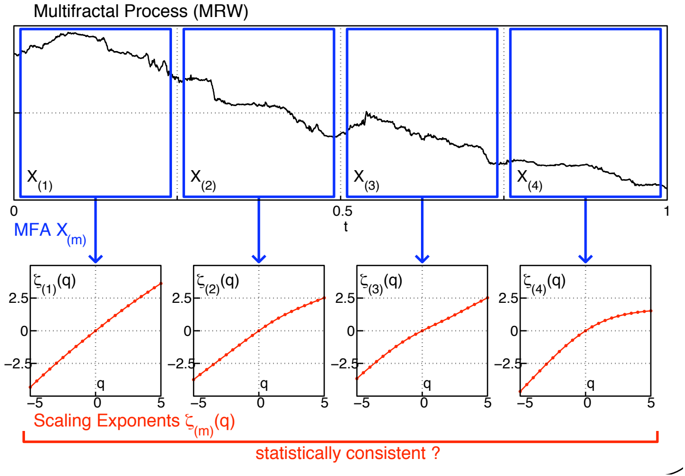
- Special Thanks : Herwig Wendt
  for great illustrations and movies !

Multifractal attribute time constancy test

- Issue ?



Multifractal attribute time constancy test

- Step1 : Split then estimate/bootstrap

  Split data into M adjacent non overlapping blocks

  M estimations $\hat{\theta}_{(m)} \to H_{\text{null}}: \quad \theta_{(1)} = \theta_{(2)} = \cdots = \theta_{(M)}$

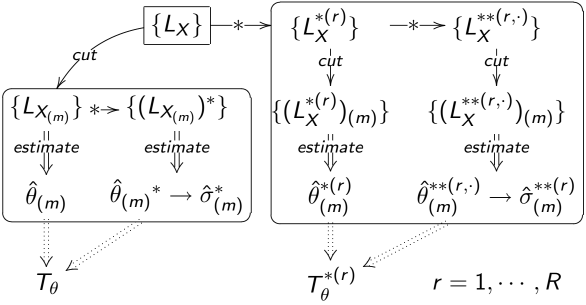
$$T_\theta = \sum_{m=1}^{M} \frac{1}{\hat{\sigma}^{2*}_{(m)}} \left(\hat{\theta}_{(m)} - \frac{\sum_{n=1}^{M} \frac{\hat{\theta}_{(m)}}{\hat{\sigma}^{2*}_{(m)}}}{\sum_{n=1}^{M} \frac{1}{\hat{\sigma}^{2*}_{(m)}}} \right)^2$$

- Step2 : Bootstrap then split/estimate

$$T^*_\theta = \sum_{m=1}^{M} \frac{1}{\hat{\sigma}^{2**}_{(m)}} \left(\hat{\theta}^*_{(m)} - \frac{\sum_{n=1}^{M} \frac{\hat{\theta}^*_{(m)}}{\hat{\sigma}^{2**}_{(m)}}}{\sum_{n=1}^{M} \frac{1}{\hat{\sigma}^{2**}_{(m)}}} \right)^2$$

Multifractal attribute time constancy test

- Bootstrap :



- Test :

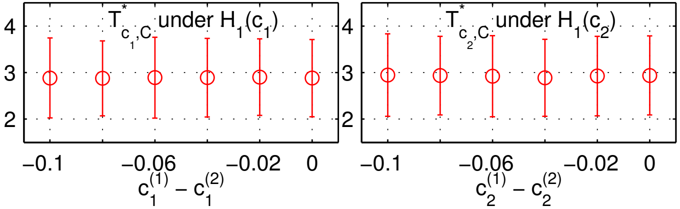
  $T^*_{\theta,C}$ - quantile $(1-\alpha)$ of empirical distribution $\{T^*_\theta\}$
  $d_\theta = 1$ if $T_\theta > T^*_{\theta,C}$ and 0 otherwise

Multifractal attribute time constancy test

- Numerical simulation : concatenate two MF processes
- Under H_{null} (MRW) : **Significance level**, $\alpha = 0.1$

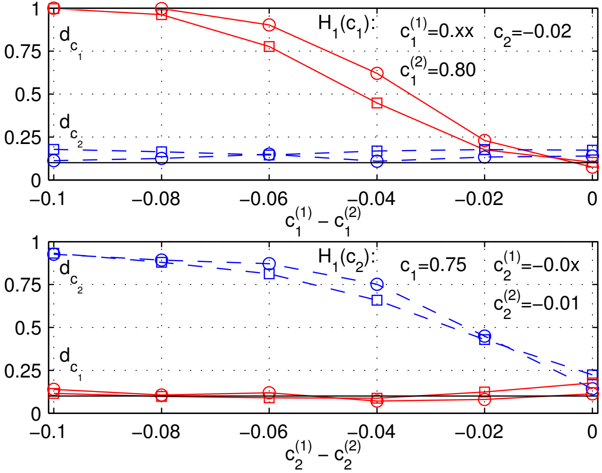
$\{c_1, c_2\}$	$\{0.75, -0.01\}$		$\{0.8, -0.02\}$	
θ	c_1	c_2	c_1	c_2
$\bar{d}_\theta^{H_0}$	0.113	0.143	0.075	0.139
$\bar{p}_\theta^{H_0}$	0.478	0.469	0.530	0.485

- Under H_A (MRW) : **p-values**



Multifractal attribute time constancy test

- Under H_A (MRW) : **Power** Wav. Coef. Wav Leaders

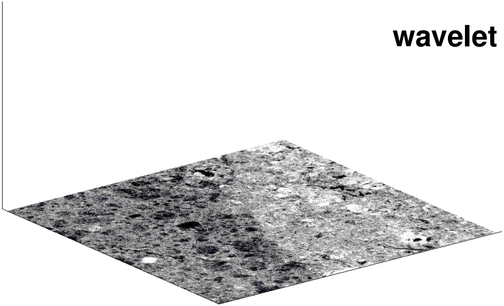


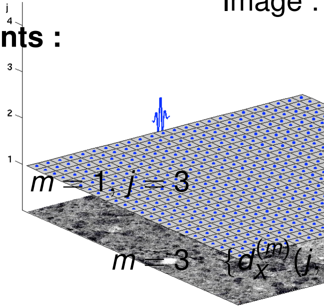
2D Discrete Wavelet Transform

- 2D Orthonormal basis

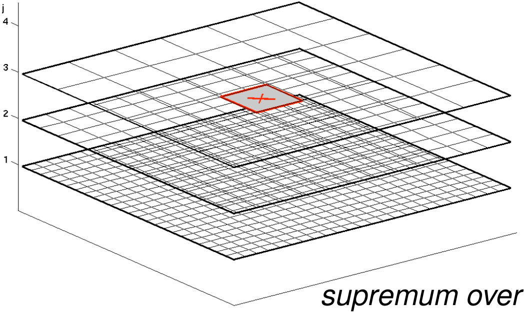
$$
\begin{array}{rrcl}
 & \phi_0(t) & & \text{scaling function (father wavelet)} \\
 & \psi_0(t) & & \text{(mother) wavelet} \\
\text{Dyadic sampling} & \phi_{j,k}(t) & = & 2^{-j}\phi_0(2^{-j}t - k) \\
 & \psi_{j,k}(t) & = & 2^{-j}\psi_0(2^{-j}t - k) \\
 & & & \\
\text{base} & \tilde{\psi}^{(1)}_{j,k_1,k_2}(x,y) & = & \phi_{j,k_1}(x)\psi_{j,k_2}(y) \\
 & \tilde{\psi}^{(2)}_{j,k_1,k_2}(x,y) & = & \psi_{j,k_1}(x)\phi_{j,k_2}(y) \\
 & \tilde{\psi}^{(3)}_{j,k_1,k_2}(x,y) & = & \psi_{j,k_1}(x)\psi_{j,k_2}(y)
\end{array}
$$

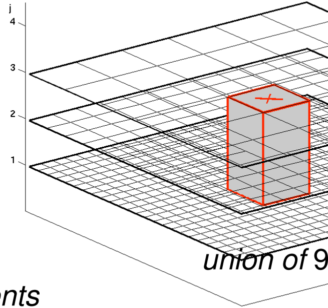
2D Discrete Wavelet Transform





2D wavelet Leaders





supremum over coefficients

$$\lambda_{j,k_1,k_2} \quad =$$
$$= \quad \{[k_1 2^j, (k_1+1)2^j), [k$$
$$3\lambda_{j,k_1,k_2} \quad =$$

Log-Cumulants

- For certain classes of processes :
  - $\mathbf{E} L_X(j,\cdot)^q = F_q |2^j|^{\zeta(q)}$
- 2^{nd} characteristic function $\ln L_X(j,\cdot)$:
  - $\ln \mathbf{E} e^{q \ln L_X(j,\cdot)} = \sum_p C_p^j \frac{q^p}{p!} = \ln F_q + \zeta(q) \ln 2^j$

    C_p^j : cumulant of order $p \geq 1$ de $\ln L_X(j,\cdot)$
- $\Rightarrow \forall p \geq 1 : \quad C_p^j = c_p^0 + c_p \ln 2^j$
  - $\ln \mathbf{E} e^{q \ln L_X(j,\cdot)} = \underbrace{\sum_{p=1}^{\infty} c_p^0 \frac{q^p}{p!}}_{\ln F_q} + \underbrace{\sum_{p=1}^{\infty} c_p \frac{q^p}{p!}}_{\zeta(q)} \ln 2^j,$
  - $\zeta(q) = \sum_{p=1}^{\infty} c_p \frac{q^p}{p!}$

Uniform Hölder regularity

- Uniform Hölder : X is a uniform Hölder function iff

  $\exists\epsilon > 0$ such that $X \in C^{\epsilon}(\mathcal{R}^d)$,

  $\exists C > 0$ such that $\forall t, s \in \mathcal{R}^d, \ |X(t) - X(s)| \leq C|t - s|^{\epsilon}$.

Hölder exponent and Wavelets (Theory - S. Jaffard)

- Uniform Hölder : X is a uniform Hölder function iff

  $\exists\epsilon > 0$ such that $X \in C^{\epsilon}(\mathcal{R}^d)$,
  $\exists C > 0 \quad$ such that $\forall t, s \in \mathcal{R}^d, \ \ |X(t) - X(s)| \leq C|t-s|^{\epsilon}$.

- Wavelet Coefficients and uniform Hölder function :

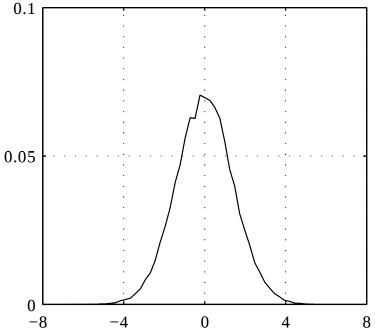
  $h > 0$, if X is $C^{h(t_0)}$, then $\exists C > 0$ such that :
  $\forall j \geq 0, \qquad |d_X(j,k)| \leq C2^{jh(t_0)}(1+|2^{-j}t_0 - k|)^{h(t_0)}$.

- Conversely, if relation above holds and if X is uniform Hölder,

  then $\exists C > 0$ and
  $\exists$ a polynomial P satisfying $deg(P) < \alpha$,
  such that, in a neighbourhood of t_0,
  $|X(t) - P(t-t_0)| \leq C|t-t_0|^{\alpha}|\log(1/|t-t_0|)$.

Limitation 1 : Negative *qs*

- Wavelet Coefficients $\Rightarrow d_X(j, k) \simeq 0$,



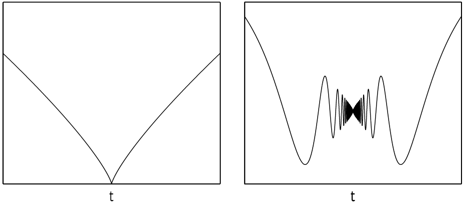
- Structure Functions are numerically instable,

$\Rightarrow$ Decreasing part of $D(h)$ cannot be measured !

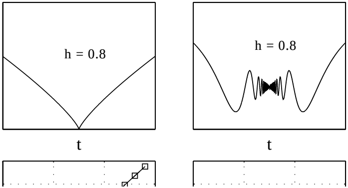
Limitation 2 : Oscillating Singularities

- Cusp Singularity : $|X(t) - X(t_0)| \sim_{|t-t_0| \to 0} |t - t_0|^h$
- Chirp (Oscillating) Singularity :

$$|X(t) - X(t_0)| \sim_{|t-t_0| \to 0} |t - t_0|^h \sin\left(\frac{1}{|t-t_0|^{\beta}}\right)$$



- Hölder exponent : $|d_X(j,k)| \sim_{|2^j| \to 0} 2^{jh}$?



Bootstrap : Principles

- Issues :

  Data $X = \{x_1, \cdots, x_N\}$,
  $x_i \stackrel{i.i.d.}{\sim} P_X(x)$, unknown !
  $\hat{\theta} = \theta(X)$

  Statistical performance of $\hat{\theta}$? pdf of $\hat{\theta}$?

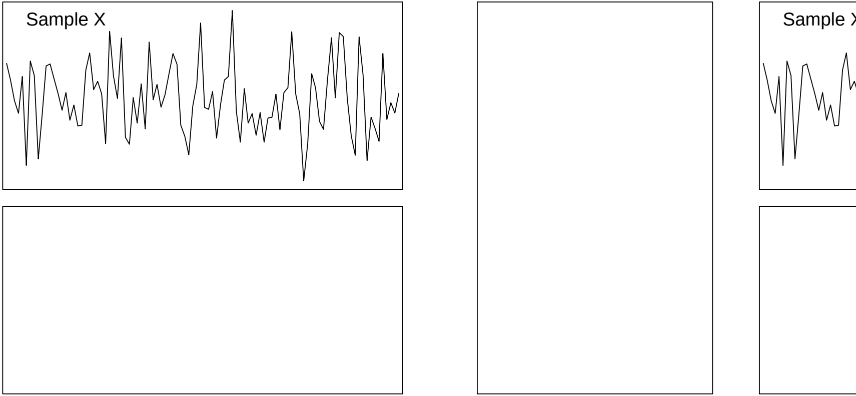
- Non parametric Bootstrap :

  Drawing with replacement procedure
  from the same observation X,

  $\rightarrow$ R copies $X^{*(r)}$,
  $\rightarrow$ R estimates $\hat{\theta}^{*(r)} = \theta(X^{*(r)})$,
  $\rightarrow$ empirical pdf $\hat{\theta}^{*(r)}$
  $\rightarrow$ use pdf of $\hat{\theta}^{*(r)}$ as pdf of $\hat{\theta}$.

Illustration

Example : $x_i \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$, $\theta(X) = \mathrm{Var}\,(X)$



Block Bootstrap

- Dependent data :

  $X = \{x_1, \cdots, x_N\}$, with $P_X(x)$, unknown !

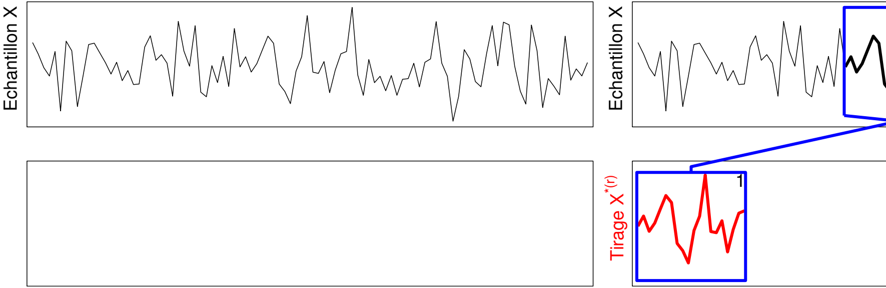
  $\hat{\theta} = \theta(X)$, Statistical performance of $\hat{\theta}$?

- Block Bootstrap : Drawing with replacement procedure,

  Drawing with replacement procedure,

  over blocks of data $\{x_i, \cdots, x_{i+l-1}\}$

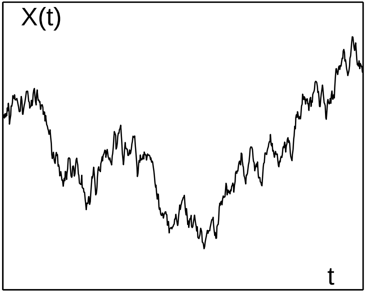
- Intuition : preserve dependence structure

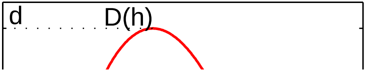


Multifractal Spectrum

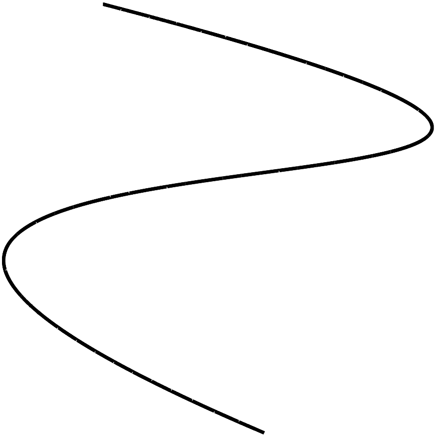
- **Multifractal Spectre** $D(h)$:
  - Irregularity : Fluctuations of regularity $h(t)$
  - Set of points that share same regularity $\{t_i | h(t_i) = h\}$
  - Fractal (or Haussdorf) Dimension of each set :

$$D(h) = \dim_H \{t : \ h(t) = h\}$$

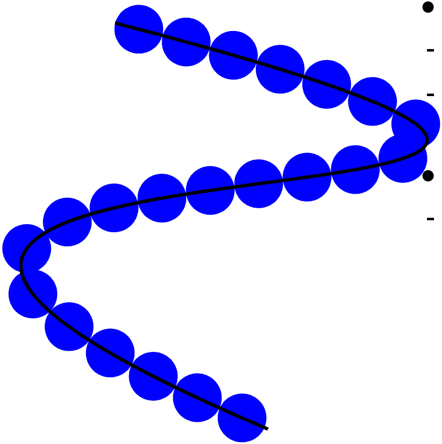




Dimension of a geometrical set

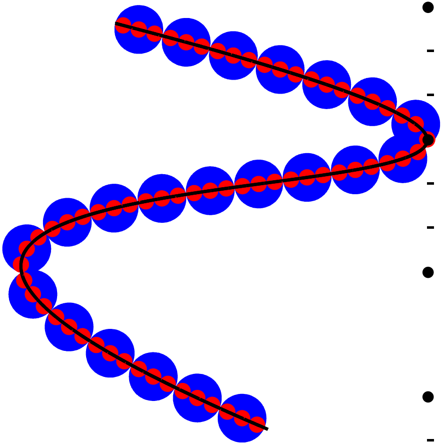


Euclidean dimension



- Let
  - $a (= 1)$ be the analysis scale,
  - N denote the number of covering boxes with size a,
- Then
  - Length is : $L = N \cdot a$

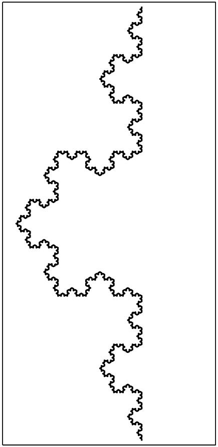
Euclidean dimension



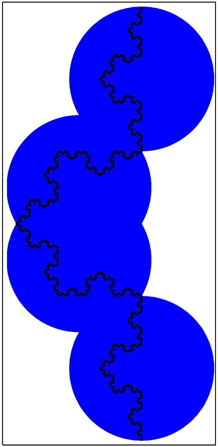
- Let
  - $a (= 1)$,
  - $a (= 1/3)$,
- hence,
  - $N = \frac{a}{a} \cdot N (= 3 \cdot N)$,
  - $L = N \cdot a = L = N \cdot a = L_0$,
- donc

  $L(a)$ does not depend on a nor on a!
- and
  - $L(a) = N(a) \cdot a = L_0$,
  - $\boxed{N(a) = L_0/a = L_0 \cdot a^{-1}}$.

Dimension of a geometrical set

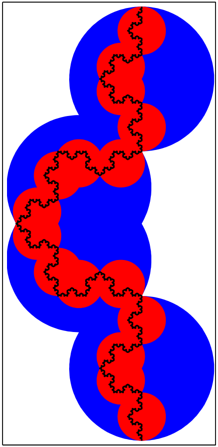


Fractal dimension



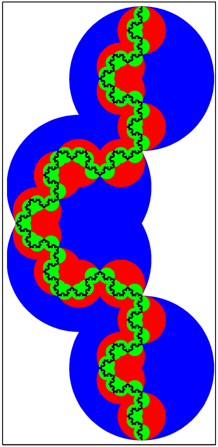
- Let
  - a, be the analysis scale
  - N denote the number of covering boxes with size a,
- Then
  - Length is : $L = N \cdot a$
- Here,
  - $a = 1/3$,
  - $N = 4$,

Fractal dimension



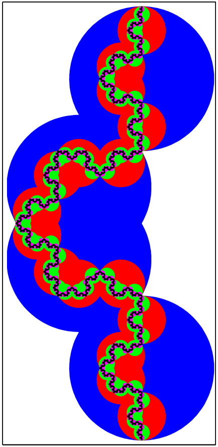
- Let
  - $a (= 1/3)$,
  - $a (= 1/9)$,
- Then,
  - $N = 4$,
  - $N = 16$,
- Hence
  - $L = N \cdot a \neq L = N \cdot a$! ,

Fractal dimension



- Let
  - $a (= 1/3)$,
  - $a (= 1/9)$,
  - $a = 1/27$,
- Then,
  - $N = 4$,
  - $N = 16$,
  - $N = 64$,
- donc
  - $L = N \cdot a \neq L = N \cdot a \neq L = N \cdot a$!

Fractal dimension



- One shows :
- $a(n) = (1/3)^n$,
- $N(n) = 4^n$,
- hence
- $L(a) = N(a) \cdot a$,
- $L(a)$ does depend on a !
- with,
- $\boxed{N(a) = a^{-D}}$,
- $L(a) = L_0 \cdot a^{1-D}$,
- D : fractal dimension,
- $1 < D < 2$,
- non integer = Frac-.

Haussdorf Dimension

- Intuition :

  Fractal dimension,
  Non integer extension of the natural *Euclidean* dimension, $0 \leq D \leq d$.
  Cover a set A with balls of size ϵ, Count how many you need $N(\epsilon)$.
  Assume a power law behaviour $N(\epsilon) \sim \epsilon^{-D}$.
  Define $D = \lim_{\epsilon \to 0} - \log N(\epsilon) / \log \epsilon$.

- Definition :

  $A \in \mathcal{R}^d$,
  $\epsilon > 0$, R ϵ-covering of A with a countable collection of bounded sets A_i, $|A_i| \leq \epsilon$,
  $\delta \in [0, d], M_\epsilon^\delta(A) = \inf_R \left(\sum_i |A_i|^\delta \right), M^\delta(A) = \lim_{\epsilon \to 0} M_\epsilon^\delta(A)$,
  D is such that $\delta > D, M^\delta(A) = 0, \delta < D, M^\delta(A) = \infty$.

Thermodynamic analogy (Parisi-Frisch, 85)

Thermodynamic	Multifractal
- $Z_\beta(U) = \sum_k e^{-\beta E_k}$,	- $S(a,q) = \sum_k \|T_X(a,k)\|^q$ $S(a,q) = \sum_k e^{q \log \|T_X(a,k)\|}$
$U = \langle E_k \rangle = \partial \log Z_\beta / \partial \beta$	- $\|T_X(a,k)\| = a^{h_k}$, - $S(a,q) = \sum_k e^{q h_k \log a}$
- β	- q
- $E_k = \epsilon_k \delta V$,	- $h_k \log a$, - $S(a,q) = a^{\zeta(q)}$,
- $F = -\ln Z_\beta$	- $\zeta(q) \log a = \log S(a,q)$,
- Entropy : $F = U - S/\beta$ (Legendre transform)	- Spectrum : $D(h) = qh - \zeta(q)$ (Legendre transform)

◂ to MF Form.

Rényi entropy

Strange attractors and chaotic systems (Kadanoff, 75)

- Rényi entropy : $Z_\alpha(a) = \sum_k P_k(a)^\alpha$,
- Rényi information : $I_\alpha(a) = \log Z_\alpha(a)/(1-\alpha)$,
- Generalized dimensions : $D_\alpha = \lim_{a\to 0} I_\alpha(a)/(-\log a)$,

$$\Rightarrow \quad (1-\alpha)D_\alpha = \lim_{a\to 0} \log Z_\alpha(a)/\log a \equiv \zeta(\alpha)\,!$$

to MF Form.