WAVELET LEADER BASED MULTIFRACTAL ANALYSIS

Bruno Lashermes$^{(1)}$, *Stéphane Jaffard*$^{(2)}$, *Patrice Abry*$^{(1)}$

(1) Laboratoire de Physique
CNRS - NS Lyon, France
{blasherm,pabry}@ens-lyon.fr

(2) Laboratoire d'Analyse et de Mathématiques Appliquées
CNRS - Univ. Paris XII, Créteil, France
jaffard@univ-paris12.fr

ABSTRACT

We introduce a new multifractal formalism based on wavelet leaders and study its properties. Comparing it against previously formulated wavelet coefficient based multifractal formalism, we show first that this wavelet leader based formalism allows first to obtain the multifractal spectrum over its entire range, and second that it does not cease to hold when applied to processes embodying unusual chirp-type (or oscillating) singularities (as opposed to the more common cusp-type ones). We illustrate these results and properties on four examples of multifractal deterministic functions or stochastic processes containing a graduation of the major difficulties. We show that this new multifractal formalism benefits from excellent theoretical and practical performance. Matlab routines implementing it are available upon request.

1. MOTIVATION

Scale invariance or scaling is a paradigm that has been largely used in the past ten years to describe/analyse/model data produced by a wide variety of applications of very different nature (heart rhythms, hydrodynamic turbulence, computer network traffic, financial markets, ...). After seminal works historically developed in the field of hydrodynamics turbulence [1, 2, 3], the *multifractal* framework became a major concept and tool involved in the practical analysis of scaling in data.

Following its original formulation in turbulence, the *multifractal formalism* has been based on the increments of the studied function. It is however now well-known that it benefits from significant theoretical and practical improvements when written with wavelet coefficients. It is usually computed either from a *discrete wavelet transform* (DWT) or from a *modulus maxima wavelet transform* (MMWT).

We propose here the formulation of a new multifractal formalism based on *wavelet leaders*. We explain how and why it overcomes two major difficulties partially or unsatisfactorily solved by previous formulations: It enables to determine the multifractal spectrum over its entire range and it works equally well for processes containing usual *cusp-type* singularities and for those presenting less common *chirp-type* singularities. Its extension to higher dimensions is straightforward and it has an extremely low computational cost. The wavelet leader based multifractal formalism brings a real enhancement in multifractal analysis both from conceptual or practical viewpoints compared to previous formulations and should definitely be systematically used.

2. MULTIFRACTAL AND WAVELET COEFFICIENTS

•Hölder Exponent and Singularity Spectrum.
Let $\{X(t)\}_{t\in\mathbb{R}}$ denote the sample path of the function or stochastic process of interest. Its local regularity is commonly studied via the notion of pointwise Hölder exponent. $X(t)$, which is assumed to be locally bounded, belongs to $C^\alpha(t_0)$ at $t_0 \in \mathbb{R}$, with $\alpha \geq 0$, if there exist a constant $C > 0$ and a polynomial P satisfying $deg(P) < \alpha$ and such that, in a neighbourhood of t_0: $|X(t) - P(t-t_0)| \leq C|t-t_0|^\alpha$. The Hölder exponent of X at t_0 is defined as $h(t_0) = \sup\{\alpha : X \in C^\alpha(t_0)\}$. A straightforward example is given by the *cusp-type* function $X(t) = A + B|t-t_0|^h$, whose Hölder exponent in t_0 is simply h (when h is not a even integer).

The fluctuations, along time t, of the Hölder exponent h are usually described through the *singularity (or multifractal) spectrum*, labelled $D(h)$ and defined as the Hausdorff dimension of the set of points where the Hölder exponent takes the value h. For the definition of the Hausdorff dimension as well as for further details on multifractals, the reader is referred to e.g., [4, 5].

•Multifractal Formalism. The determination of the multifractal spectrum from an observed sample path $X(t)$ is a crucial practical issue. However, a direct numerical determination based on the definition of the Hölder exponent turns out to be impossible. This is first because by definition the Hölder exponent of a multifractal path varies widely from one time to another and second because actual empirical data come with practical limitations such as discrete time sampling and finite resolution. To overcome such difficulties, Parisi and Frisch, in a seminal work in turbulence [2], proposed to overcome such difficulties by introducing a *multifractal formalism*: it consists in reaching the multifractal spectrum through auxiliary easily computable quantities: the structure functions (see below). Their original proposition was based on increments $X(t+\tau) - X(t)$ of $X(t)$, they are now advantageously replaced by wavelet coefficients $d_X(j,k)$.

•Wavelet coefficients and Hölder Regularity. It has long been recognised that wavelet coefficients constitute ideal quantities to study the regularity of a path (see e.g., [6, 4, 5]). Let us briefly recall how and why. Let $\psi_0(t)$ denote a reference pattern with fast exponential decay and called the *mother-wavelet*. It is also characterised by a strictly positive integer $N \geq 1$, called its *number of vanishing moments*, defined as $\forall k = 0, 1, \ldots, N-1$, $\int_\mathbb{R} t^k \psi_0(t)dt \equiv 0$ and $\int_\mathbb{R} t^N \psi_0(t)dt \neq 0$. Let $\{\psi_{j,k}(t) = 2^{-j}\psi_0(2^{-j}t-k), j \in \mathbb{N}, k \in \mathbb{N}\}$ denote templates of ψ_0 dilated to scales 2^j and translated to time positions $2^j k$. The discrete wavelet transform (DWT) of X is defined through its coefficients:

$$d_X(j,k) = \int_\mathbb{R} X(t)\; 2^{-j}\psi_0(2^{-j}t-k)\,dt. \qquad (1)$$

For further details on the wavelet transforms, the reader is referred to e.g., [7]. It has been shown that on condition that $N > h$, when $X \in C^\alpha(t_0)$ then there exists a constant $C > 0$ such that

$|d_X(j,k)| \le C2^{jh}(1+|2^{-j}t_0-k|^h)$. Loosely speaking, it is commonly read as the fact that when X has Hölder exponent h at $t_0 = 2^j k$, the corresponding wavelet coefficients $d_X(j,k)$ are of the order of magnitude $|d_X(j,k)| \sim 2^{jh}$. This is precisely the case of the cusp like function mentioned above. Further results relating the decrease along scales of wavelet coefficients and Hölder exponent can be found in e.g., [8].

•Wavelet Coefficient based Multifractal Formalism.
Wavelet coefficient based structure functions and scaling exponents are defined as:

$$S_d(q,j) = \frac{1}{n_j}\sum_{k=1}^{n_j} |d_X(j,k)|^q, \tag{2}$$

$$\zeta_d(q) = \liminf_{j\to 0}\left(\frac{\log_2 S_d(q,j)}{j}\right), \tag{3}$$

where n_j is the number of available $d_X(j,k)$ at octave j: $n_j \simeq n_0 2^{-j}$. By definition of the multifractal spectrum, there are about $(2^j)^{-D(h)}$ points with Hölder exponent h, hence with wavelet coefficients of the order $|d_X(j,k)| \simeq (2^j)^h$. They contribute to $S_d(q,j)$ as $\sim 2^j(2^j)^{qh}(2^j)^{-D(h)} = (2^j)^{1+qh-D(h)}$. Therefore, $S_d(q,j)$ will behave as $\sim c_q(2^j)^{\zeta_d(q)}$ and a standard steepest descent argument yields a *Legendre transform* relationship between the multifractal spectrum $D(h)$ and the scaling exponents $\zeta_d(q)$: $\zeta_d(q) = \inf_h(1+qh-D(h))$. The Wavelet Coefficient based Multifractal Formalism (hereafter **WCMF**) is standardly said to hold when the following equality is valid:

$$D(h) = \inf_{q\neq 0}\left(1+qh-\zeta_d(q)\right). \tag{4}$$

•Limitations. However, the wavelet coefficient based multifractal formalism suffers from two major drawbacks, illustrated in Section 4 below.
First, by definition, wavelet decompositions necessarily yield a large number of *close to* 0 coefficients. This implies that the computation of $S_d(q,j)$ for negative qs will be numerically instable. In a stochastic framework, it would translate into the fact that the wavelet coefficients are random variables with a strictly positive probability density function at the origin and hence infinite moments of order $q < -1$. In both cases, it implies practically that wavelet coefficient based structure functions cannot be used for $q < -1$. The Legendre transform indicates that it will prevent us from measuring the, roughly speaking, right part of $D(h)$ (i.e., the part of D such that $h > h_0$, where h_0 corresponds to the value of h where $\inf_q(1+qh-\zeta_d(q))$ is the largest).
Second and foremost, while valid for cusp-type singularities, the wavelet coefficient based multifractal formalism fails to hold for instance when the analysed $X(t)$ embodies *oscillating (or chirp-type) singularities*, i.e., when $X(t)$ is of the form: $X(t) = |t-t_0|^\alpha \sin\left(1/|t-t_0|^\beta\right)$.

3. MULTIFRACTAL AND WAVELET LEADERS

•Wavelet Leaders. To overcome those two drawbacks, it has recently been suggested that the relevant quantities the multifractal formalism should be based on are not wavelet coefficients but *wavelet leaders* [9].
From now on, let us assume that ψ_0 is a compact support mother wavelet and that the $\{2^{-j/2}\psi_0(2^{-j}t-k), j\in\mathbb{N}, k\in\mathbb{N}\}$ form an orthonormal basis. Let us define an alternative indexing for the dyadic intervals $\lambda\ (=\lambda_{(j,k)}) = \left[k2^j,(k+1)2^j\right)$, so that $d_\lambda \equiv d_X(j,k)$. Finally, let 3λ denote the union of the interval λ and its 2 adjacent dyadic intervals: $3\lambda_{j,k} = \lambda_{j,k-1}\cup\lambda_{j,k}\cup\lambda_{j,k+1}$. One has $|d_\lambda| \le \int |X(t)||\psi_{j,k}(t)|dt \le C \parallel \psi_0 \parallel_{L^1} \parallel X \parallel_{L^\infty}$ as soon as $X\in L^\infty$, and the quantities,

$$L_X(j,k) \equiv L_\lambda = \sup_{\lambda'\subset 3\lambda} |d_{\lambda'}| \tag{5}$$

are thus finite. They are referred to as *wavelet leaders*. In all generality, we now have that $X \in C^\alpha(t=2^jk)$ is equivalent to the fact that the wavelet leaders $L_X(j,k)$ decay as power laws of the scales $\sim 2^{jh}$ (up to a logarithmic correction), cf. [4].

•Multifractal Formalism. Let $S_L(q,j)$ denote the wavelet leader based structure functions and $\zeta_L(q)$ the corresponding scaling exponents:

$$S_L(q,j) = \frac{1}{n_j}\sum_{k=1}^{n_j} |L_X(j,k)|^q, \tag{6}$$

$$\zeta_L(q) = \liminf_{j\to 0}\left(\frac{\log_2 S_L(q,j)}{j}\right). \tag{7}$$

Arguments similar to those of the previous section yield

$$\zeta_L(q) = \inf_h\left(1+qh-D(h)\right). \tag{8}$$

This leads to a Wavelet Leader based Multifractal Formalism (hereafter **WLMF**), stated as:

$$D(h) = \inf_{q\neq 0}\left(1+qh-\zeta_L(q)\right). \tag{9}$$

It is mathematically proven that $\inf_{q\neq 0}(1+qh-\zeta_L(q))$ acts as a sharp upper bound for $D(h)$ for all functions or processes (on condition that they satisfy some mild uniform regularity conditions) [9]. This is in contrast with the WCMF for which much weaker mathematical results hold.

•WCMF vs WLMF. As opposed to the WCMF, the WLMF overcomes the two drawbacks mentioned above: It holds both for positive and negative qs and when the function or process under study embodies chirp-type singularities and enables to obtain in any case the whole range of the multifractal spectrum [9]. This will be illustrated in Section 4. Furthermore, tracking, for $q > 0$, the discrepancies between $\zeta_d(q)$ and $\zeta_L(q)$ or between their Legendre transforms is likely to enable us to detect whether the process under study contains chirp-like singularities or simply cusp-like singularities.
However, numerically deriving the $\zeta_L(q)$s requires the knowledge of wavelet coefficients on a wider range of scales than that necessary to get the $\zeta_d(q)$s: indeed, in order to be meaningful, the computation of L_λ at a given scale requires that of the wavelet coefficients d_λ over several scales below.

•Higher dimensions. For sake of simplicity, the presentation was proposed here for $1D$ processes or functions. However, the wavelet leader multifractal formalism can be straightforwardly extended to arbitrary higher dimensions $d\ge 1$, $\{X(t)\}_{t\in\mathbb{R}^d}$, simply by adapting the definitions of the wavelet coefficients and leaders as well as that of the Legendre transform, $\inf_q(d+hq-\zeta_L(q))$. An example in the case $d=2$ will be given in Section 4.

•Modulus Maxima of the Wavelet Transform. The wavelet leader approach is highly reminiscent of the Modulus Maxima of the Wavelet Transform technique (MMWT) initially introduced by S. Mallat [7] and developed in the context of multifractal analysis by Arneodo et al.[6]. It is based on the continuous wavelet transform (CWT) defined in the upper half plane $\{(a,t):\ a>0,\ t\in\mathbb{R}\}$:

$$T_X(a,t) = \frac{1}{a}\int X(u)\psi\left(\frac{t-u}{a}\right)du. \tag{10}$$

It consists in finding local maxima of the functions $t \rightarrow |T_X(a,t)|$ along time t for each given scale a and in chaining maxima along scales at given time positions. Structure functions are then based on this skeleton. This MMWT based multifractal formalism is known to solve the $q < 0$ issue [6, 10], it has also been shown to work on examples containing oscillating singularities [10]. The main difference between the leader and the modulus maxima approaches lies in the fact that in this latter method, the spacing between local maxima need not be of the order of magnitude of the scale a or even be regularly spaced. Therefore, the MMWT scaling exponents can differ from those obtained with leaders (see [4] where counterexamples are constructed). It follows that no mathematical result such as the one in Eq. (9) is expected to hold for the MMWT method.
On a more practical side, the computation of the MMWT involves that of a CWT plus maxima tracking and chaining operations. This results in a high computational cost. The WLMF can be implemented using the fast pyramidal algorithm underlying the DWT and has thus a significantly lower computational cost. Furthermore, as already mentioned, the wavelet leader approach can be easily theoretically and practically generalised to higher dimensions. This is far less the case for the MMWT method.

4. ILLUSTRATIONS AND EXAMPLES

•**Methodology.** The goal of this section is to illustrate the wavelet coefficient and wavelet leader multifractal formalisms, on 4 well-chosen reference examples. They consist of both deterministic functions and stochastic processes and contain either cusp-like or chirp-like singularities. Their multifractal spectra $D(h)$ are known theoretically and so are their $\zeta_L(q)$ through Eq. (7) or Eq. (8). The DWT is computed using least asymmetric compact support orthonormal Daubechies wavelets with $N = 3$ [7]. The scaling exponents and multifractal spectra obtained from the WCMF and WLMF will be denoted $\hat{\zeta}_d(q), \hat{\zeta}_L(q), \hat{D}_d(h)$ and $\hat{D}_L(h)$ respectively. The $\hat{\zeta}_d(q)$ and $\hat{\zeta}_L(q)$ are computed via linear regressions in log of structure functions versus log of scales diagrams, as thoroughly described in [11, 12], a standard Legendre transform algorithm is then used to get $\hat{D}_d(h)$ and $\hat{D}_L(h)$. The orders qs are chosen in a range that avoid the occurrence of the so-called *linearisation effect* that necessarily takes place in any scaling exponent estimation procedure (cf. [11, 12]). Results reported and compared in Fig. 1 and 2, are obtained on sample paths of duration $nbpoints$ and, for the case of stochastic processes, by averaging over $nbreal = 1000$ replications to ensure statistical convergence of the measures. Matlab routines implementing both the process synthesis procedures and the WCMF and WLMF analysis procedures were developped by ourseleves and are available upon request.

•**Fractional Brownian Motion (FBM).** Our first example consists of the FBM with self-similarity parameter H (see e.g., [5]). FBM is a stochastic process that is almost surely everywhere singular with constant Hölder exponent $h(t) = H, \forall t \in \mathbb{R}$. Therefore, its $D(h)$ reduces to a single point ($h = H, D = 1$) ($\zeta_L(q) = qH, \forall q \in \mathbb{R}$). Moreover, it contains only cusp-like singularities. Fig. 1 (first row) shows that the $\hat{\zeta}_d(q)$ depart from qH when $q < -1$, the corresponding Legendre transform yields the inaccurate $\hat{D}_d(h) = 1-h+H, h \in [H, H+1]$. The WCMF hence produces an incorrect determination of $D(h)$ while the WLMF results in a perfect one (here, $nbpoints = 2^{20}$).

•**Canonical Mandelbrot's Cascade (CMC).** CMC constitute the historical archetype for stochastic multifractal processes [3, 1]. They are built on an iterative *split/multiply* cascade scheme (see e.g., [5, 3, 1]). They contain only cusp-like singularities and are everywhere singular with bell-shaped $D(h)$. We chose here a fractionally integrated 2D cascade, (this is why its $D(h)$ ranges from 0 to 2) with *log-normal* characteristics, i.e., $D(h) = 2 - (h - H)^2/(2\sigma^2)$ and hence $\zeta_L(q) = qH - \sigma^2 q^2/2$. The 2D fractional integration (cf. [9]) of parameter 1 ensures that the analysed process contains singularities with $h > 0$ only. Again, the WCMF yields an incorrect determination of the scaling exponents for $q < -1$ and of $D(h)$ for its upper (or right) part while the WLMF produces a very relevant one (cf. Fig. 1, second row, here $nbpoints = 2^{10} \times 2^{10}$). Note that the MMWT would also produce a correct determination of $D(h)$ at the prices of significant conceptual difficulties (for the 2D extension) and of a huge increase of the computational cost (2D CWT plus local maxima detection and chaining).

•**Riemann's Function.** It is a deterministic function, defined as $X(t) = \sum_{n\in\mathbb{N}} \frac{sin(2\pi n^2 t)}{n^2}$, providing us with a reference example for chirp-like singularities. Its $D(h)$ consists of $D(h) = 4(h - 0.5), h \in [0.5, 0.75]$ plus an isolated point ($h = 1.5, D = 0$). This latter point constitute a signature of the existence of chirp-like singularities. Fig. 1, third row, shows that the WCMF and the WLMF coincide on the linear part of the spectrum but that the WCMF totally misses the isolated point while the WCMF correctly recovers it. Equivalently, this corresponds to the discrepancy between $\hat{\zeta}_d(q)$ and $\hat{\zeta}_L(q)$ for $q < 0$ (here, $nbpoints = 2^{20}$, the summation is practically truncated to 100 terms).

•**Random Wavelet Series (RWS).** RWS are multifractal stochastic processes that were very recently introduced in [13] and that are not based on an iterative multiplicative cascade. They are everywhere singular and contain chirp-like singularities (note that this cannot be reached with iterative multiplicative constructions). We chose here RWS such that their $D(h)$ consists of a piece of parabola together with a linear part (see solid line in Fig. 1, bottom row, left). The fact that $D(h)$ departs from the parabola when approaching its abruptly ending point at its maximum $D = 1$ constitutes a signature for the chirp-type nature of the singularities in RWS. Fig. 1, bottom row shows that the WCMF *creates* a *right* part of $D(h)$ that does not actually exists (again, it corresponds to the discrepancy between $\hat{\zeta}_d(q)$ and $\hat{\zeta}_L(q)$ for $q < 0$) while $\hat{D}_L(h)$ satisfactorily recovers the theoretical $D(h)$. Notably, the ending point of $D(h)$ is clearly captured by $\hat{D}_L(h)$ while totally missed by $\hat{D}_d(h)$ that continues toward its right (here, $nbpoints = 2^{15}$).
Fig. 2 shows an important difference between the CMC example (cusp-type singularities) and the RWS one (chirp-type singularities). For CMC, the left parts of $\hat{D}_L(h)$ and $\hat{D}_d(h)$ (i.e., the parts obtained from the positive qs of the $\zeta_L(q)$ and $\zeta_d(q)$ via the Legendre transform) are close. For RWS, they significantly differ, in particular $\hat{D}_d(h)$ clearly misses the linear part of $D(h)$ and the end point at $D = 1$ while $\hat{D}_L(h)$ does not. Such a discrepancy between the left parts of $\hat{D}_L(h)$ and $\hat{D}_d(h)$ may therefore be used to assess the presence of chirp-type singularities in the analysed data.

5. CONCLUSION AND PERSPECTIVE

We gave here the definition of a new multifractal formalism based on the leaders of the wavelet coefficients of a discrete wavelet transform. It enables a precise determination of the multifractal spectrum over its full range for very large classes of processes, including those containing oscillating (or chirp-like) singularities. It has a very low computational cost and can be easily implemented for processes in any dimension. Compared to other previously existing multifractal formalisms, it benefits from more accurate theoretical results and brings a real enhacement from a practical point of view. Its ability to detect the existence of chirp-like singularities is under current investigation. The statistical performance of estimation procedures, based on wavelet leaders, for the scaling exponents and multifractal spectra of stochastic processes will be further studied. The occurence of the linearisation effect [11, 12] will specifically be investigated. Applications to the analysis of data coming from hydrodynamic turbulence and internet traffic are considered. Matlab routines implementing practically this wavelet leader multifractal formalism are available upon request.

6. REFERENCES

[1] B. B. Mandelbrot, "Intermittent turbulence in self similar cascades: Divergence of high moments and dimension of the carrier," *J. Fluid. Mech.*, vol. 62, pp. 331, 1974.

[2] G. Parisi and U. Frisch, "On the singularity structure of fully developed turbulence, *appendix to* fully developed turbulence and intermittency by U. Frisch," in *Proc. Int. Summer school Phys. Enrico Fermi, North Holland*, 1985.

[3] A.M. Yaglom, "Effect of fluctuations in energy dissipation rate on the form of turbulence characteristics in the inertial subrange," *Dokl. Akad. Nauk. SSR*, vol. 166, pp. 49–52, 1966.

[4] S. Jaffard, "Multifractal formalism for functions," *S.I.A.M. J. Math. Anal.*, vol. 28(4), pp. 944–998, 1997.

[5] R. H. Riedi, "Multifractal processes," *in: "Theory and applications of long range dependence ", eds. Doukhan, Oppenheim and Taqqu*, pp. 625–716, 2003.

[6] A. Arneodo, E. Bacry, and J.F. Muzy, "The thermodynamics of fractals revisited with wavelets," *Physica A*, vol. 213, pp. 232–275, 1995.

[7] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, San Diego, CA, 1998.

[8] S. Jaffard, "Exposants de hölder en des points donnés et coefficients d'ondelettes," *C. R. Acad. Sci. Sér. I Math.*, vol. 308, pp. 79–81, 1989.

[9] S. Jaffard, "Wavelet tecnhiques in multifractal analysis," *to appear in: "Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot", eds. M. Lapidus et M. van Frankenhuysen, Proc. of Symp. in Pure Mathematics*, 2004.

[10] A. Arneodo, E. Bacry, S. Jaffard, and J.F. Muzy, "Oscillating singularities on cantor sets: a grand-canonical multifractal formalism," *J. Stat. Phys.*, vol. 87(1–2), pp. 179–209, 1997.

[11] B. Lashermes, P. Abry, and P. Chainais, "New insights on the estimation of scaling exponents," *Int. J. of Wavelets, Multiresolution and Information Processing, to appear*, 2004.

[12] B. Lashermes, P. Abry, and P. Chainais, "Scaling exponents estimation for multiscaling processes," in *ICASSP 2004, Montréal, Canada*, 2004.

[13] J.-M. Aubry and S. Jaffard, "Random wavelet series," *Comm. Math. Phys.*, vol. 227, pp. 483–514, 2002.

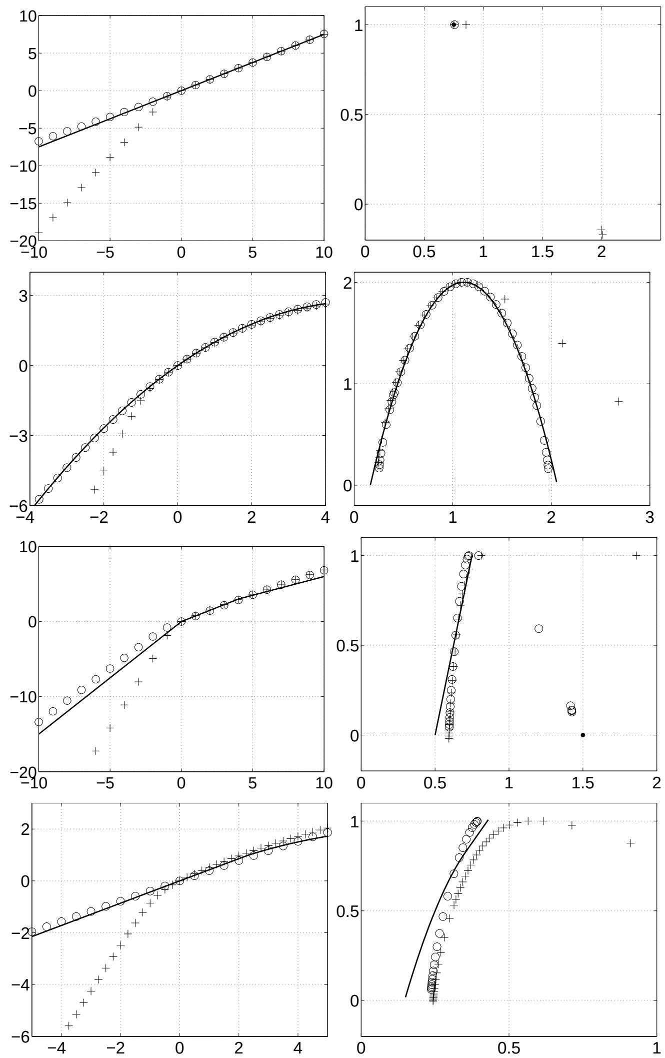

Fig. 1. **Illustration of the MultiFractal Formalisms.** From top to bottom, Fractional Brownian Motion (FBM), Canonical Mandelbrot's Cascade (CMC), Riemann's function, Random Wavelet Series (RWS); left column: scaling exponents $\zeta_L(q), \hat{\zeta}_d(q), \hat{\zeta}_L(q)$, right column: multifractal spectra $D(h), \hat{D}_d(h), \hat{D}_L(h)$; solid line and isolated points: theory, (+) and (o) wavelet coefficient and wavelet leader based multifractal spectra, respectively.

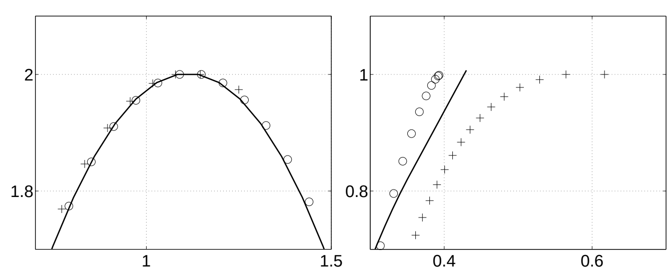

Fig. 2. **Cusp-like vs Chirp-like singularities.** Left, for a CMC (with cusp-like singularities only), both $\hat{D}_d(h)$ and $\hat{D}_L(h)$ are equally satisfactorily close to $D(h)$. Right, for a RWS (with chirp-like singularities), significant discrepancies between $\hat{D}_d(h)$ and $\hat{D}_L(h)$ can be observed (with $\hat{D}_L$ much closer to D) and can be interpreted as a way to detect chirp-type singularities.