From stationarity to self-similarity, and back

(Variations on the Lamperti transformation)

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A brief historical sketch

Lamperti, 1962: seminal result on self-similar processes, often quoted (e.g., in Vervaat, 1987 or Samorodnitsky & Taqqu, 1994), but rarely discussed *per se* (until Burnecki *et al.*, 1997).

Gray & Zhang, 1988; Yazici & Kashyap, 1995–1997; Vidács & Virtamo, 1999: independent re-introductions of Lamperti's warping idea.

Nuzman & Poor, 1999–2000: systematic use of the Lamperti transform for processing self-similar processes.

Borgnat *et al.*, 2001: extension and application to stochastic discrete scale invariance.

Outline of the talk

1. Stationarity and self-similarity
2. **The Lamperti transformation**: Definition, consequences, examples and applications
3. A variation related to *stochastic discrete scale invariance*
4. Concluding remarks

Shifts and dilations

Definition 1 *Given* $\tau \in \mathbb{R}$*, the shift operator* $\mathcal{S}_\tau$ *operates on processes* $\{Y(t), t \in \mathbb{R}\}$ *according to:*

$$(\mathcal{S}_\tau Y)(t) := Y(t+\tau).$$

Definition 2 *Given* $H > 0$ *and* $\lambda > 0$*, the renormalized dilation operator* $\mathcal{D}_{H,\lambda}$ *operates on processes* $\{X(t), t > 0\}$ *according to:*

$$(\mathcal{D}_{H,\lambda} X)(t) := \lambda^{-H} X(\lambda t).$$

Stationarity and self-similarity

Definition 3 *A process* $\{Y(t), t \in \mathbb{R}\}$ *is said to be stationary if*

$$\{(\mathcal{S}_\tau Y)(t), t \in \mathbb{R}\} \stackrel{d}{=} \{Y(t), t \in \mathbb{R}\}$$

for any $\tau \in \mathbb{R}$.

Definition 4 *A process* $\{X(t), t > 0\}$ *is said to be self-similar of index* H *(or "*H*-ss") if*

$$\{(\mathcal{D}_{H,\lambda} X)(t), t > 0\} \stackrel{d}{=} \{X(t), t > 0\}$$

for any $\lambda > 0$.

The Lamperti transformation

Definition 5 *Given some number* $H > 0$, *the Lamperti transform* $\mathcal{L}_H$ *operates on processes* $\{Y(t), t \in \mathbb{R}\}$ *according to:*

$$(\mathcal{L}_H Y)(t) := t^H Y(\log t), t > 0,$$

whereas the corresponding inverse Lamperti transform $\mathcal{L}_H^{-1}$ *operates on processes* $\{X(t), t > 0\}$ *according to:*

$$(\mathcal{L}_H^{-1} X)(t) := e^{-Ht} X(e^t), t \in \mathbb{R}.$$

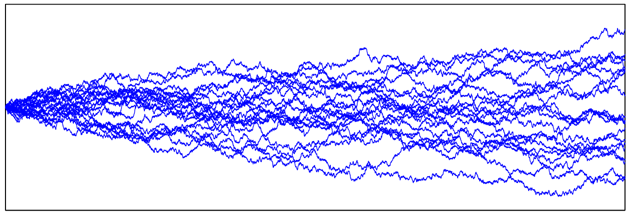
Lamperti's theorem

Lemma 1 *The Lamperti transform guarantees an equivalence between shifts and renormalized dilations in the sense that, for any* $\lambda > 0$:

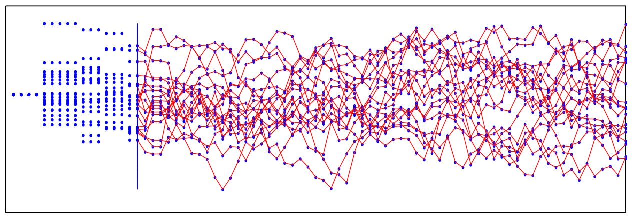
$$\mathcal{L}_H^{-1} \mathcal{D}_{H,\lambda} \mathcal{L}_H = \mathcal{S}_{\log \lambda}.$$

Theorem 1 *If* $\{Y(t), t \in \mathbb{R}\}$ *is stationary, its Lamperti transform* $\{(\mathcal{L}_H Y)(t), t > 0\}$ *is* H*-ss. Conversely, if* $\{X(t), t > 0\}$ *is* H*-ss, its inverse Lamperti transform* $\{(\mathcal{L}_H^{-1} X)(t), t \in \mathbb{R}\}$ *is stationary.*

Brownian motion



Lamperti-stationarized Brownian motion



Consequences — 1.

Covariances — Statistical properties of self-similar processes can be inferred from those of their Lamperti counterparts, and vice-versa. Introducing the notation $\mathbf{R}_X(t,s) := \mathbb{E}X(t)X(s)$, we have

$$\mathbf{R}_{\mathcal{L}_H^{-1}X}(t,s) = e^{-H(t+s)}\,\mathbf{R}_X(e^t, e^s)$$

$$\mathbf{R}_{\mathcal{L}_H Y}(t,s) = (ts)^H\,\mathbf{R}_Y(\log t, \log s)$$

Stationarity — In the case where $\{Y(t), t \in \mathbb{R}\}$ is stationary, $\mathbf{R}_Y(t,s) = \gamma_Y(t-s)$ and

$$\mathbf{R}_{\mathcal{L}_H Y}(t,s) = (ts)^H\,\gamma_Y(\log(t/s)).$$

Consequences — 2.

Corollary 1 *Any second-order H-ss process $\{X(t), t > 0\}$ has necessarily a covariance function of the form*

$$\mathbf{R}_X(t,s) = (ts)^H\,\mathbf{c}_H(t/s)$$

for any $t, s > 0$, with $\mathbf{c}_H(\exp(.))$ a non-negative definite function.

Corollary 2 *Given a second-order H-ss process $\{X(t), t > 0\}$, the spectrum of its stationary counterpart $\{(\mathcal{L}_H^{-1}X)(t), t \in \mathbb{R}\}$ is the Mellin transform of the scale-covariant function $\mathbf{c}_H(.)$:*

$$\mathbf{\Gamma}_{\mathcal{L}_H^{-1}X}(f) = (\mathcal{M}\mathbf{c}_H)(i2\pi f),$$

with

$$(\mathcal{M}X)(s) := \int_0^{+\infty} X(t)\,t^{-s-1}\,dt.$$

Scale-covariant systems — 1.

Definition 6 *A linear operator $\mathcal{G}$ is said to be scale-covariant if it commutes with any renormalized dilation, i.e., if*

$$\mathcal{G}\mathcal{D}_{H,\lambda} = \mathcal{D}_{H,\lambda}\mathcal{G}$$

for any $H > 0$ and any $\lambda > 0$.

Proposition 1 *If an operator $\mathcal{G}$ is scale-covariant, it necessarily acts on processes $\{X(t), t > 0\}$ as a multiplicative convolution:*

$$(\mathcal{G}X)(t) = \int_0^{+\infty} g(t/s)\,X(s)\,ds/s.$$

Corollary 3 *Scale-covariant operators preserve self-similarity.*

Scale-covariant systems — 2.

Corollary 4 *The Lamperti transform maps linear filters onto scale-covariant systems.*

Proposition 2 *Any H-ss process $\{X(t), t > 0\}$ can be represented as the output of a linear scale-covariant system of impulse response $g(.)$:*

$$X(t) = \int_0^{+\infty} g(t/s)\,dV(s)/s,$$

with $\mathbb{E}dV(t)dV(s) = \sigma^2\,t^{2H+1}\,\delta(t-s)\,dt\,ds$.

Corollary 5 *The spectrum of $\{(\mathcal{L}_H^{-1}X)(t), t \in \mathbb{R}\}$, stationary counterpart of the H-ss process $\{X(t), t > 0\}$, is given by*

$$\mathbf{\Gamma}_{\mathcal{L}_H^{-1}X}(f) = \sigma^2\,|(\mathcal{M}g)(H + i2\pi f)|^2.$$

Applications

From stationarity to self-similarity — Classes and models of self-similar processes can be obtained by "lampertizing" corresponding classes and models of stationary processes.

From self-similarity to stationarity — Conversely, "delampertizing" self-similar processes can render their processing easier, by making them amenable to classical tools aimed at stationary processes.

A general framework

A transformation $\mathcal{T}$ on an H-ss process $\{X(t), t > 0\}$ can be equivalently achieved as $\mathcal{T} = \mathcal{L}_H \tilde{\mathcal{T}} \mathcal{L}_H^{-1}$, according to the commutative diagram:

$$\begin{array}{ccccc} X(t) & \stackrel{?}{\longrightarrow} & (\mathcal{T}X)(t) & & (self-similar\,world) \\ \Big\downarrow \text{inverse Lamperti} & & \Big\uparrow \text{Lamperti} & & \\ (\mathcal{L}_H^{-1}X)(t) & \longrightarrow & (\tilde{\mathcal{T}}\mathcal{L}_H^{-1}X)(t) & & (stationary\,world) \end{array}$$

Example 1. — Tones and chirps

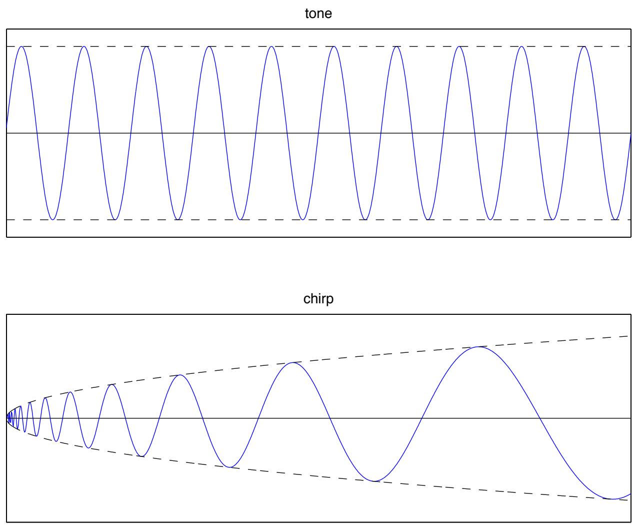
The (stationary) random phase "tone"

$$Y_0(t) := a \cos(2\pi f_0 t + \varphi), t \in \mathbb{R},$$

with $a, f_0 > 0$ and $\varphi \in \mathcal{U}(0, 2\pi)$, is "lampertized" into the (self-similar) random phase "chirp"

$$X_0(t) := (\mathcal{L}_H Y_0)(t) = a\, t^H \cos(2\pi f_0 \log t + \varphi), t > 0.$$

Remark — $X_0(t) = \text{Re}\{a\, e^{i\varphi}\, m_s(t)\}$, with $s = H + i2\pi f_0$ and $m_s(t) := t^s$ the basic building block of the Mellin transform.



Example 2. — From fBm to gOU

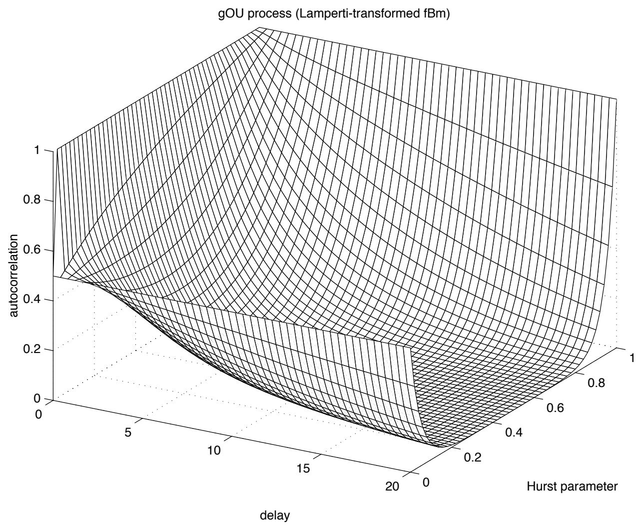
H-ss processes $\{X(t), t > 0\}$ with stationary increments (or, "H-sssi" processes) have a covariance function of the form

$$\mathbf{R}_X(t,s) = \frac{\sigma^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

H-sssi + Gaussian $\Rightarrow$ *fractional Brownian motion* (fBm) $B_H(t)$.

The inverse Lamperti transform $\{Y_H(t) := (\mathcal{L}_H^{-1} B_H)(t), t \in \mathbb{R}\}$ is a *generalized Ornstein-Uhlenbeck* (gOU) process of (stationary) covariance function

$$\gamma_{Y_H}(\tau) = \sigma^2\left(\cosh(H|\tau|) - 2^{2H-1}\sinh^{2H}(|\tau|/2)\right).$$



Example 2. — From fBm to gOU (cont'd)

Bm & OU — If $H = 1/2$, $\{B_{1/2}(t), t > 0\}$ is the ordinary Brownian motion of covariance function $\mathbf{R}_{B_{1/2}}(t,s) = \sigma^2 \min(t,s)$, and its Lamperti image $\{Y_{1/2}(t), t \in \mathbb{R}\}$ is the ordinary OU process of (stationary) covariance function:

$$\gamma_{Y_{1/2}}(\tau) = \sigma^2 e^{-|\tau|/2}.$$

Long-range vs. short-range dependence — The (stationary) increment process of fBm (or fractional Gaussian noise, fGn) is long-range dependent if $1/2 < H < 1$, whereas the gOU process $Y_H(t)$ is short-range dependent for any $H \in (0,1)$, since

$$\gamma_{Y_H}(\tau) \propto \sigma^2 e^{-\min(H,1-H)\tau}$$

when $\tau \to \infty$.

Example 2. — From fBm to gOU, and back

The spectrum density of gOU processes reads

$$\Gamma_{Y_H}(f) = \frac{\sigma^2}{H^2 + 4\pi^2 f^2}\left|\frac{\Gamma((1/2) + i2\pi f)}{\Gamma(H + i2\pi f)}\right|^2,$$

and it can be factorized so that $\Gamma_{Y_H}(f) = |\Phi_+(f)|^2$, with $\Phi_+(f)$ the transfer function of a causal filter.

Whitening — Whitening gOU is equivalent to transforming fBm into Bm $\Rightarrow$ innovations representations for fBm.

Prediction — Observing a self-similar process on some finite interval $[0, T]$ is equivalent to observing its (stationary) Lamperti counterpart on the real half-line $[0, \infty)$ $\Rightarrow$ linear prediction of fBm from Wiener-type prediction of gOU (Nuzman & Poor, 2000).

Example 3. — From OU to ssOU

The OU process $\{Y_{1/2}(t), t \in \mathbb{R}\}$ is solution of the Langevin equation

$$dY(t) + \alpha\, Y(t)\, dt = dB(t),$$

with $\alpha = 1/2$. Given $\alpha > 0$, the general solution is

$$Y_\alpha(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)}\, dB(s),$$

whose Lamperti transform (or, ssOU process)

$$X_{\alpha,H}(t) := (\mathcal{L}_H Y_\alpha)(t) = t^{H-\alpha} \int_0^t s^\alpha\, dB(\log s), t > 0,$$

is solution of

$$t\, dX(t) + (\alpha - H)\, X(t)\, dt = dV(t),$$

with $\mathbb{E} dV(t) dV(s) = \sigma^2\, t^{2H+1}\, \delta(t-s)\, dt\, ds$.

Example 3. — From OU to ssOU (cont'd)

Scale-covariant representation — Noting that $dB(\log t)$ is covariance-equivalent to $t^{-1/2} dB(t)$, we have

$$X_{\alpha,H}(t) = \int_0^{+\infty} \underbrace{[(t/s)^{H-\alpha}\, u(t/s-1)]}_{g(t/s)}\, \underbrace{[s^{H+1/2}\, dB(s)]}_{dV(s)} /s,$$

with $u(.)$ the unit-step function.

ssOU covariance function — The (nonstationary and H-ss) ssOU process generalizes Bm according to

$$\mathbf{R}_{X_{\alpha,H}}(t,s) = \sigma^2\, (\min(t,s))^{H+\alpha}\, (\max(t,s))^{H-\alpha},$$

leading to $\mathbf{R}_{X_{H,H}}(t,s) = (\min(t,s))^{2H}$, and $X_{1/2,H}(t) = t^{H-1/2}\, B(t)$.

Example 4. — From ARMA to EC

Stochastic differential equation — Ignoring non-differentiability issues, the Langevin equation of OU processes can be written as

$$\frac{dY}{dt}(t) + \alpha\, Y(t) = W(t),$$

with $W(t)$ "white noise" such that $\mathbb{E} W(t)\, W(s) = \sigma^2\, \delta(t-s)\, dt\, ds$.

ARMA — This can be generalized to ARMA(p,q) processes of the form

$$\sum_{n=0}^{p} \alpha_n\, Y^{(n)}(t) = \sum_{n=0}^{q} \beta_n\, W^{(n)}(t),$$

with the notation $Y^{(n)}(t) := (d^n Y/dt^n)(t)$.

Example 4. — From ARMA to EC (cont'd)

Proposition 3 *Stationary ARMA processes have an H-ss Lamperti counterpart, referred to as Euler-Cauchy processes, which is solution of an equation of the form*

$$\sum_{n=0}^{p} \alpha'_n\, t^n\, X^{(n)}(t) = \sum_{n=0}^{q} \beta'_n\, t^n\, \tilde{W}^{(n)}(t), t > 0,$$

with $\tilde{W}(t) = t^{H+1/2}\, W(t)$.

Variations on Lamperti's theorem

Applying the Lamperti transformation to weakened forms of stationarity leads to weakened forms of self-similarity.

Multiplicative harmonizability — Harmonizable nonstationary processes $\{Y(t), t \in \mathbb{R}\}$ have a Lamperti counterpart which admits the Mellin representation

$$(\mathcal{L}_H Y)(t) = \int_{-\infty}^{+\infty} t^{H+i2\pi f}\, d\xi(f), t > 0,$$

with $\mathbb{E} d\xi(f)\overline{d\xi(\nu)} \neq 0$ for $f \neq \nu$.

Example — Spectral increments may be periodically correlated, i.e., $\mathbb{E} d\xi(f)\overline{d\xi(\nu)} \neq 0$ for $f = \nu + k/T, k \in \mathbb{Z}$.

Cyclostationarity and DSI

Definition 7 *A process $\{Y(t), t \in \mathbb{R}\}$ is said to be periodically correlated (PC) of period T_0 (or "T_0-cyclostationary") if*

$$\{(\mathcal{S}_{T_0} Y)(t), t \in \mathbb{R}\} \stackrel{d}{=} \{Y(t), t \in \mathbb{R}\}.$$

Definition 8 *A process $\{X(t), t > 0\}$ is said to possess a discrete scale invariance of index H and of scaling factor $\lambda_0 > 0$ (or to be "(H, λ_0)-DSI") if*

$$\{(\mathcal{D}_{H,\lambda_0} X)(t), t > 0\} \stackrel{d}{=} \{X(t), t > 0\}.$$

It follows from these definitions that T_0-cyclostationary processes are also T-cyclostationary for any $T = kT_0, k \in \mathbb{Z}$, and that (H, λ_0)-DSI processes are also (H, λ)-DSI for any $\lambda = \lambda_0^k, k \in \mathbb{Z}$.

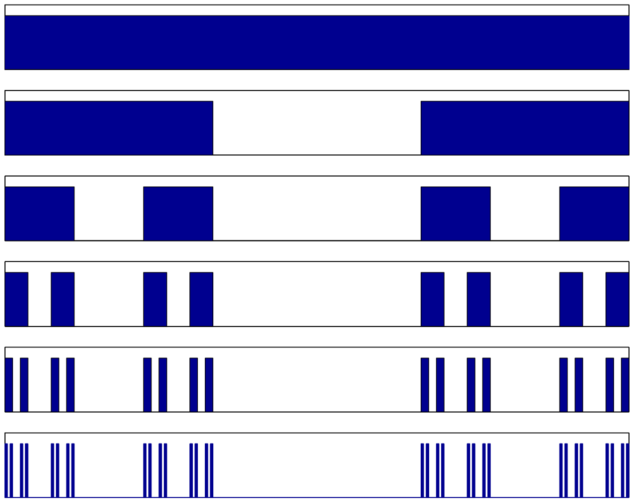
Discrete scale invariance

Deterministic DSI — The concept of DSI has been introduced in a *deterministic* sense in Saleur & Sornette, 1996.

Ubiquity — DSI has been *theoretically* shown to naturally occur in many critical systems, and it has been *experimentally* evidenced in a number of situations: earthquakes, financial crashes, etc.

Evidence — Power laws attached to usual scale invariance are decorated with *log-periodic oscillations*.

Example — The simplest example is given by the *middle-third Cantor set*, which is *deterministically* $(0, 1/3)$-DSI.



Cyclostationarity, DSI and Lamperti

Theorem 2 *If $\{Y(t), t \in \mathbb{R}\}$ is T_0-cyclostationary, then its Lamperti transform $\{(\mathcal{L}_H Y)(t), t > 0\}$ is (H, e^{T_0})-DSI. Conversely, if $\{X(t), t > 0\}$ is (H, e^{T_0})-DSI, its inverse Lamperti transform $\{(\mathcal{L}_H^{-1} X)(t), t \in \mathbb{R}\}$ is T_0-cyclostationary.*

Synthesis and *analysis* of DSI processes can therefore be achieved:

- either by "lampertizing" cyclostationary tools (PC world $\rightarrow$ DSI world),
- or by "delampertizing" self-similar tools (DSI world $\rightarrow$ PC world).

DSI and multiplicative harmonizability

Covariance — If a process $\{X(t), t > 0\}$ is (H, λ)-DSI, its covariance function $\mathbf{R}_X(t, s)$ can be expanded on a *Mellin* basis:

$$\mathbf{R}_X(t, kt) = k^H \sum_{n=-\infty}^{\infty} C_n(k)\, t^{2H + i2\pi n/\log\lambda}$$

Spectral distribution function — Spectral increments of (H, λ)-DSI processes are such that $\mathbb{E} d\xi(f)\overline{d\xi(\nu)} = \Phi_X(f, \nu)\, df\, d\nu$, with

$$\Phi_X(f, \nu) = \sum_{n=-\infty}^{\infty} (\mathcal{M}C_n)(f)\, \delta(f - \nu - n/\log\lambda).$$

Example 1. Weierstrass

Given i.i.d. phases $\varphi_n \in \mathcal{U}(0, 2\pi)$, the Weierstrass-like functions

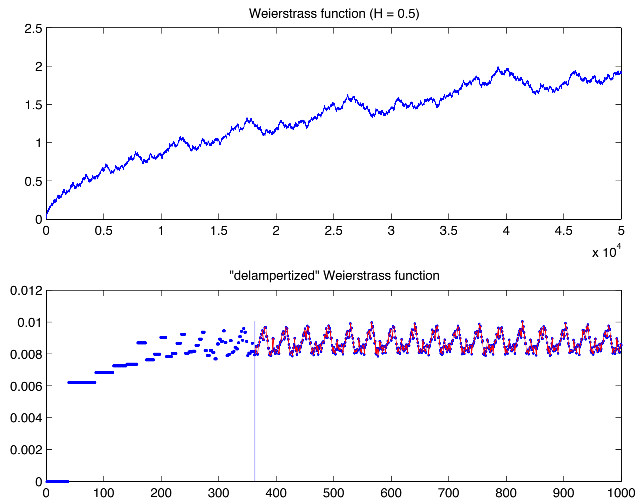
$$W_{H,\lambda}(t) = \sum_{n=-\infty}^{\infty} \lambda^{-Hn}\, g(\lambda^n t)\, e^{i\varphi_n},$$

define (H, λ)-DSI processes whose (inverse) Lamperti image is $\log\lambda$-cyclostationary, according to:

$$(\mathcal{L}_H^{-1} W_{H,\lambda})(t) = \sum_{n=-\infty}^{\infty} (\mathcal{L}_H^{-1} g)(t + n\log\lambda)\, e^{i\varphi_n}.$$

Weierstrass-Mandelbrot — In the specific case $g(t) = 1 - \exp it$, the process $W_{H,\lambda}(t)$ has furthermore stationary increments, and

$$(\mathcal{L}_H^{-1} g)(t) = e^{-Ht}\,(1 - \exp ie^t).$$

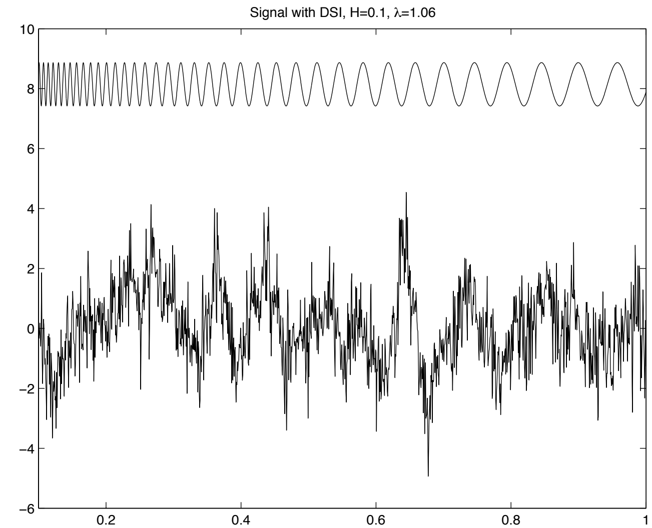


Example 2. DSI sequences

From EC to DSI — Continuous-time: DSI can be obtained by introducing log-periodic time-varying coefficients in an EC model.

Discrete-time: discretize EC by integration of its evolution + log-periodic coefficients.

Another model — Fractional difference operator + AR with log-periodic time-varying coefficients.



Analysis 1. From DSI to PC

Theory — "Delampertizing" a DSI process turns it into a PC process amenable to classical cyclostationary tools, such as *cyclic spectrum analysis*.

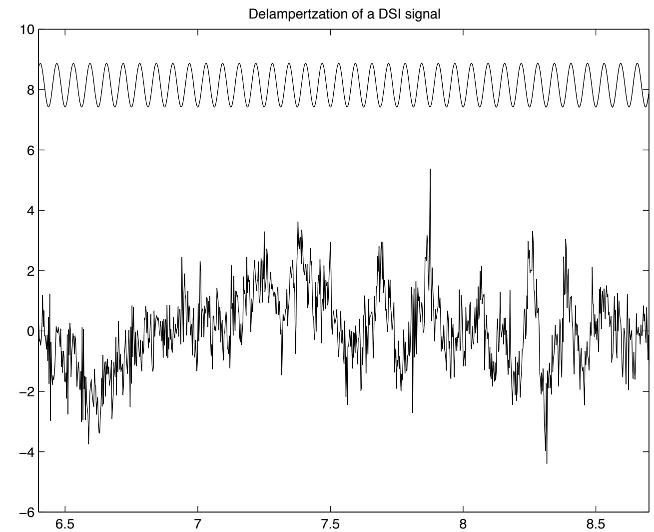
Practice — Effective analysis decomposes as:

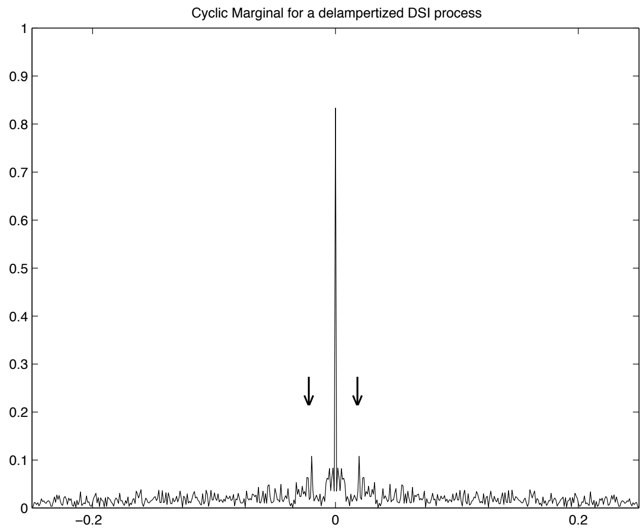
1. *geometrical sampling* of the data (given or interpolated)

2. *inverse Lamperti transform* (with H guessed or estimated)

3. *cyclic periodogram* (function of f and cyclic frequency ν)

4. *marginalization* in cyclic frequency ν





Analysis 2. From PC to DSI

A reversed perspective — Another way of analyzing DSI processes would be to operate *in the data space* directly, by using "lampertized" cyclostationary tools.

Mellin strikes back — In this respect, a central tool is the (Mellin-based) *scale-invariant Wigner spectrum* (F., 1990) :

$$\underline{\mathbf{W}}_X(t,\sigma) := \int_0^{+\infty} \mathbf{R}_X(t\tau^{+1/2}, t\tau^{-1/2})\, \tau^{-i2\pi\sigma-1}\, d\tau,$$

from which $\Phi_X(f,\nu)$ can be recovered and marginalized.

Conclusion

- **Lamperti**, from stationarity to self-similarity, and back
- A framework for stochastic DSI
- *Applications*? (DLA)

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