"Drawing sounds, listening to images"
The art of time-frequency analysis

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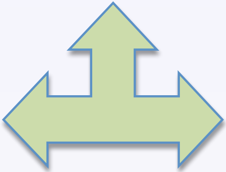
a general framework

« physics »

(laws of Nature, real world applications)

« mathematics »

(models, proofs)



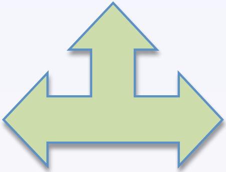
« computer science »

(algorithms)

the Fourier example

« physics »

(heat equation)



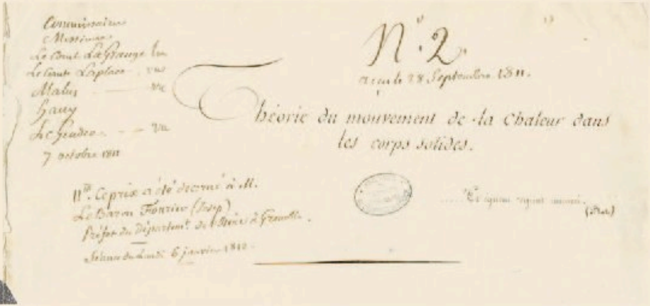
« mathematics »

(harmonic analysis)

« computer science »

(Fast Fourier Transform)

Fourier (1811)



2 quotations and 2 references

- 2 **quotations** from Fourier
  - “*The profound study of nature is the most fertile source of mathematical discoveries.*”
  - “*The [proposed] method does not leave anything vague and indefinite in its solutions; it drives them to their ultimate numerical applications, necessary condition for any research, and without which we would only end up with useless transformations.*”
- 2 **references** on Fourier (in French)
  - Jean Dhombres et Jean-Bernard Robert, *Fourier, créateur de la physique mathématique*, Belin 1999.
  - Jean-Pierre Kahane, “Fourier, un mathématicien inspiré par la physique”, *Images des Mathématiques*, CNRS 2009.

analysis/synthesis

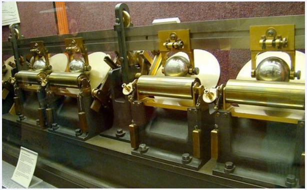
Fourier decomposition based on $e_f(t) := \exp\{i2\pi ft\}$

$$x(t) \to X(f) = \langle x, e_f \rangle, \; s.t. \; x(t) = \int \langle x, e_f \rangle \, e_f(t) \, df$$

- **mathematics**: "any" waveform is made of the superimposition of a (possibly infinite) number of harmonic modes which are *everlasting, undamped* and with a *fixed frequency*
- **physics**: keyrole played by the concept of *frequency* in relation with vibrations and waves
- **computer science**: further development of efficient algorithms (FFT = 1965) which favoured its *practical use*

cycles

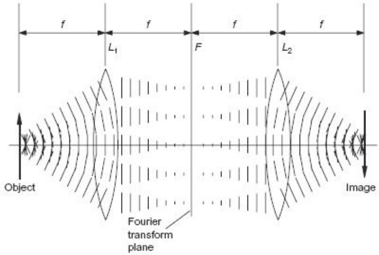
- **physics "of Nature"**, from *macrophysics* (celestial mechanics, tides, . . .) to *microphysics* (Quantum Mechanics)
- **physics "of engineers"** (rotating machines, modal analysis, surveillance of vibrating structures, . . .)

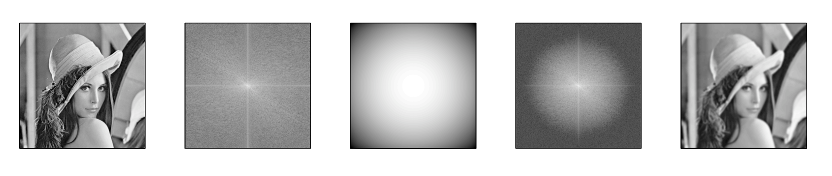


W. Thomson (Lord Kelvin), 1876-1878

lenses

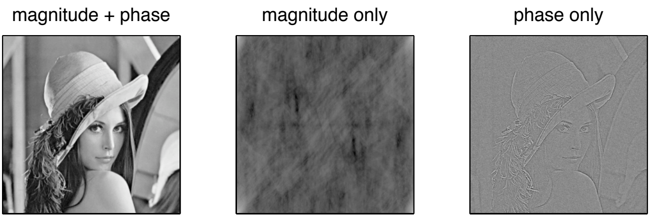
- diffracted field
- Fourier image in the focal plane
- spatial filtering



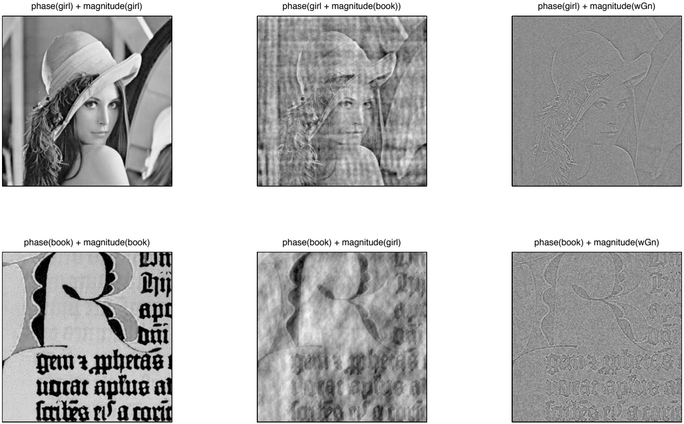


magnitude and phase

Fourier reconstruction with…



magnitude and phase



tones

- eigenmodes of cavities
- Helmholtz resonators
- inner ear (cochlea)

Appareil de Kœnig pour l'analyse du timbre des sons. Document Laboratoire de Mécanique et d'Acoustique, CNRS, Marseille.

beyond Fourier

back to the definition

$$x(t) \to X(f) = \langle x, e_f \rangle, \; s.t. \; x(t) = \int \langle x, e_f \rangle \, e_f(t) \, df,$$

$$e_f(t) := \exp\{i2\pi ft\}$$

Fourier

1. spectrum without any time dependence
2. localization on fixed frequencies
3. harmonic modes

beyond Fourier

back to the definition

$$x(t) \rightarrow X(f) = \langle x, e_f \rangle, \; s.t. \; x(t) = \int \langle x, e_f \rangle \, e_f(t) \, df,$$

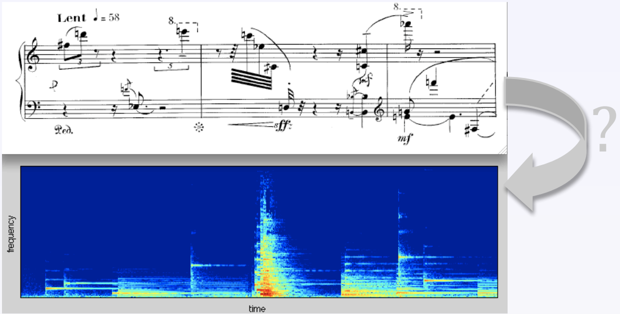
$$e_f(t) := \exp\{i2\pi ft\}$$

Fourier "+"

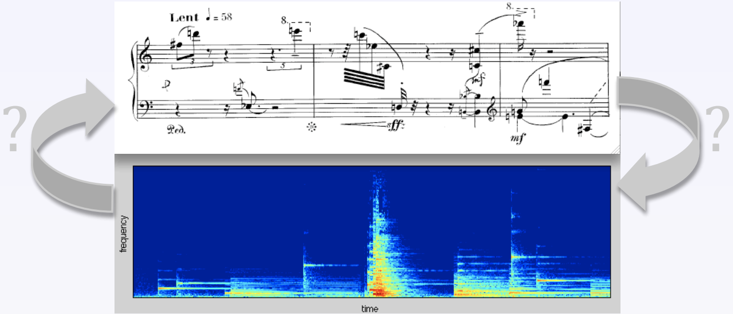
1. spectrum **with** time dependence
2. localization on **varying** frequencies
3. **non**-harmonic modes

music notation

from sounds to images. . .



. . . and back



mathematical notes

Issue

"localized modes" $\Rightarrow$ *switch to a 2-parameter transformation group that includes time*

$$x(t) \rightarrow T(t,\lambda) = \langle x, h_{t,\lambda}\rangle,\ s.t.\ x(t) = \iint \langle x, h_{s,\lambda}\rangle\, h_{s,\lambda}(t)\, d\mu(s,\lambda)$$

1. time-frequency: $\lambda = f$ and $h_{s,f}(t) = h(t-s)\, e_f(t)$

   $\rightarrow$ **short-time Fourier transform**

2. time-scale: $\lambda = a$ and $h_{s,a}(t) = |a|^{-1/2}\, h((s-t)/a)$

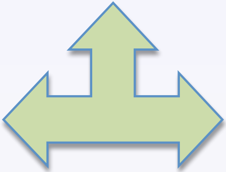
   $\rightarrow$ **wavelet transform**

the wavelet connection ($\sim$ 1980-90)

« physics »

vibroseismics for oil exploration

(Morlet)



« mathematics »

CWT, MRA, bases, etc.

(Grossmann, Meyer, Daubechies)

« computer science »

filter banks, fast algorithms

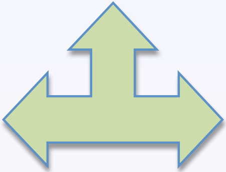
(Mallat, Cohen, Vetterli)

exclusion principles

« physics »

joint measurement of position and momentum

(Heisenberg, 1925)



« mathematics »

any Fourier pair of variables

(Weyl, 1927)

« computer science »

time and frequency

(Gabor, 1946 + ...)

classical formulation

"*A fast jig on the lowest register of an organ is in fact not so much bad music but no music at all.*" (N. Wiener in *I am a mathematician*)

Localization trade-off

based on a second order (variance-type) measure:
$\Delta t_x = (\int t^2 \, |x(t)|^2 \, dt)^{1/2}$ *and* $\Delta f_x = (\int f^2 \, |X(f)|^2 \, df)^{1/2} \Rightarrow$

$$\Delta t_x \, \Delta f_x \geq \frac{\|x\|}{4\pi} \; (> 0)$$

- no perfect pointwise localization
- variations: same limitation with other spreading measures, e.g., entropy (Hirschman, 1957)
- common denominator: minimum achieved with **Gaussians**

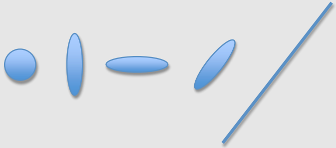
extension

no pointwise localization does not mean no localization

Stronger uncertainty relation (Schrödinger, 1935)

$$\Delta t_x \, \Delta f_x \geq \frac{\|x\|}{4\pi} \sqrt{1 + 16\pi^2 \left(\int t \, (\partial_t \arg x(t)) \, |x(t)|^2 \, dt \right)^2}$$

*bound achieved for **"squeezed states"*** $\{\exp(\alpha t^2 + \beta t + \gamma)\}$, *with **linear "chirps"** as a limit when* $\mathrm{Re}\{\beta\} = 0$ *and* $\mathrm{Re}\{\alpha\} \to 0_-$



local methods and localization

- **Back to the short-time FT** — One defines the **local** quantity

$$F_x^{(h)}(t,f) = \int x(s)\,\overline{h(s-t)}\,e^{-i2\pi fs}\,ds,$$

where $h(t)$ is some short-time observation window.

- **Measurement** — The representation results from an interaction between the signal and a **measurement device** (the window $h(t)$).
- **Trade-off** — A short window favors the "resolution" in time at the expense of the "resolution" in frequency, and vice-versa.

adaptation

- **Chirps —** Adaptation to **pulses** if $h(t) \to \delta(t)$ and to **tones** if $h(t) \to 1 \Rightarrow$ adapting the analysis to arbitrary **chirps** suggests to make $h(t)$ **(locally) depending on the signal.**
- **Linear chirp —** In the linear case $f_x(t) = f_0 + \alpha t$, the equivalent frequency width δf_S of the **spectrogram** $S_x^{(h)}(t,f) := |F_x^{(h)}(t,f)|^2$ behaves as:

$$\delta f_S \approx \sqrt{\frac{1}{\delta t_h^2} + \alpha^2\, \delta t_h^2}$$

for a window $h(t)$ with an equivalent time width $\delta t_h \Rightarrow$ minimum for $\delta t_h \approx 1/\sqrt{\alpha}$ (but α **unknown**. . .).

self-adaptation and Wigner-Ville distribution

- **Matched filtering —** If one takes for the window $h(t)$ the **time-reversed** signal $x_-(t) := x(-t)$, one readily gets that $F_x^{(x_-)}(t,f) = W_x(t/2, f/2)/2$, where

$$W_x(t,f) := \int x(t+\tau/2)\,\overline{x(t-\tau/2)}\,e^{-i2\pi f\tau}\,d\tau$$

  is the **Wigner-Ville Distribution** (Wigner, '32; Ville, '48).
- **Linear chirps —** The WVD **perfectly** localizes on **straight lines** of the plane:

$$x(t) = \exp\{i2\pi(f_0 t + \alpha t^2/2)\} \Rightarrow W_x(t,f) = \delta\left(f - (f_0 + \alpha t)\right).$$

- **Remark —** Localization via self-adaptation leads to a **quadratic** transformation (energy distribution).

interferences

- **Quadratic superposition** — For any pair of signals $\{x(t), y(t)\}$ and coefficients (a, b), one gets

$$W_{ax+by}(t,f) = |a|^2\, W_x(t,f) + |b|^2\, W_y(t,f) + 2\, Re\left\{a\,\overline{b}\, W_{x,y}(t,f)\right\},$$

with

$$W_{x,y}(t,f) := \int x(t+\tau/2)\, \overline{y(t-\tau/2)}\, e^{-i2\pi f\tau}\, d\tau$$

- **Drawback** — Interferences between **disjoint** component reduce readability.
- **Advantage** — Inner interferences between **coherent** components guarantee localization.

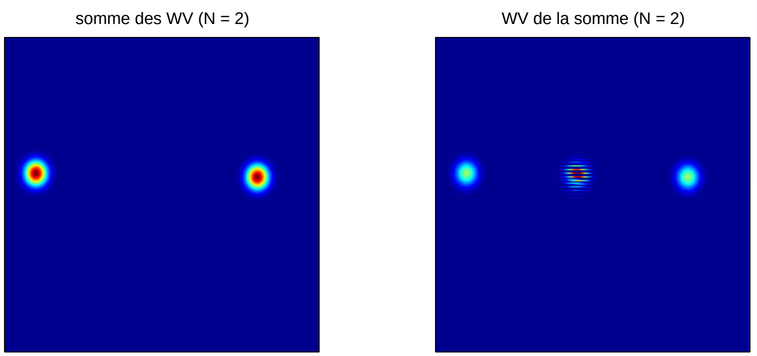
interferences

- **Janssen's formula (Janssen, '81)** — It follows from the **unitarity** of $W_x(t,f)$ that:

$$|W_x(t,f)|^2 = \iint W_x\left(t+\frac{\tau}{2}, f+\frac{\xi}{2}\right) W_x\left(t-\frac{\tau}{2}, f-\frac{\xi}{2}\right) d\tau\, d\xi$$

- **Geometry (Hlawatsch & F., '85)** — Contributions located in any two points of the plane plan interfere to create a third contribution
  1. midway of the segment joining the two components
  2. oscillating (positive and negative values) in a direction perpendicular to this segment
  3. with a "frequency" proportional to their "time-frequency distance".

interferences and readability



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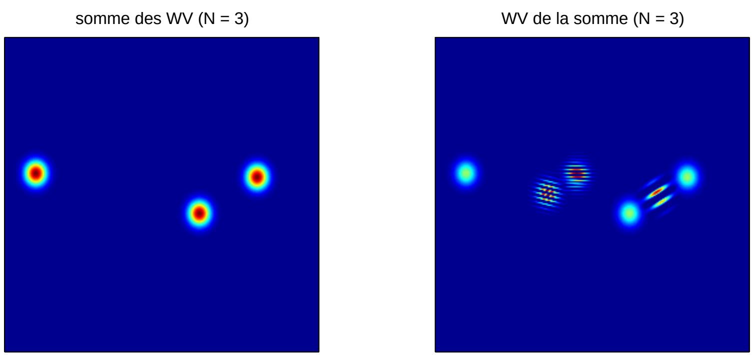

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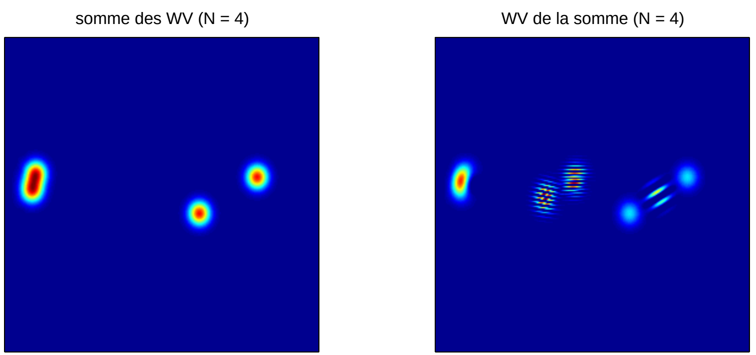

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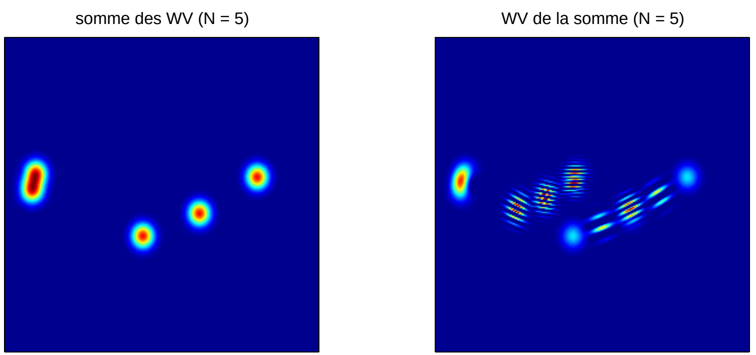

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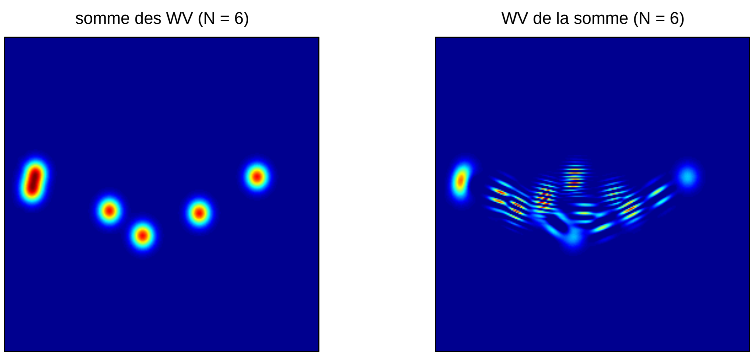

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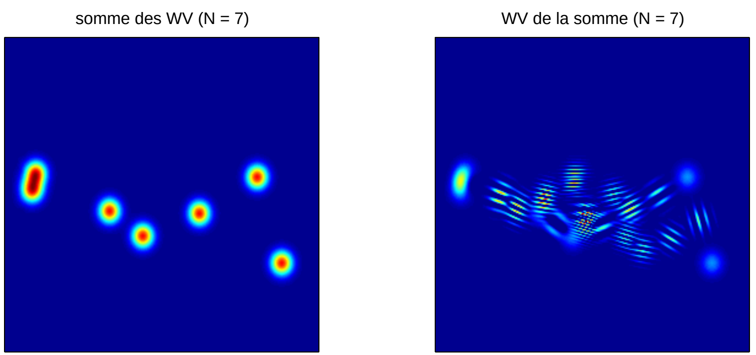

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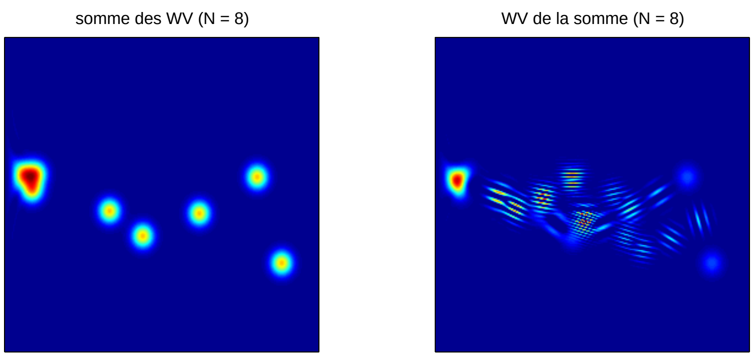

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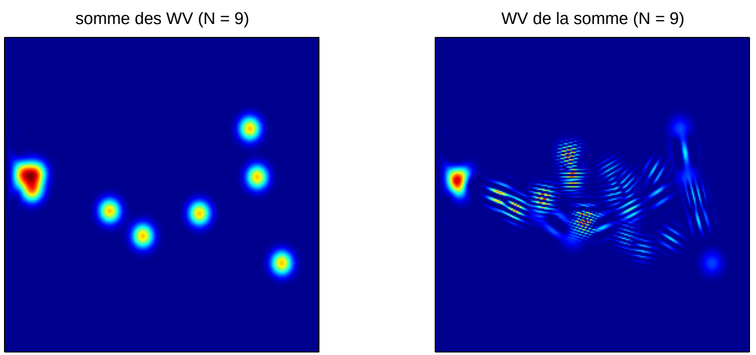

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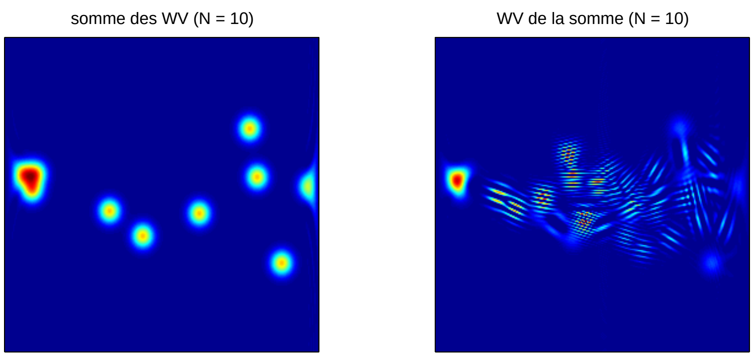

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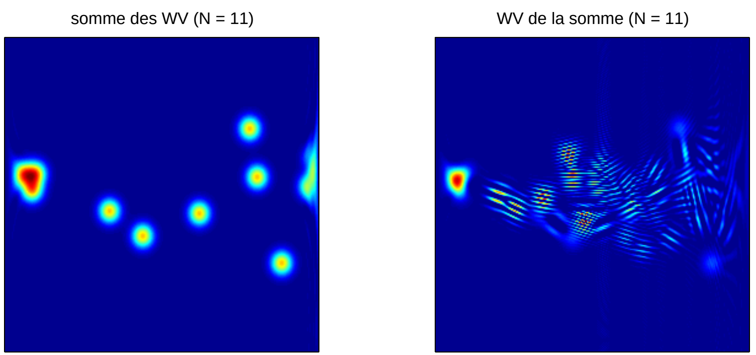

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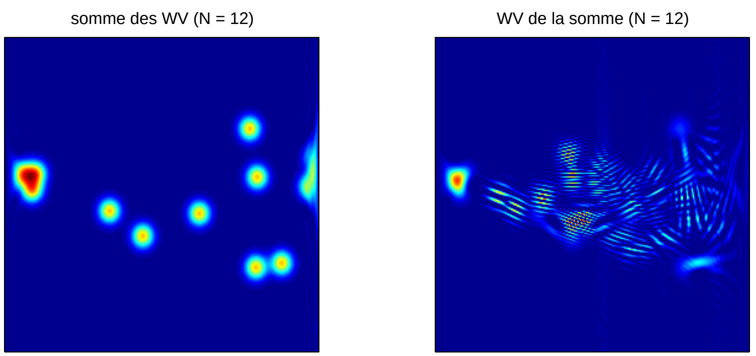

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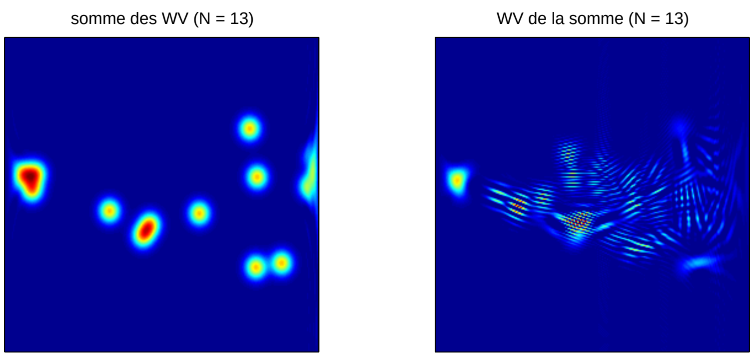

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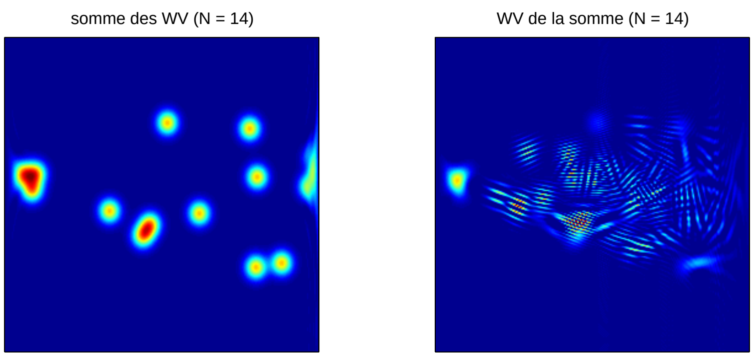

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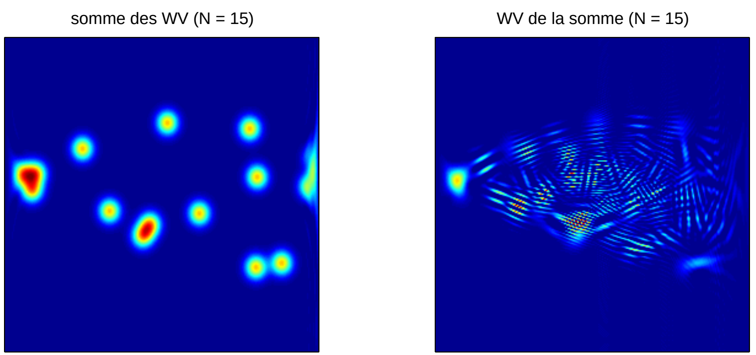

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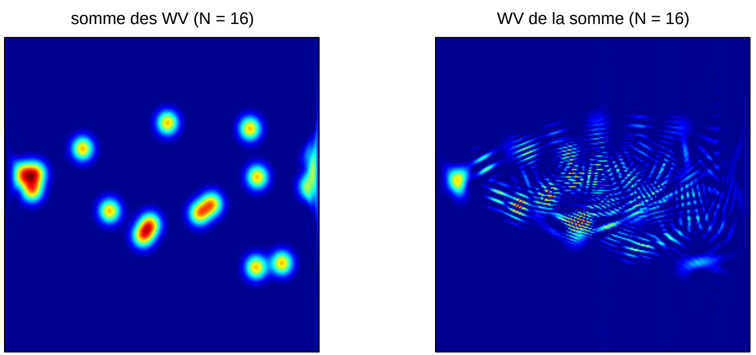
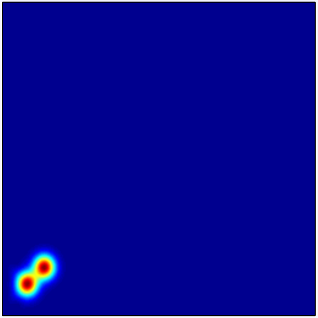
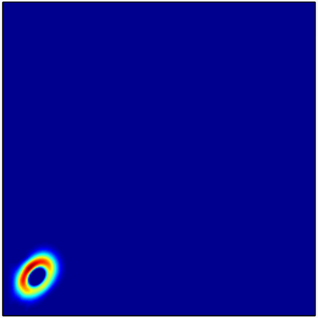


interferences and localization

sum(WV) (N = 2)

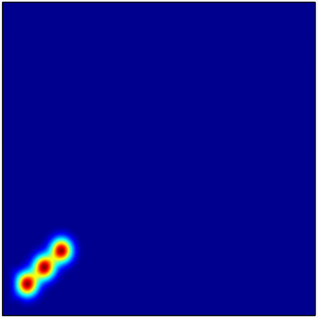


WV(sum) (N = 2)

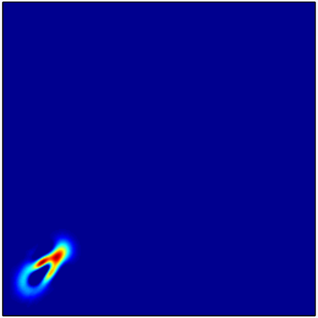


interferences and localization

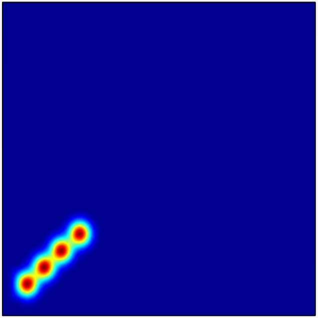
sum(WV) (N = 3)

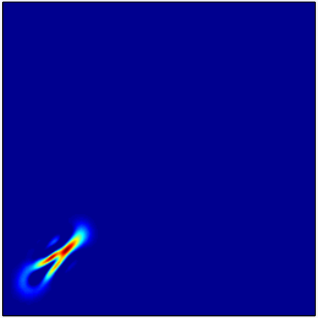


WV(sum) (N = 3)



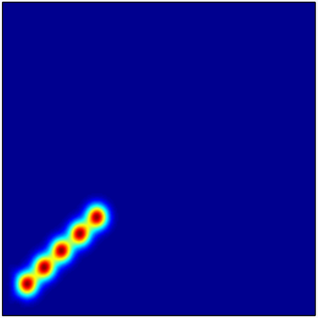
interferences and localization



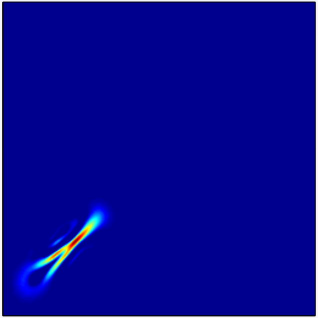


interferences and localization

sum(WV) (N = 5)

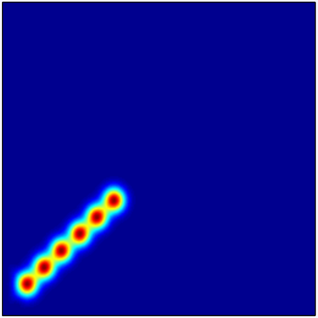


WV(sum) (N = 5)

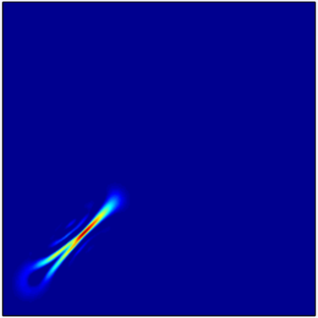


interferences and localization

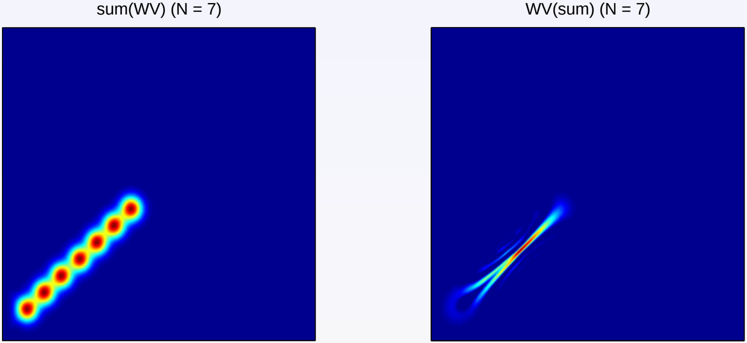
sum(WV) (N = 6)



WV(sum) (N = 6)



interferences and localization



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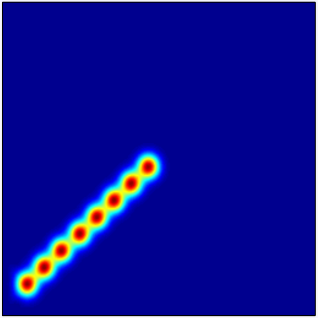

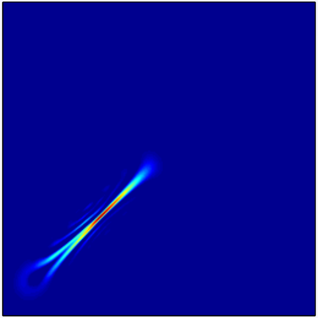

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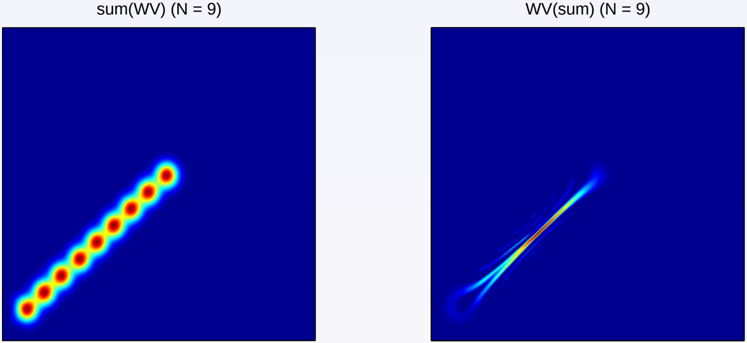

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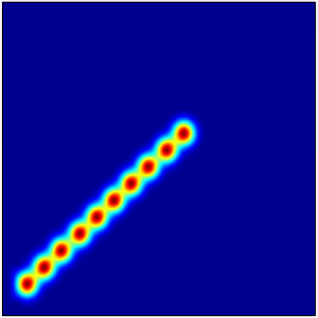

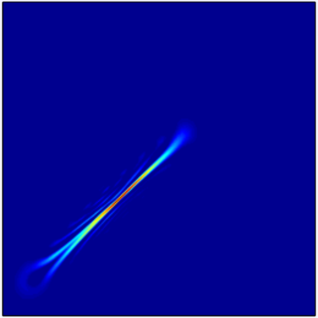

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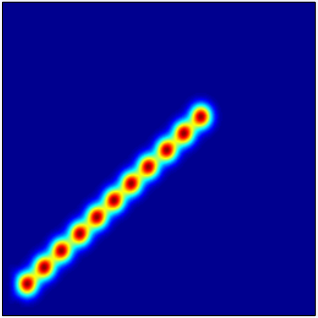

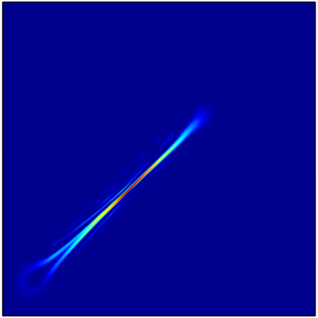

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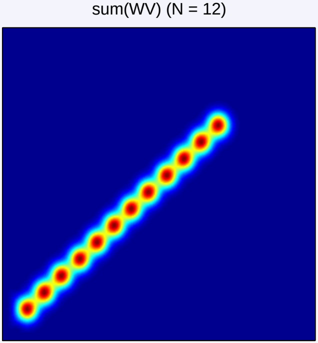
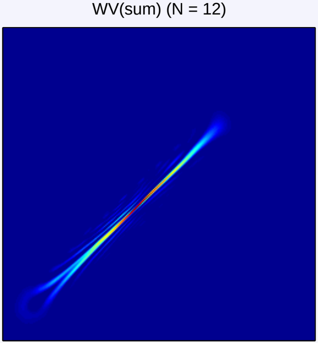
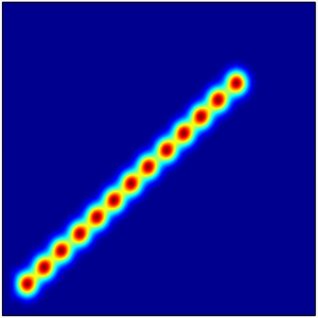
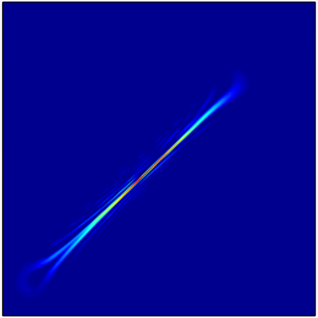


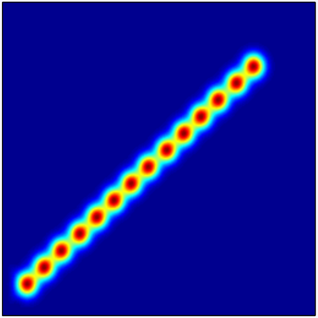


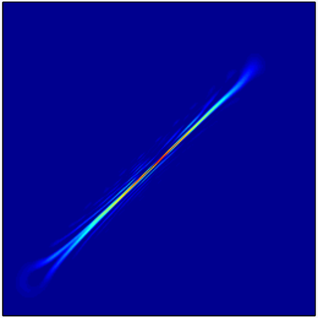
interferences and localization





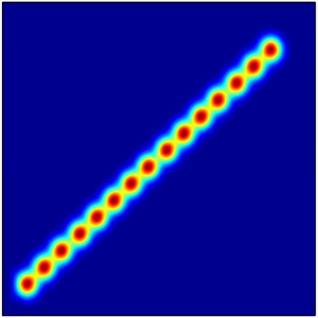
interferences and localization



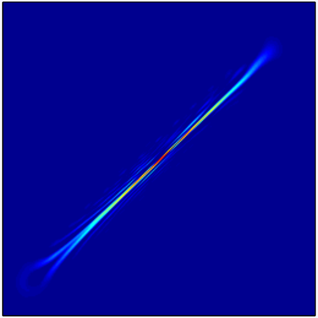


interferences and localization

sum(WV) (N = 15)

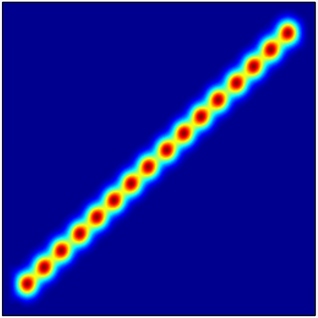


WV(sum) (N = 15)

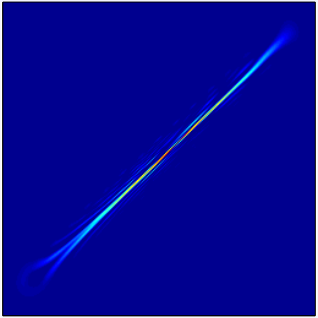


interferences and localization

sum(WV) (N = 16)



WV(sum) (N = 16)



spectrogram and Wigner-Ville: what else?

Observation

Many quadratic distributions have been proposed in the literature since more than half a century: ***none fully extends the notion of spectrum density to the nonstationary case****.*

Principle of conditional unicity — Classes of quadratic distributions of the form $\rho_x(t,f) = \langle x, \mathbf{K}_{t,f} x \rangle$ can be constructed based on **covariance requirements** :

$$\begin{array}{ccc} x(t) & \rightarrow & \rho_x(t,f) \\ \downarrow & & \downarrow \\ (\mathbf{T}x)(t) & \rightarrow & \rho_{\mathbf{T}x}(t,f) = (\tilde{\mathbf{T}}\rho_x)(t,f) \end{array}$$

classes of quadratic distributions

- **Cohen's class** — Covariance wrt **shifts** $(\mathbf{T}_{t_0,f_0}x)(t) = x(t-t_0)\exp\{i2\pi f_0 t\}$ leads to **Cohen's class** (Cohen, '66) :

$$C_x(t,f) := \iint W_x(s,\xi)\,\Pi(s-t,\xi-f)\,ds\,d\xi,$$

  with $\Pi(t,f)$ “arbitrary" (and to be specified via additional constraints).
- **Variations** — Other choices possibles, e.g., $(\mathbf{T}_{t_0,f_0}x)(t) = (f/f_0)^{1/2}x(f(t-t_0)/f_0) \rightarrow$ **affine class** (Rioul & F, '92), etc.

Cohen's class and smoothing

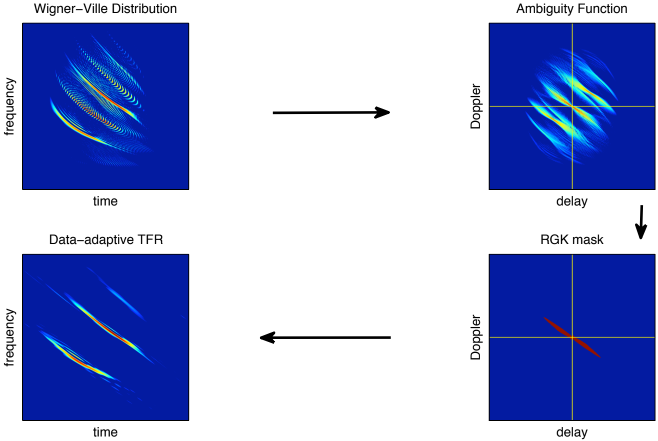
- **Spectrogram** — Given a low-pass window $h(t)$, one gets the **smoothing** relation:

$$S_x^{(h)}(t,f) := |F_x^{(h)}(t,f)|^2 = \iint W_x(s,\xi)\, W_h(s-t,\xi-f)\, ds\, d\xi$$

- **From Wigner-Ville to spectrograms** — A generalization amounts to choose a smoothing function $\Pi(t,f)$ allowing for a **continuous** and **separable** transition between Wigner-Ville and a spectrogram (**smoothed pseudo-Wigner-Ville** distributions) :

$$\begin{array}{ccccc} Wigner-Ville & \ldots \rightarrow & PWVL & \ldots \rightarrow & spectrogram \\ \\ \delta(t)\,\delta(f) & & g(t)\,H(f) & & W_h(t,f) \end{array}$$

extension: data-dependent smoothing



global vs. local

- **Global approach** — The Wigner-Ville Distribution localizes perfectly on **straight lines** of the plane (linear chirps). One can construct other distributions localizing on more general **curves** (ex.: **Bertrand**'s distributions adapted to hyperbolic chirps).
- **Local approach** — A different possibility consists in revisiting the smoothing relation defining the spectrogram and in considering localization wrt the instantaneous frequency as it can be measured **locally**, at the scale of the short-time window $\Rightarrow$ **reassignment**.

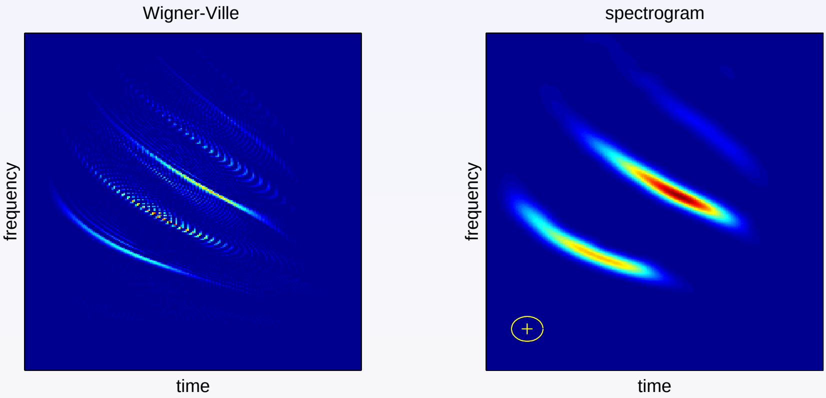
reassignment

- **Principle** — The key idea is (1) to replace the **geometrical** center of the smoothing time-frequency domain by the **center of mass** of the WVD over this domain, and (2) to **reassign** the value of the smoothed distribution to this local centroïd:

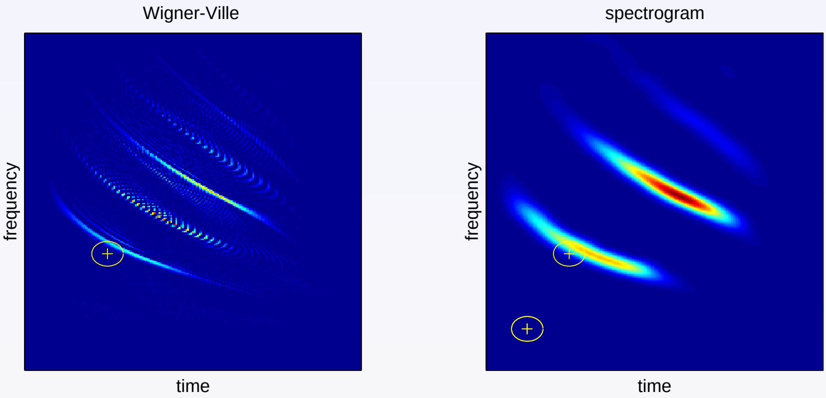
$$S_x^{(h)}(t,f) \mapsto \iint S_x^{(h)}(s,\xi)\,\delta\left(t-\hat{t}_x(s,\xi), f-\hat{f}_x(s,\xi)\right)\,ds\,d\xi.$$

- **Remark** — Reassignment has been first introduced for the only spectrogram (Kodera *et al.*, ’76), but its principle has been further generalized to **any** distribution resulting from the smoothing of a localizable mother-distribution (Auger & F., ’95).

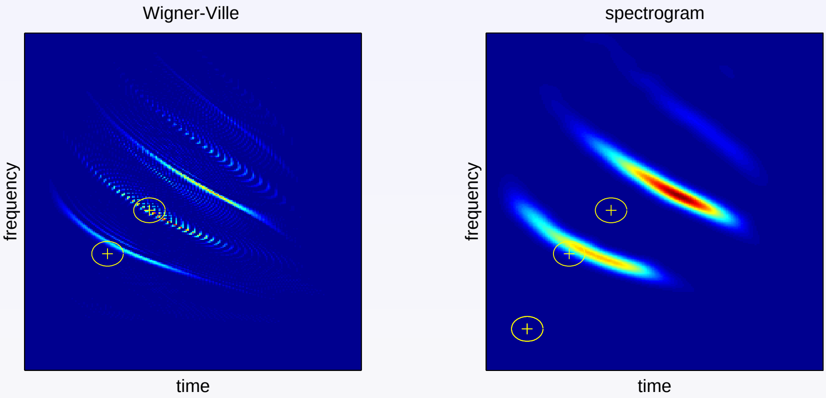
spectrogram = smoothed Wigner



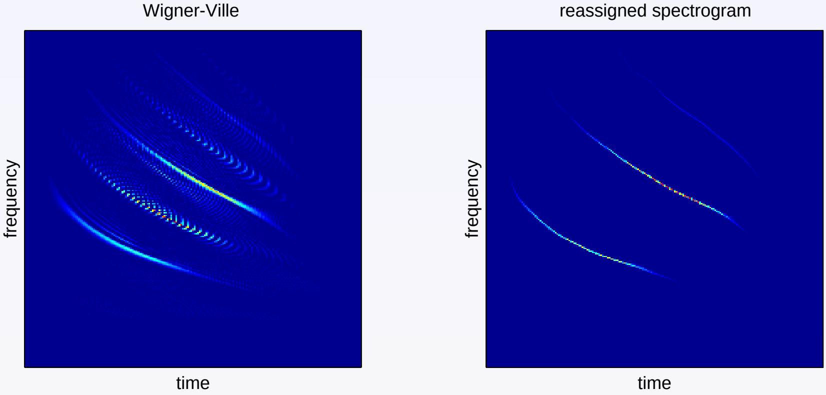
spreading of auto-terms



cancelling of cross-terms



reassignment (Kodera *et al.*, 1976, Auger & F., 1995)



reassignment in action

- **Spectrogram — Implicit** computation of the local centroïds (Auger & F., '95) :

$$\hat{t}_x(t,f) = t + Re\left\{\frac{F_x^{(\mathcal{T}h)}}{F_x^{(h)}}\right\}(t,f)$$

$$\hat{f}_x(t,f) = f - Im\left\{\frac{F_x^{(\mathcal{D}h)}}{F_x^{(h)}}\right\}(t,f),$$

with $(\mathcal{T}h)(t) = t\,h(t)$ and $(\mathcal{D}h)(t) = (dh/dt)(t)/2\pi$.

- **Beyond spectrograms — Possible generalizations to other smoothings (smoothed pseudo-Wigner-Ville, scalogram, etc.).**

reassignment in action

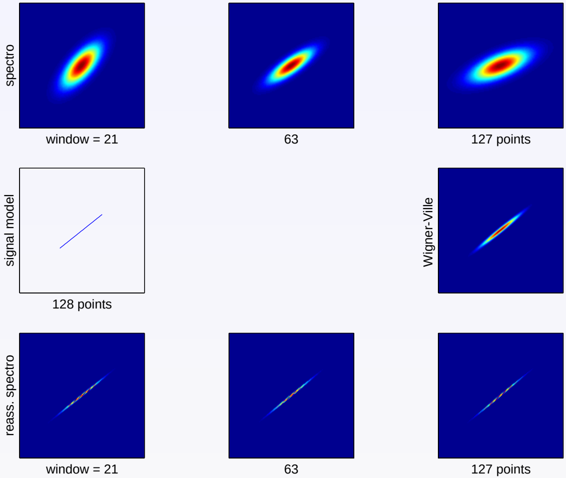
- **Gaussian spectrogram — Implicit** computation of the local centroïds (Auger, Chassande-Mottin & F., '12) :

$$\hat{t}_x(t,f) = t + \frac{\partial}{\partial t} \log M_x^{(h)}(t,f)$$

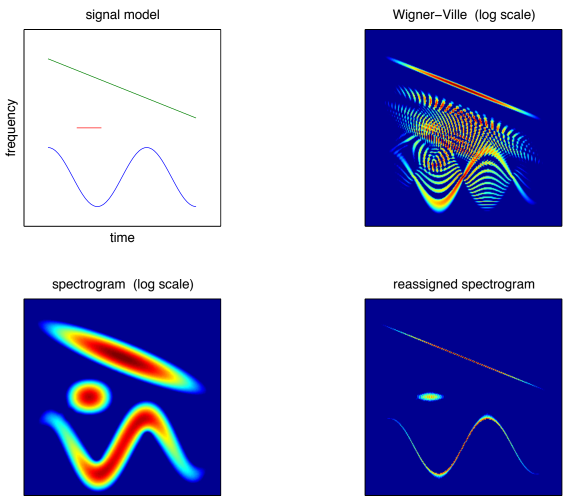
$$\hat{f}_x(t,f) = f + \frac{\partial}{\partial f} \log M_x^{(h)}(t,f),$$

with $M_x^{(h)}(t,f) = |F_x^{(h)}(t,f)|$.

independence wrt window size



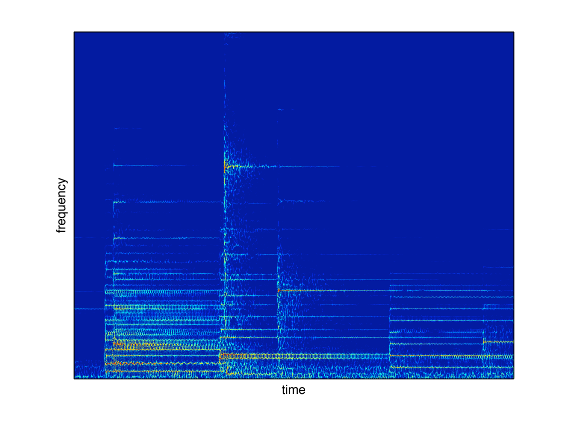
an example of comparison



music

tones & transients

"visual score"

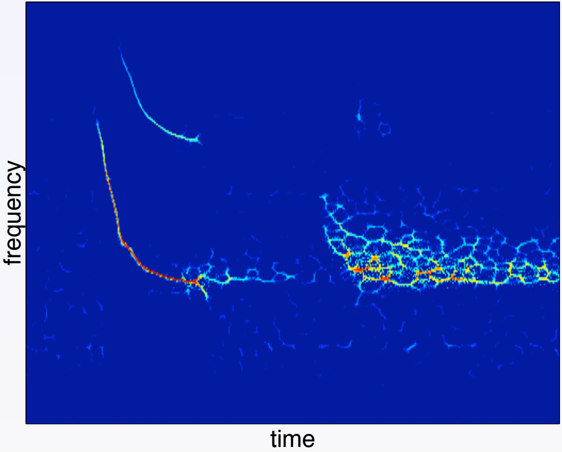


echolocation

bats

“animal sonar”

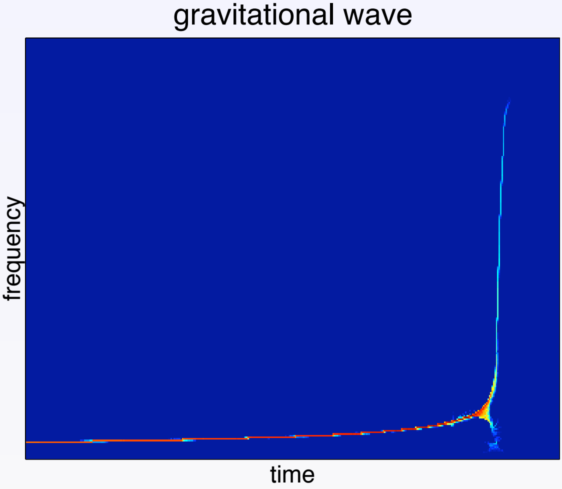
bat echolocation call + echo



gravitational waves

VIRGO

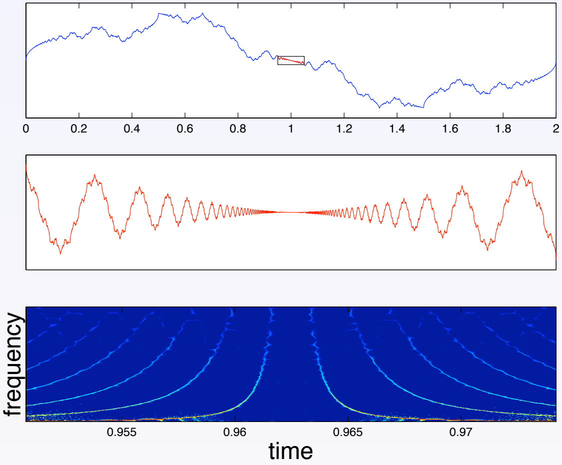
“coalescence of binaries”



Riemann’s function

B. Riemann

$\sigma(t) = \sum_{n=1}^{\infty} n^{-2} \sin \pi n^2 t$



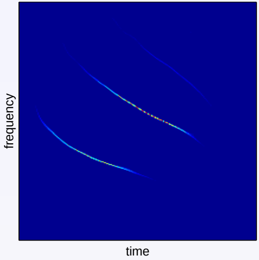
a "compressed sensing" approach

Discrete time

signal of dimension $N \Rightarrow$ TF distribution of dimension $\approx N^2$

Few components

$K \ll N \Rightarrow$ at most $KN \ll N^2$ non zero values in the TF plane



Sparsity

*minimizing the ℓ_0 quasi-norm not feasible, but almost optimal solution by **minimizing the ℓ_1 norm***

a "compressed sensing" approach"

Idea (F. & Borgnat, 2008-10)

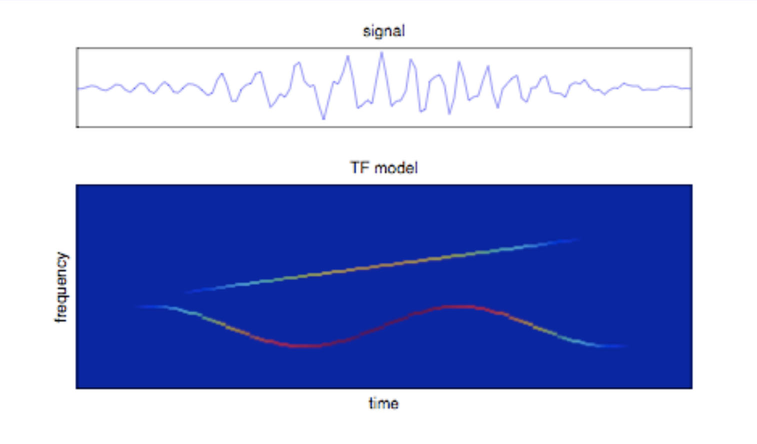
1. *choose a domain Ω neighbouring the origin of the AF plane*
2. *solve the program*

$$\min_{\rho} \|\rho\|_1 \,;\, \mathcal{F}\{\rho\} - A_x = 0|_{(\xi,\tau)\in\Omega}$$

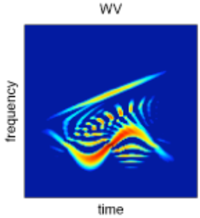
3. *the exact equality over Ω can be relaxed to*

$$\min_{\rho} \|\rho\|_1 \,;\, \|\mathcal{F}\{\rho\} - A_x\|_2 \leq \epsilon|_{(\xi,\tau)\in\Omega}$$

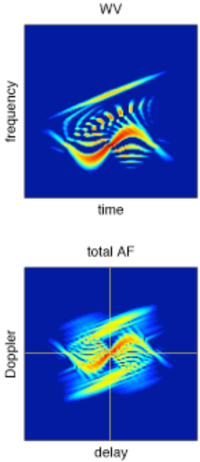
a toy example



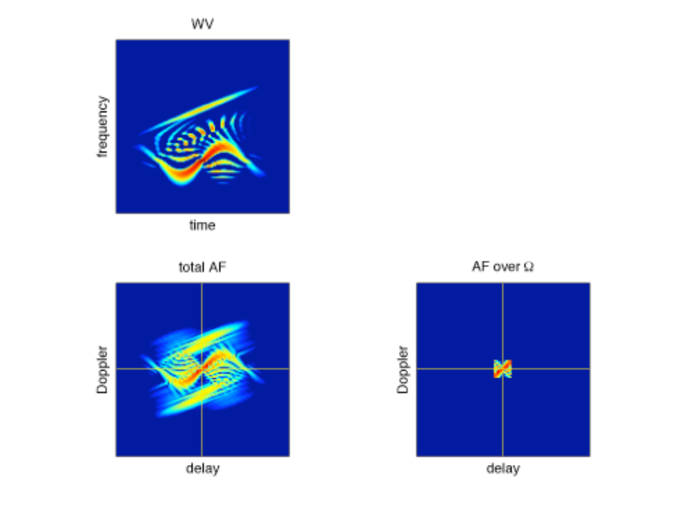
Wigner



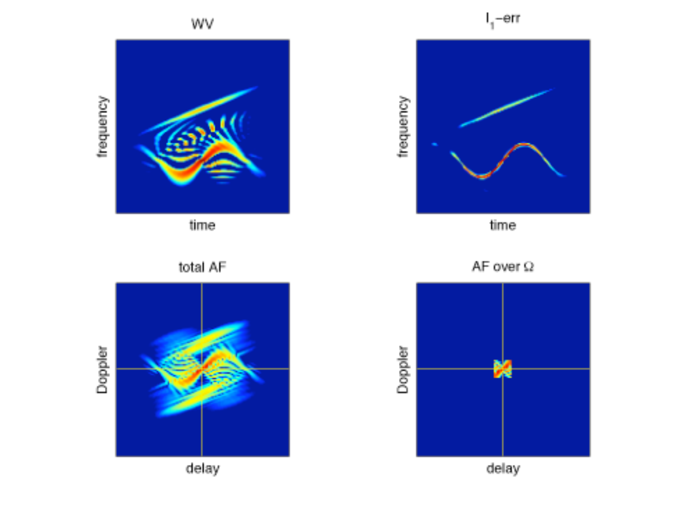
ambiguity



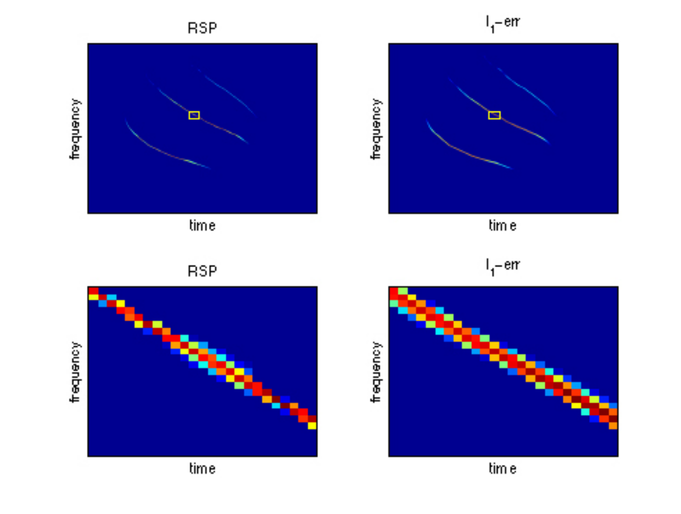
selection


WV
frequency
time
total AF
Doppler
delay
AF over Ω
Doppler
delay

sparse solution



comparison sparsity vs. reassignment



instantaneous frequency

Aim

model a signal $x(t) \in \mathbb{R}$ *as* $x(t) = a_x(t) \cos 2\pi \int^t f_x(s)\, ds$

- for a given t, "1 equation and 2 unkowns" $\Rightarrow$ **no unique representation**
- **multiplicity of solutions** under constraints
  - global
  - local
  - non harmonic

"global" approach (Gabor, 1946 ; Ville, 1948)

1. $e_f(t) = \cos 2\pi ft + i\,\mathcal{H}\{\cos 2\pi ft\}$, with $\mathcal{H}$ Hilbert

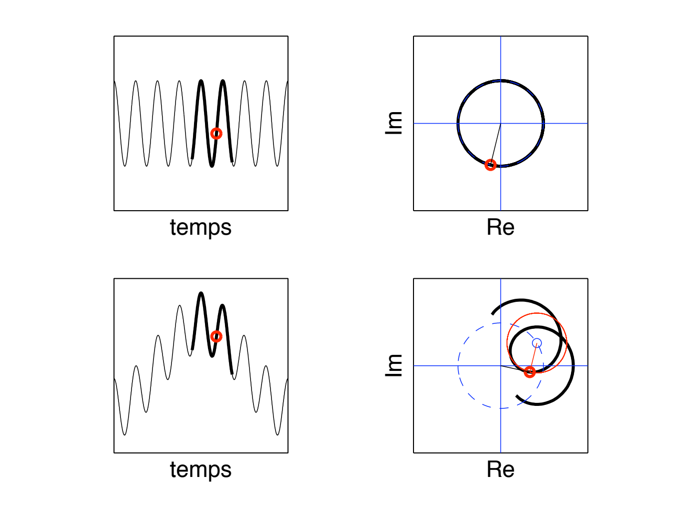
   monochromatic wave = **circle** in the complex plane + constant speed

2. $x(t) \to z_x(t) = x(t) + i\,\mathcal{H}\{x(t)\}$ (analytic signal)

   modulated "AM-FM" signal: circle $\to$ **"any" loop** around the origin of the complex plane + varying speed

3. **amplitude** : $a_x(t) = |z_x(t)|$
   instantaneous frequency : $f_x(t) = \frac{1}{2\pi}\partial_t \arg z_x(t)$

variation (Equis, Jacquot & F., 2011)

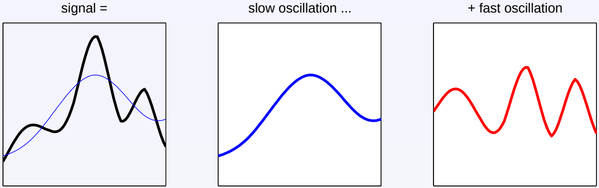


“local” approach (Teager, 1980 ; Kaiser, 1990)

1. $x(t) = a\cos 2\pi ft \Rightarrow \Psi(x) := (\partial_t x)^2 - x.\partial_t^2 x = 4\pi^2 a^2 f^2$

   $\Psi(x)$ **energy operator** taking the form
   $E(x) = x^2[n] - x[n-1]x[n+1]$ in discrete-time
2. similar local properties when $a \rightarrow a_x(t)$ and $f \rightarrow f_x(t)$
3. **instantaneous amplitude** : $a_x(t) = \Psi(x)/\sqrt{|\Psi(\partial_t x)|}$
   instantaneous frequency : $f_x(t) = \frac{1}{2\pi}\sqrt{|\Psi(\partial_t x)/\Psi(x)|}$

“non harmonic” approach (Huang *et al.*, 1998)



Idea of Empirical Mode Decomposition (EMD)

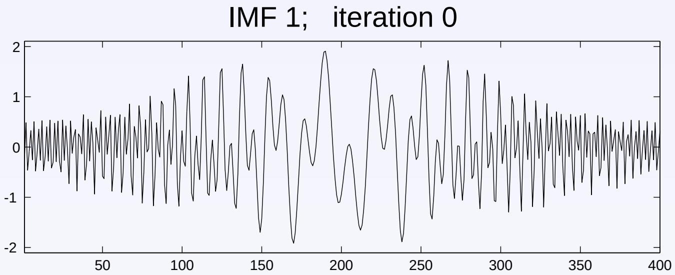
signal = fast oscillation + slow oscillation [& iteration]

- **data-driven** “fast vs. slow” disentanglement
- **“local"** analysis based on neighbouring extrema
- **oscillation** rather than frequency

algorithm

1. identify local maxima and minima
2. interpolate (cubic splines) to get an upper and a lower envelope
   1. subtract the mean of the envelopes from the signal
   2. iterate until “mean envelope $\approx$ 0" (*sifting*)
3. subtract the so-obtained mode from the signal
4. iterate on the residual

$$
\begin{aligned}
x(t) &= c_1(t) + r_1(t) \\
&= c_1(t) + c_2(t) + r_2(t) \\
&= \ldots\ldots\ldots\ldots\ldots\ldots \quad = \sum_{k=1}^{K} c_k(t) + r_K(t)
\end{aligned}
$$


IMF 1; iteration 0
2
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-1
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350
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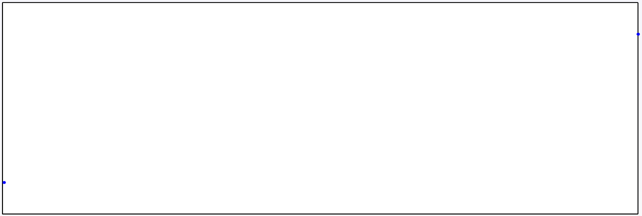

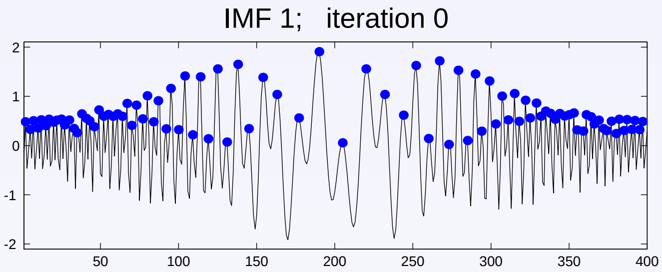
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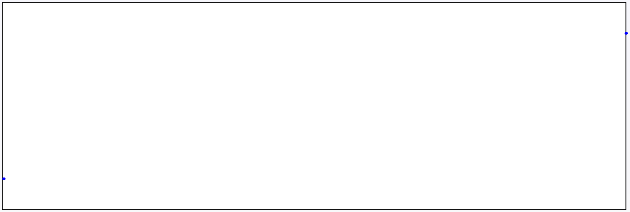
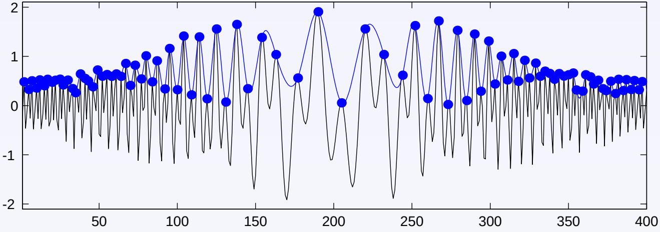
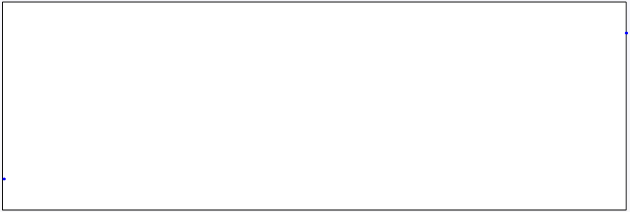
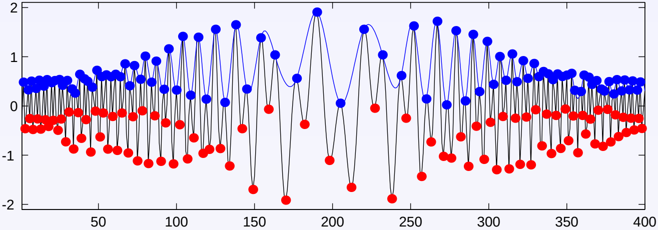


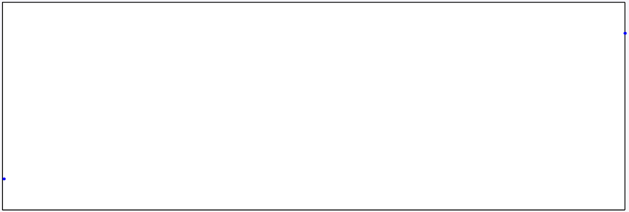
IMF 1; iteration 0





IMF 1; iteration 0





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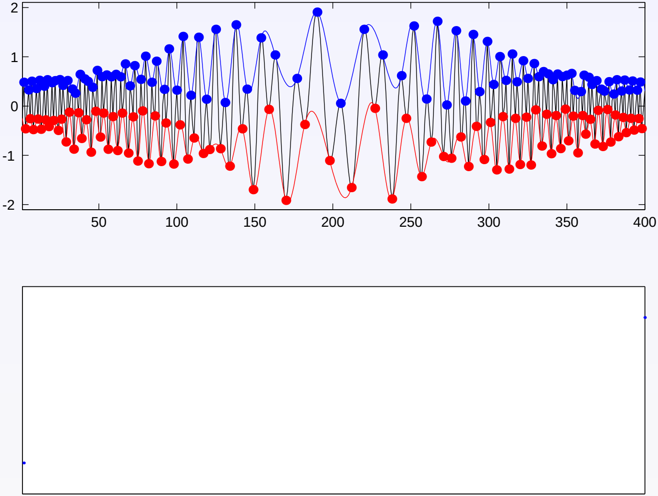

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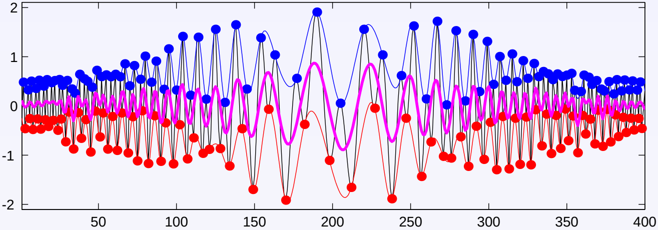

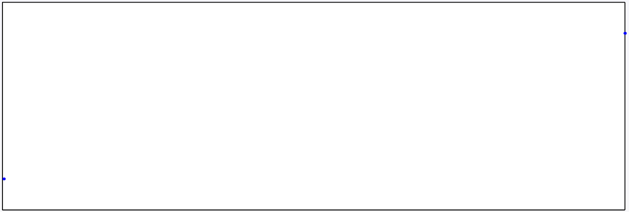

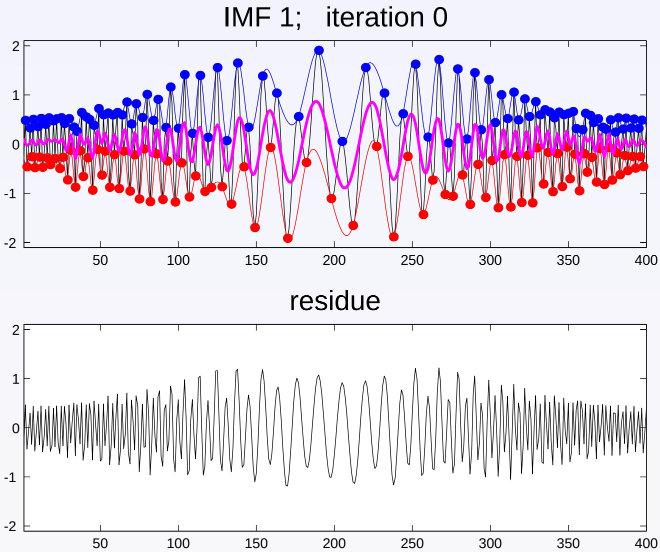
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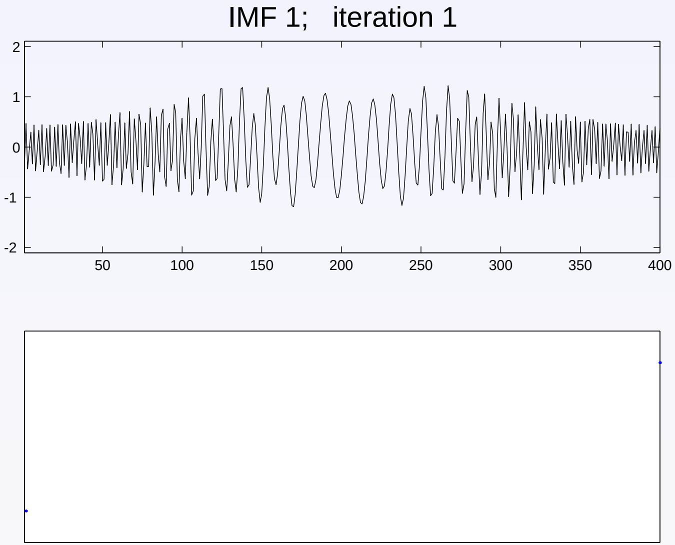
IMF 1; iteration 1
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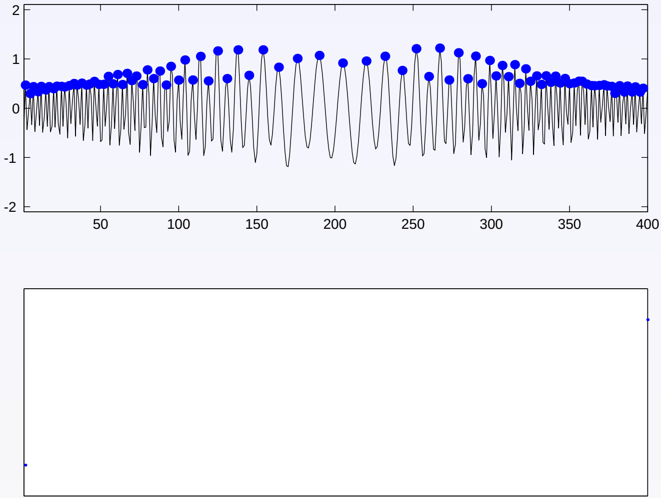
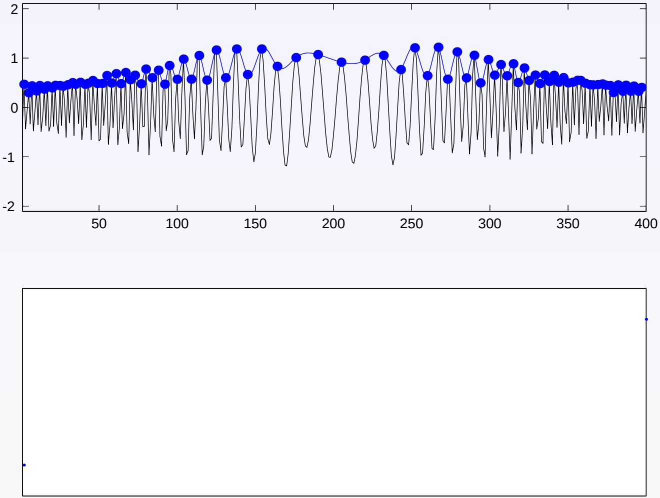
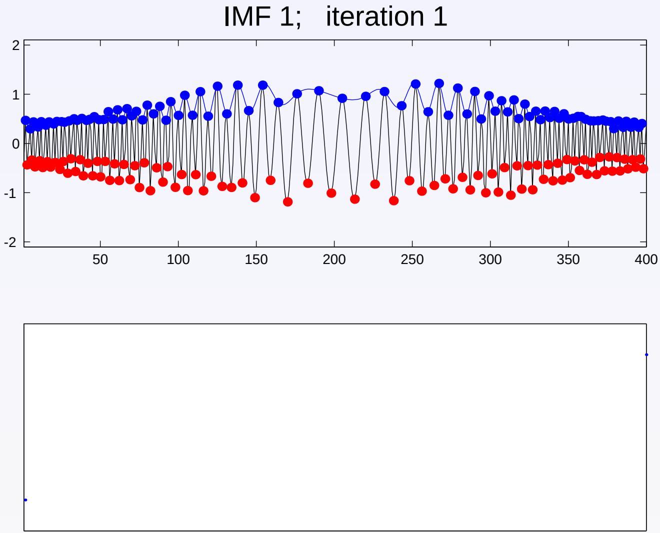
IMF 1; iteration 1



IMF 1; iteration 1

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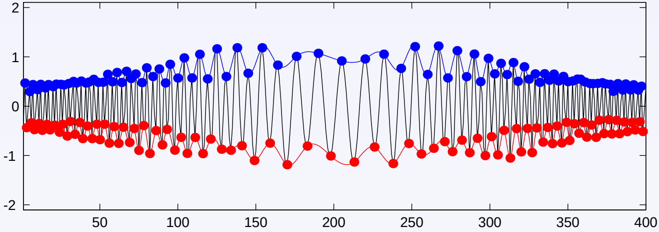
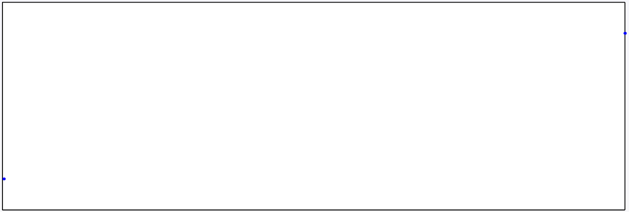
IMF 1; iteration 1





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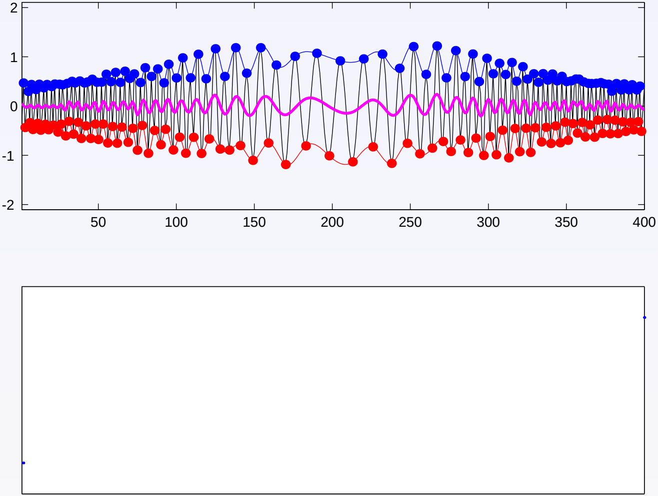

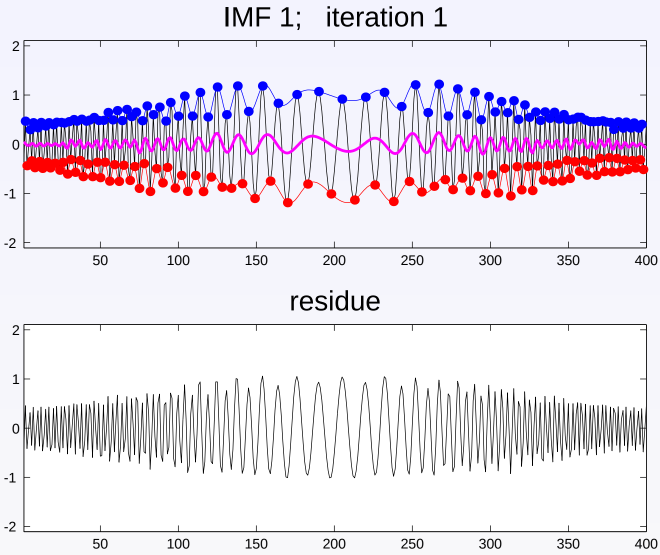
IMF 1; iteration 1
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50 100 150 200 250 300 350 400
residue
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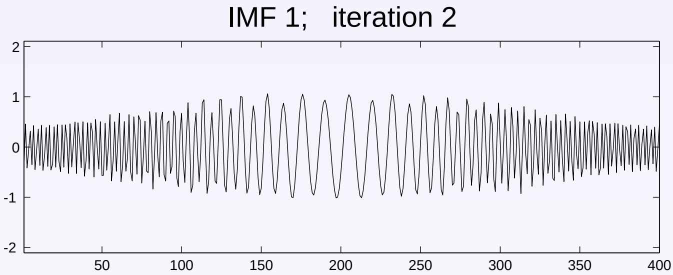
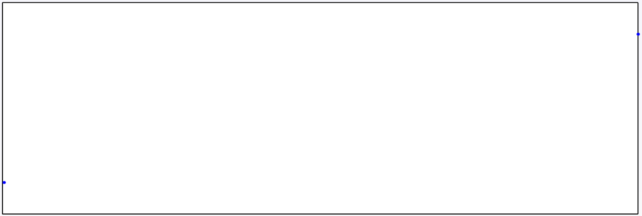
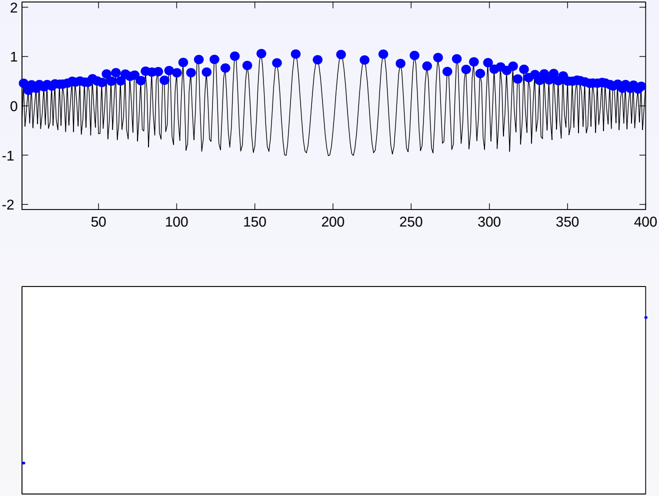
-2
50 100 150 200 250 300 350 400

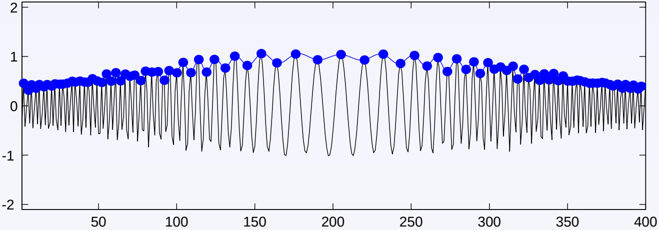

IMF 1; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400

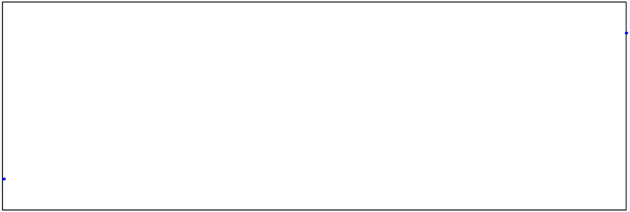


IMF 1; iteration 2

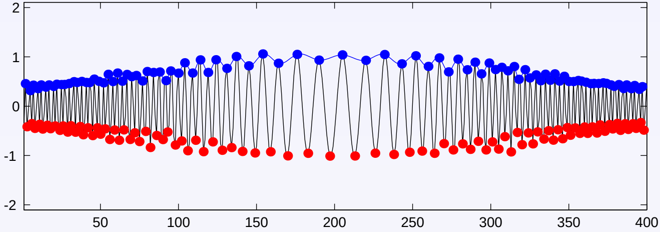


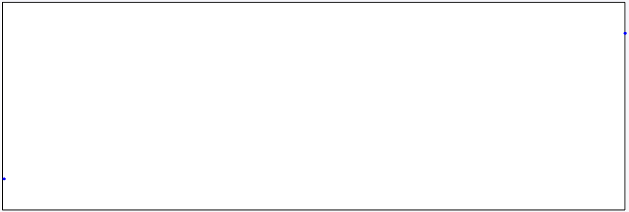
IMF 1; iteration 2

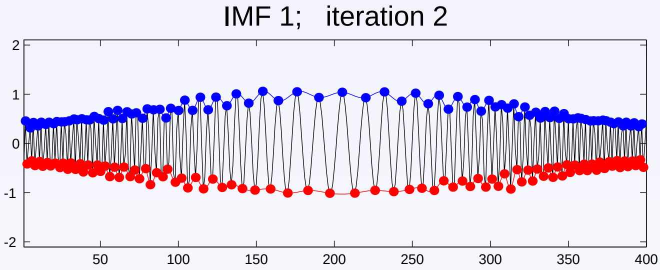


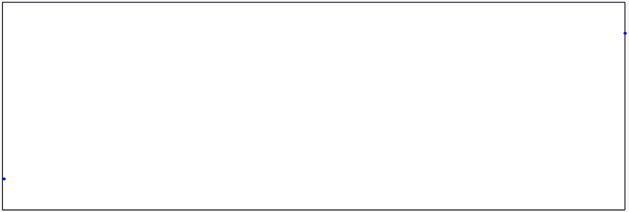


IMF 1; iteration 2

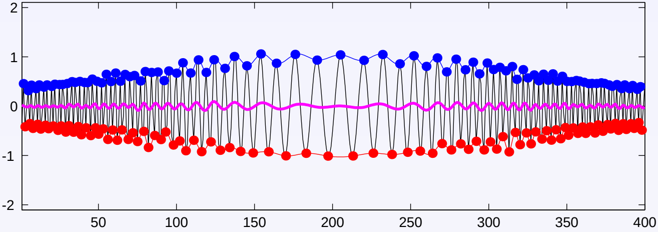


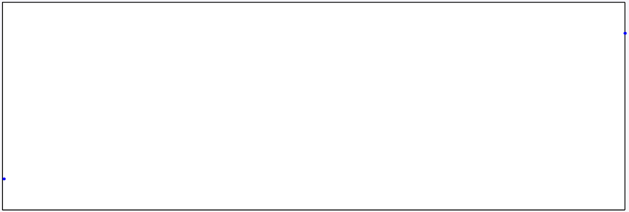



IMF 1; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400

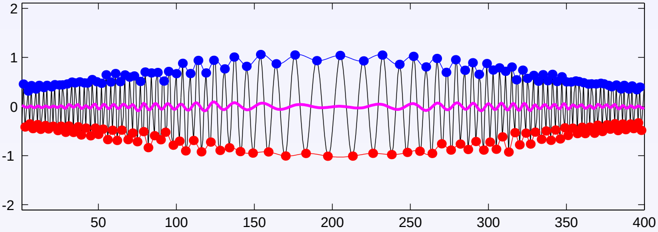


IMF 1; iteration 2

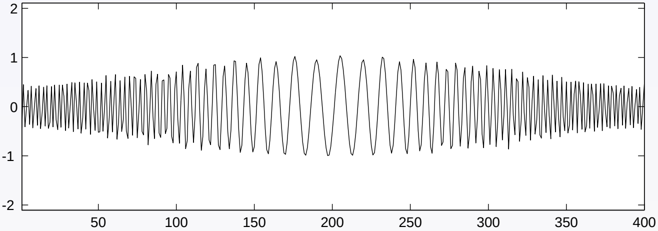


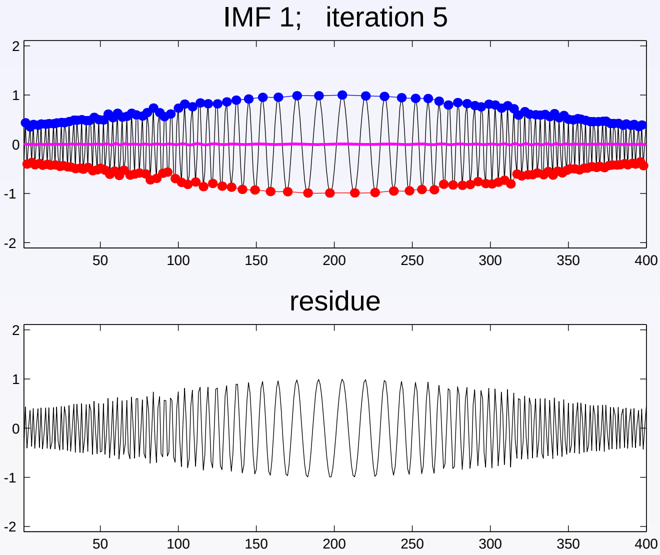


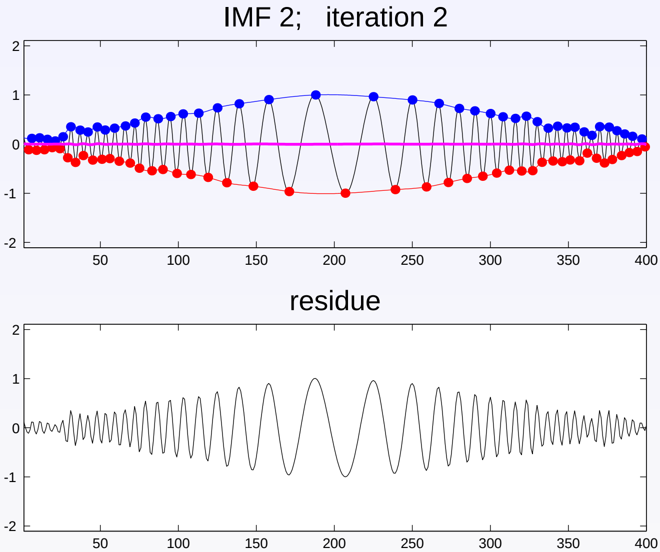
IMF 1; iteration 2



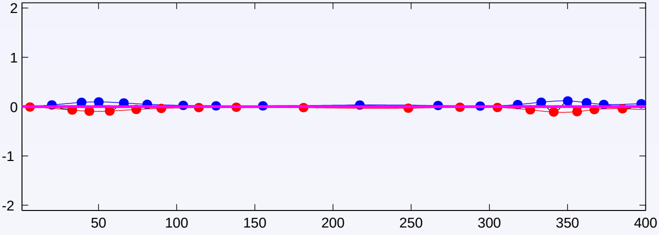
residue



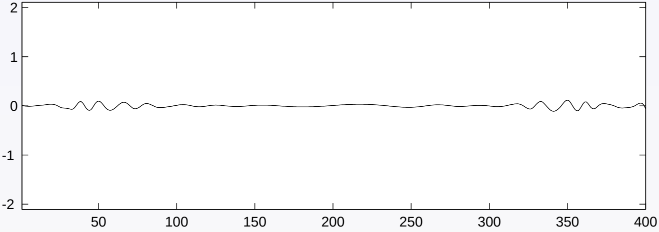

IMF 1; iteration 5
2
1
0
-1
-2
50 100 150 200 250 300 350 400
residue
2
1
0
-1
-2
50 100 150 200 250 300 350 400

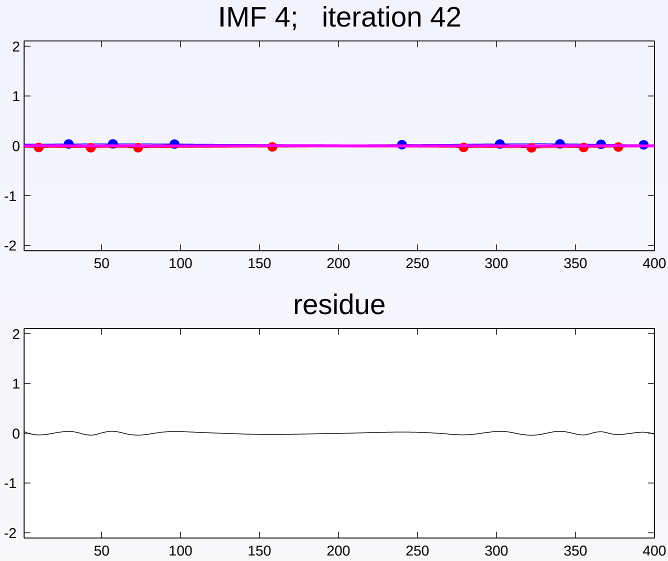

IMF 2; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

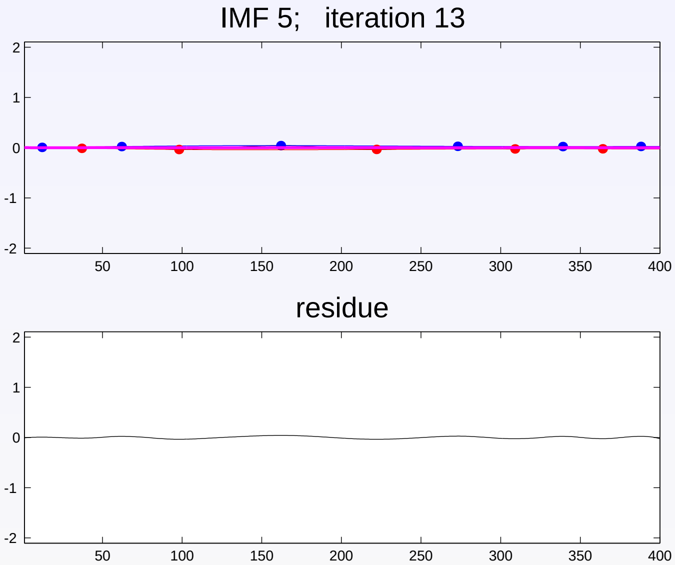
IMF 3; iteration 14



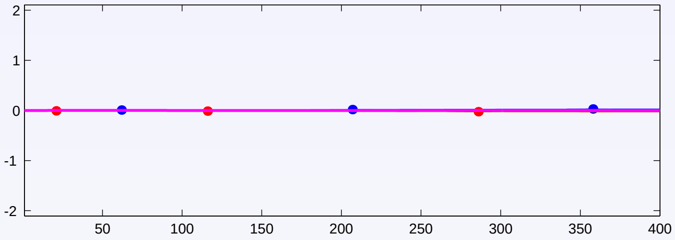
residue



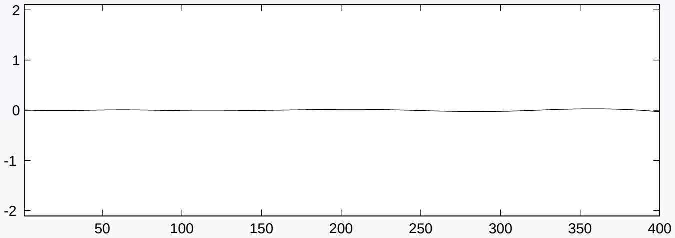

IMF 4; iteration 42
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

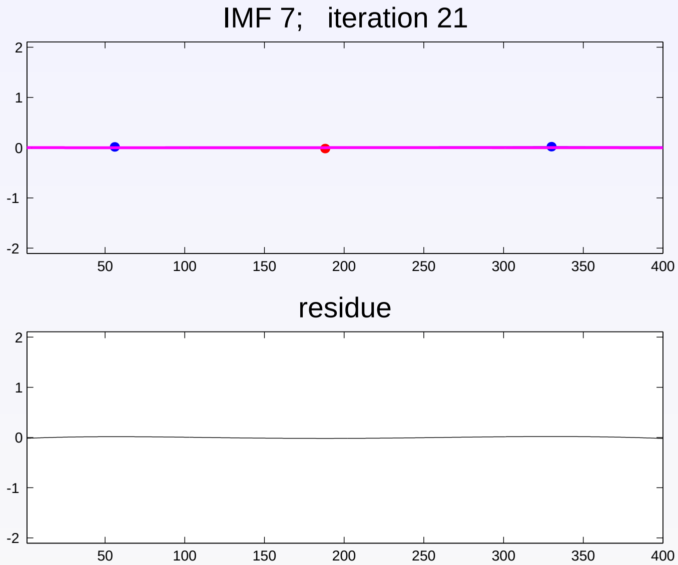

IMF 5; iteration 13
2
1
0
-1
-2
50 100 150 200 250 300 350 400
residue
2
1
0
-1
-2
50 100 150 200 250 300 350 400

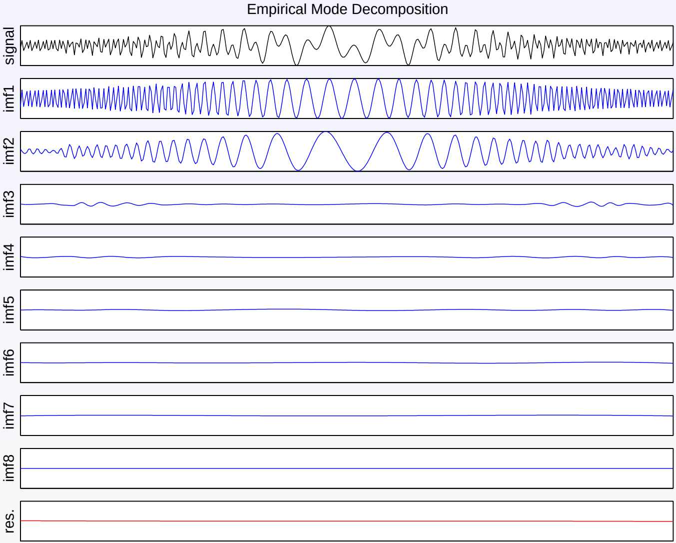
IMF 6; iteration 8

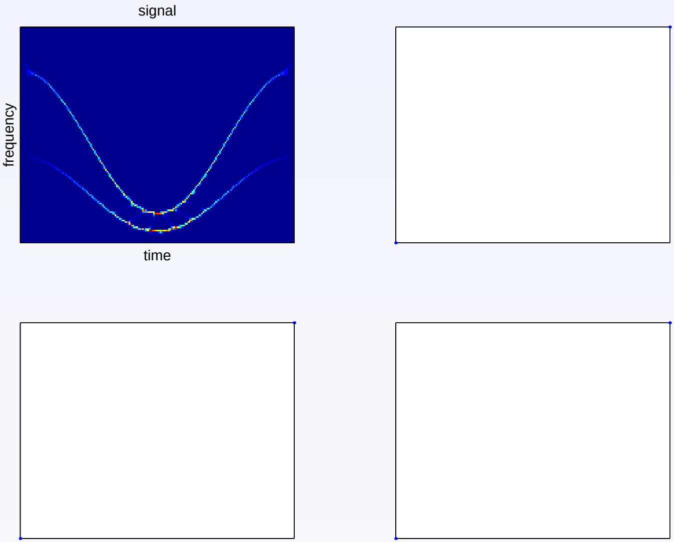


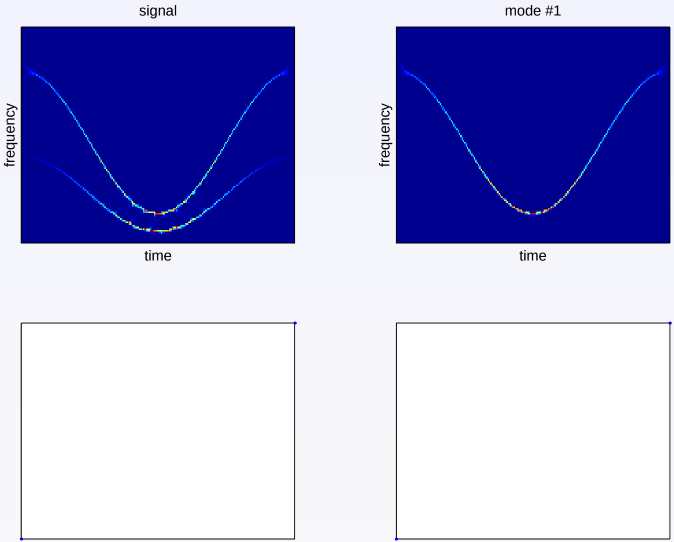
residue

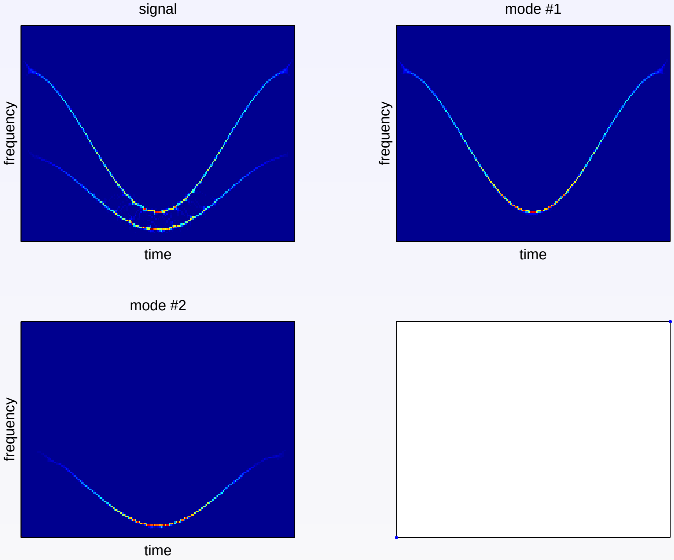


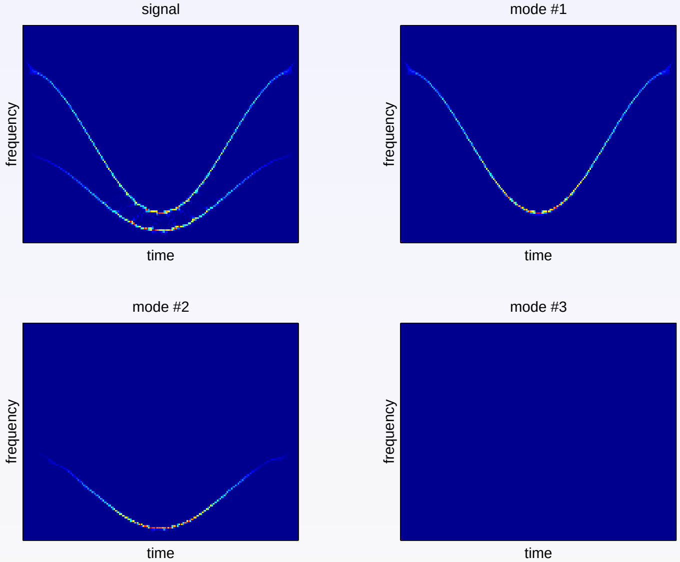

IMF 7; iteration 21
2
1
0
-1
-2
50 100 150 200 250 300 350 400
residue
2
1
0
-1
-2
50 100 150 200 250 300 350 400


Empirical Mode Decomposition
signal
imf1
imf2
imf3
imf4
imf5
imf6
imf7
imf8
res.


signal
frequency
time


signal
frequency
time
mode #1
frequency
time


signal
frequency
time
mode #1
frequency
time
mode #2
frequency
time

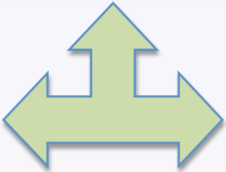

signal
frequency
time
mode #1
frequency
time
mode #2
frequency
time
mode #3
frequency
time

one or two components?

$$« \cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1+\omega_2}{2} t\right) \cos\left(\frac{\omega_1-\omega_2}{2} t\right) »$$

« physics »

(production, perception)



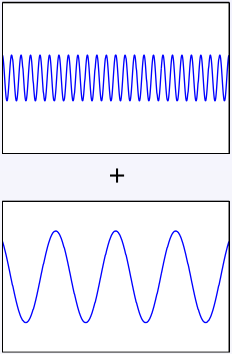
« mathematics »

(equivalent descriptions)

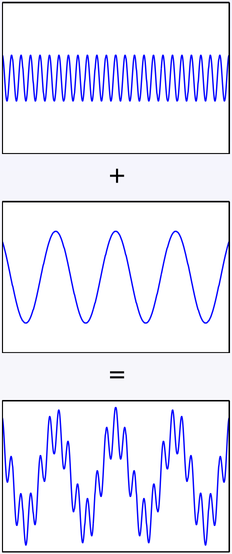
« computer science »

(model-based? data-driven?)

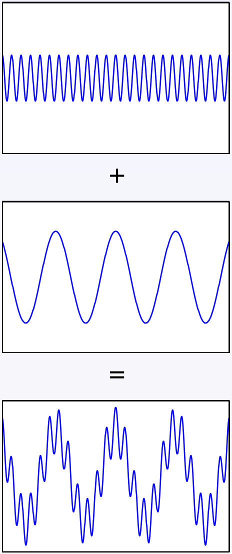
one or two components?

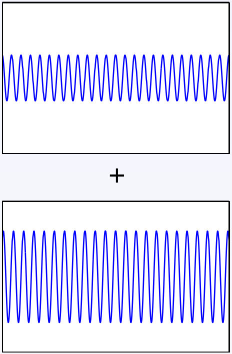


one or two components?

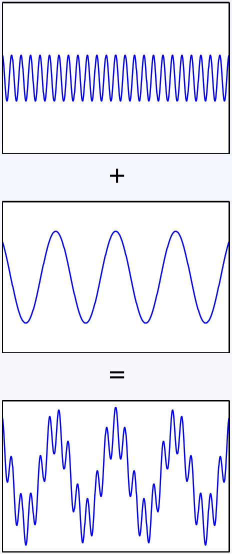


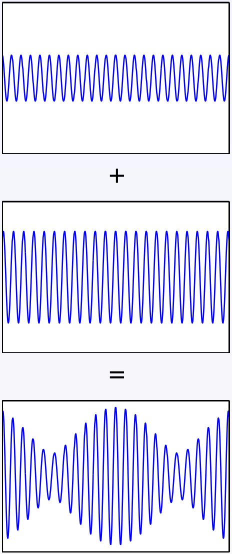
one or two components?



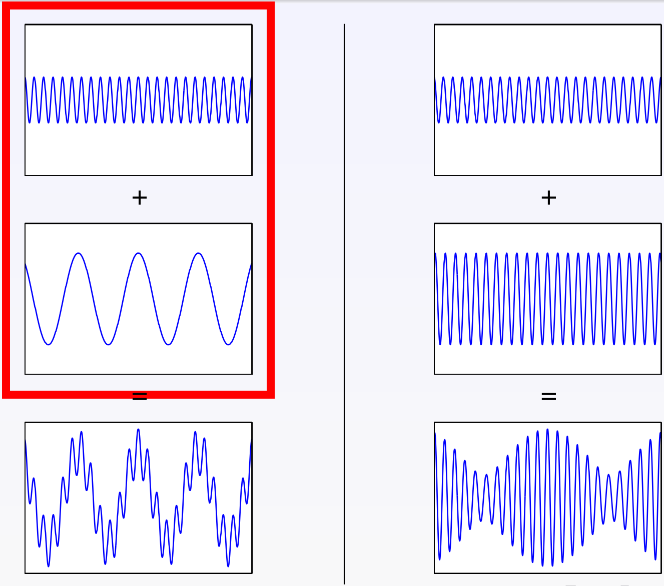


one or two components?

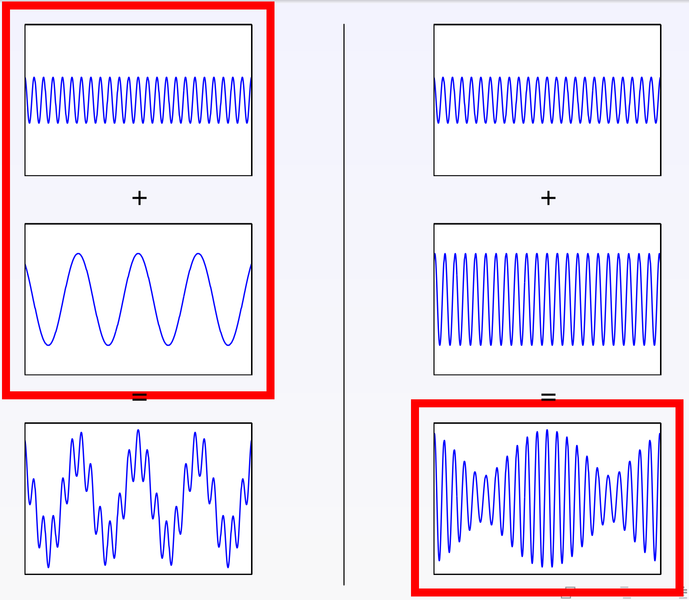




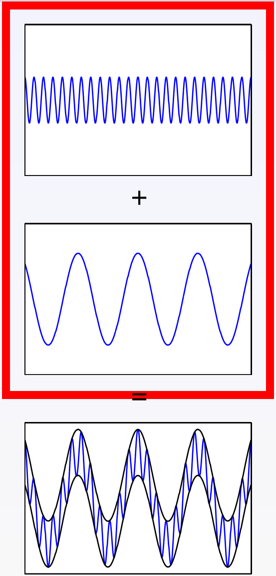
one or two components?

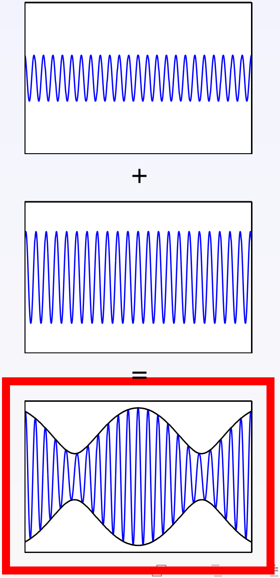


one or two components?

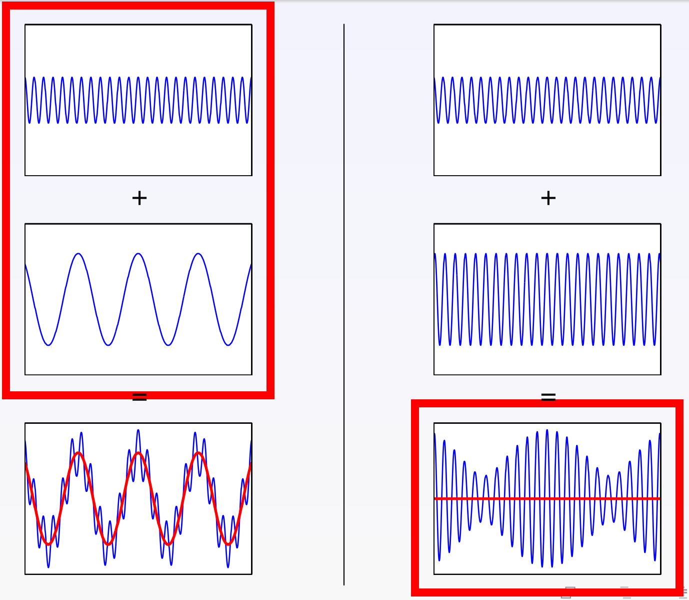


one or two components?





one or two components?



simulations

Signal

$$x(t) = \underbrace{a_1 \cos(2\pi f_1 t)}_{x_1(t)} + \underbrace{a_2 \cos(2\pi f_2 t + \varphi)}_{x_2(t)}, \quad f_1 > f_2$$

Analysis of its EMD

- only the **first IMF** is computed: if separation, it should be equal to the highest frequency component $x_1(t)$
- **criterion** (= 0 if separation) :

$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

- sampling effects are **neglected** : $f_1, f_2 \ll f_s$, with f_s the sampling frequency

simulations

Signal

$$x(t) = \underbrace{a_1 \cos(2\pi f_1 t)}_{x_1(t)} + \underbrace{a_2 \cos(2\pi f_2 t + \varphi)}_{x_2(t)}, \quad f_1 > f_2$$

Analysis of its EMD

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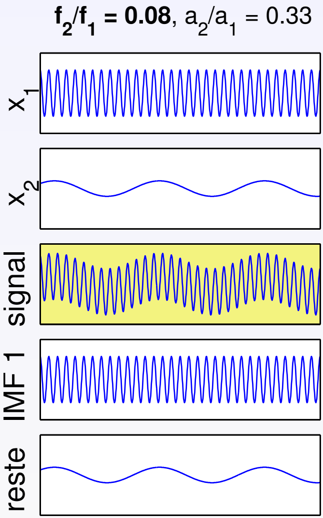
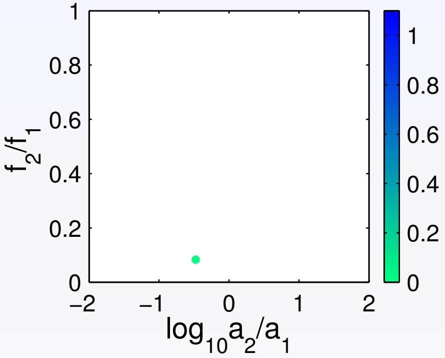
sum of two tones

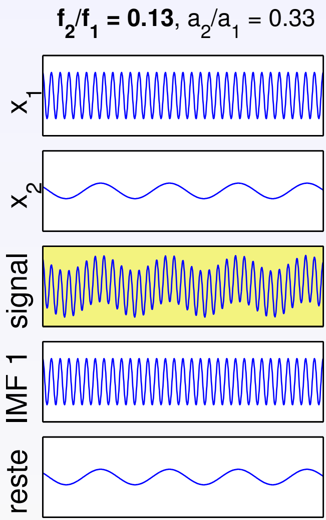


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

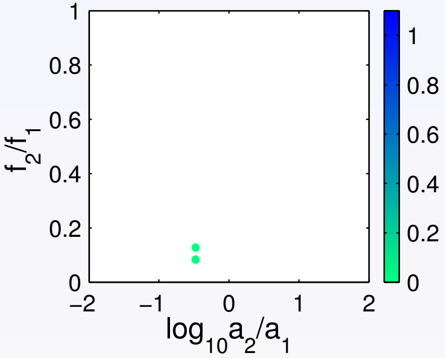


sum of two tones

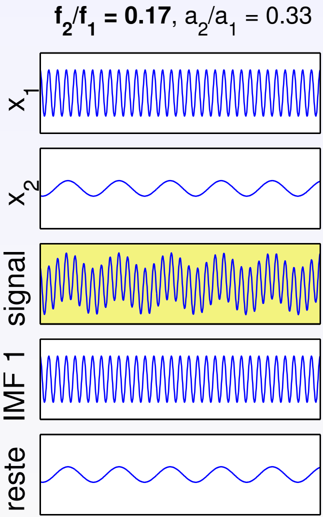


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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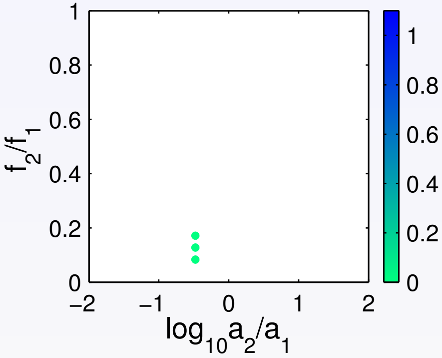
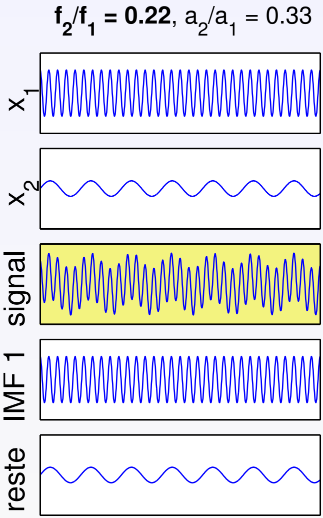


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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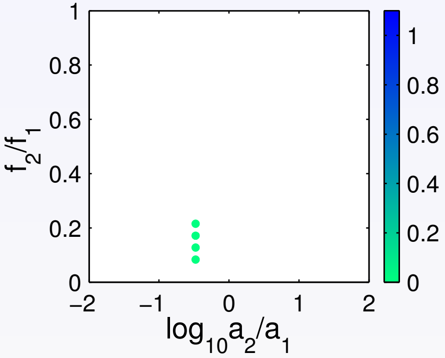
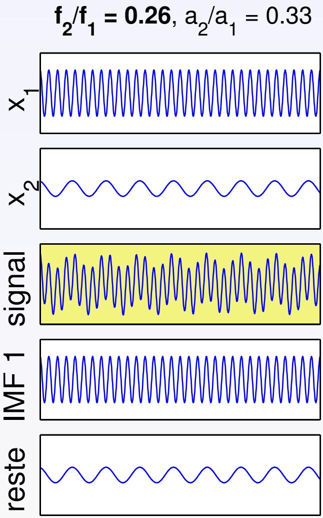


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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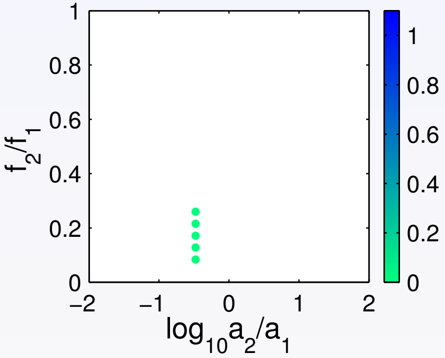
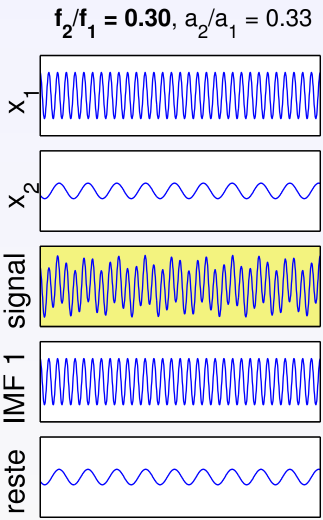
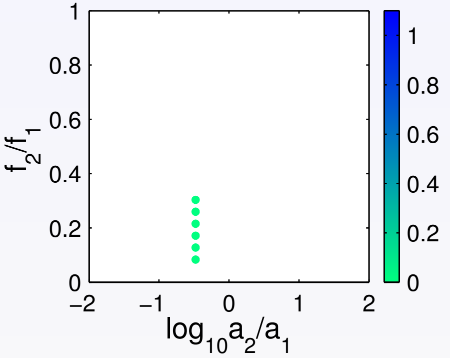


sum of two tones

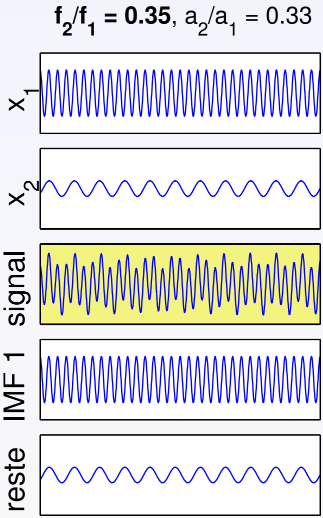


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

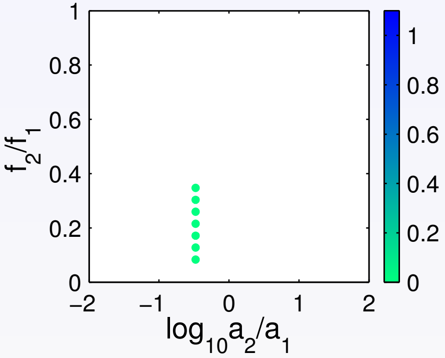


sum of two tones

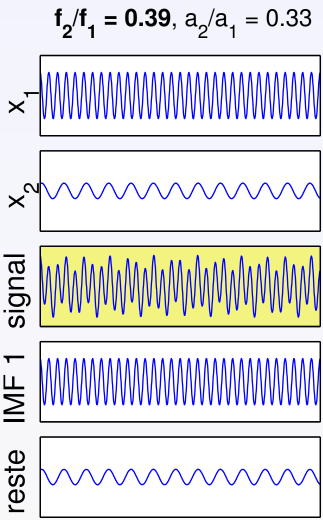


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

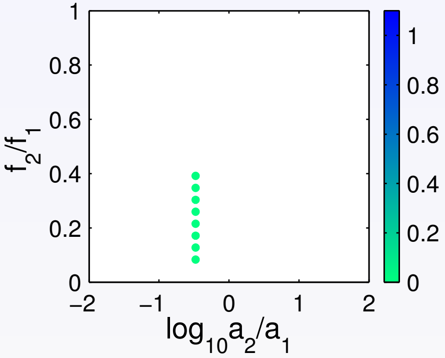


sum of two tones

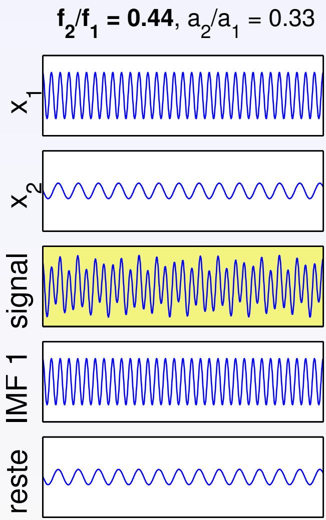


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

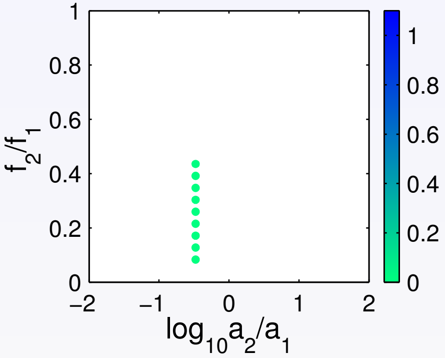


sum of two tones

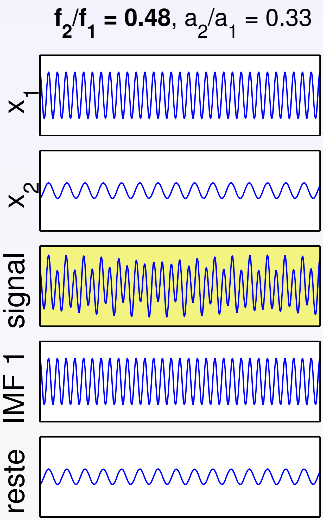


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation

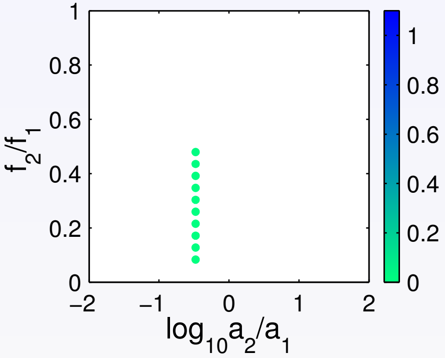


sum of two tones

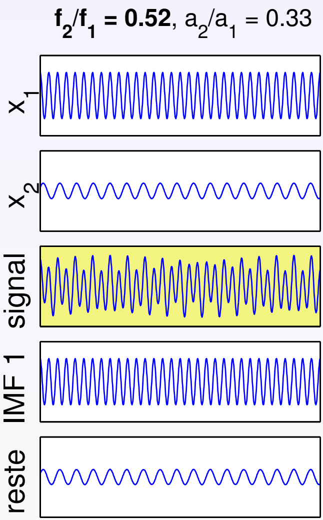


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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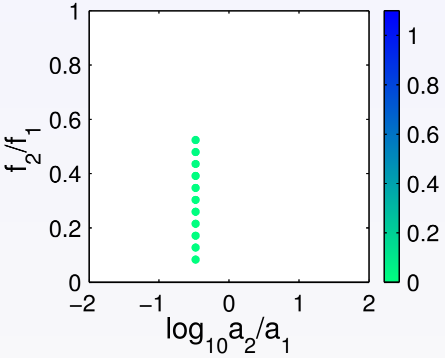
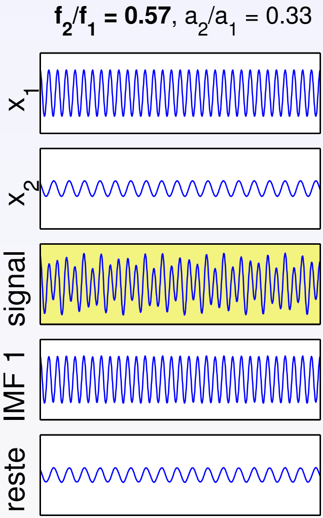


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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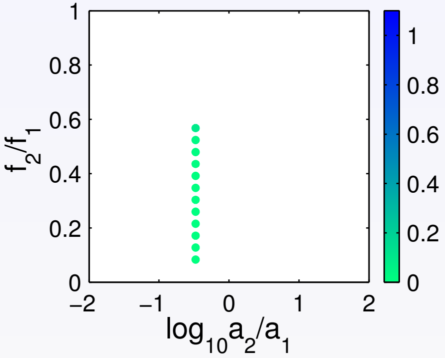
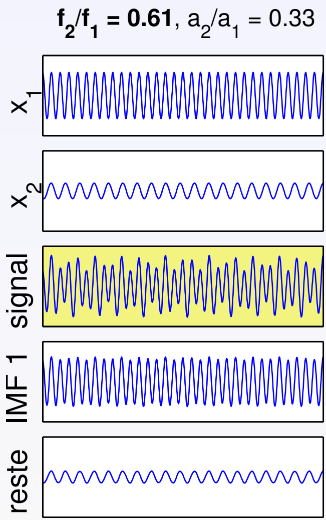


sum of two tones



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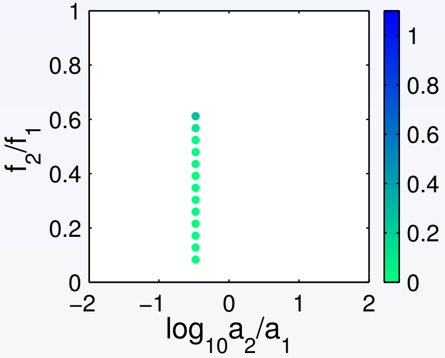

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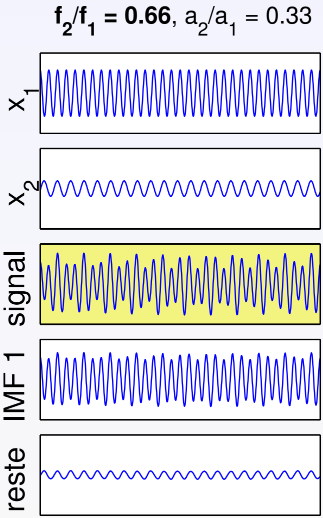

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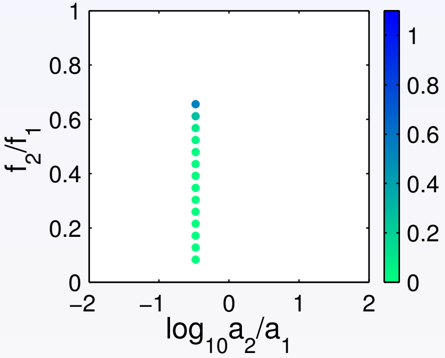

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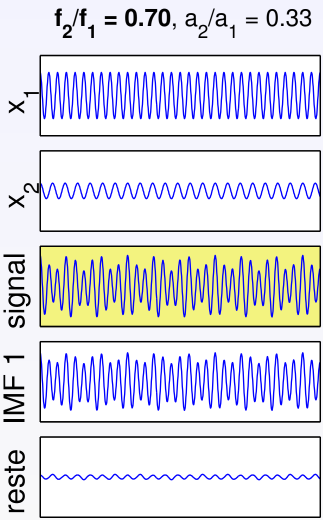

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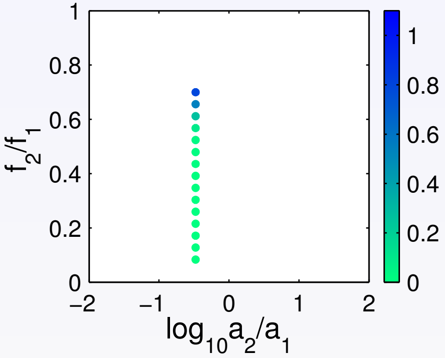

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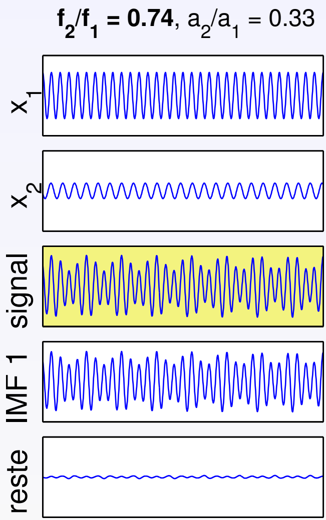

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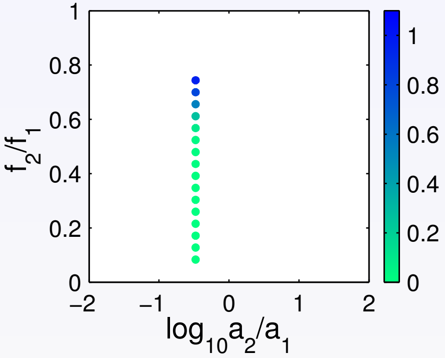
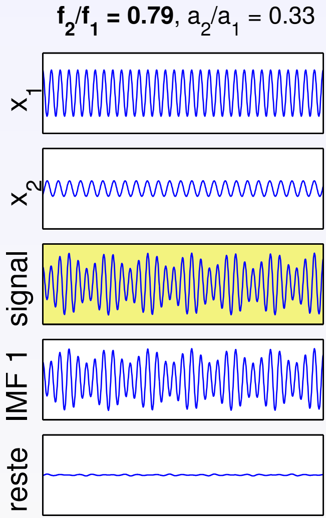


sum of two tones



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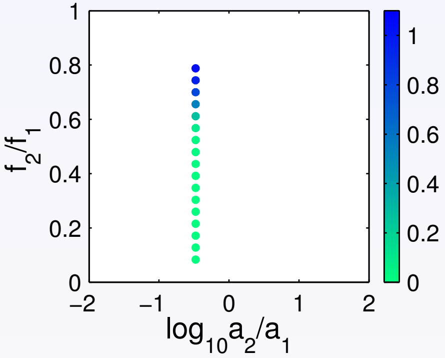
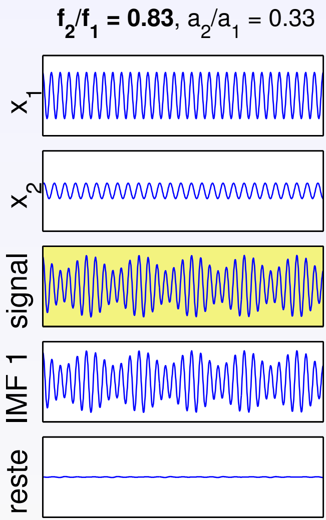


sum of two tones



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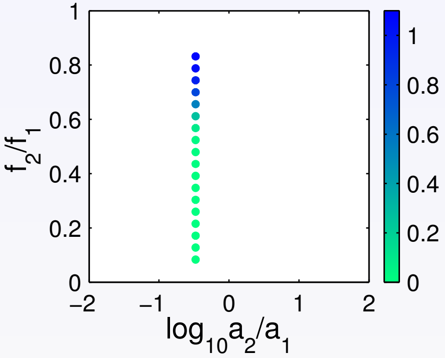
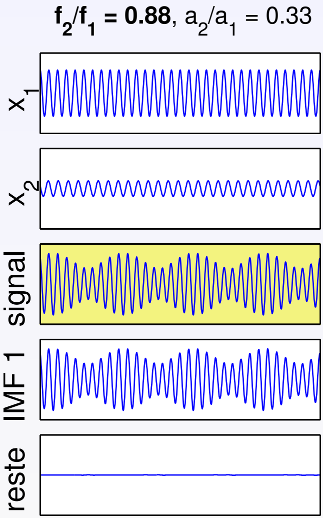


sum of two tones



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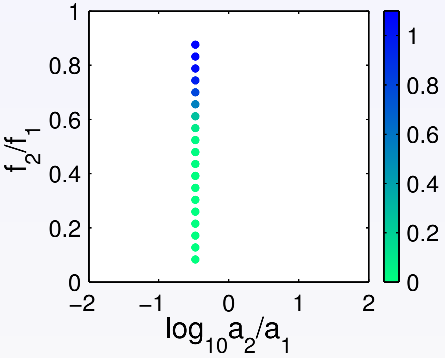
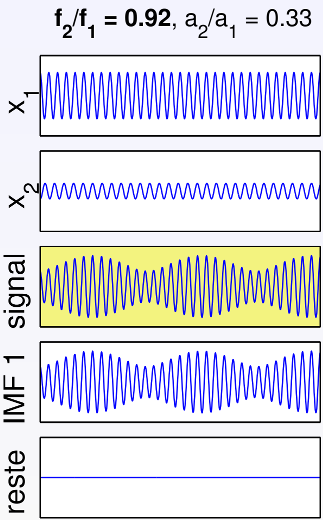


sum of two tones



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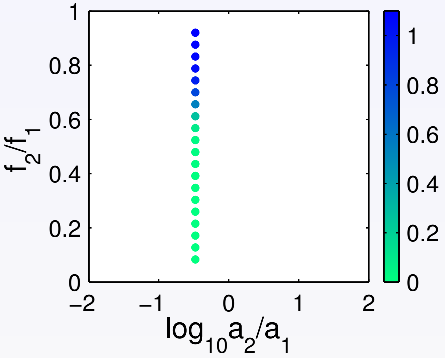

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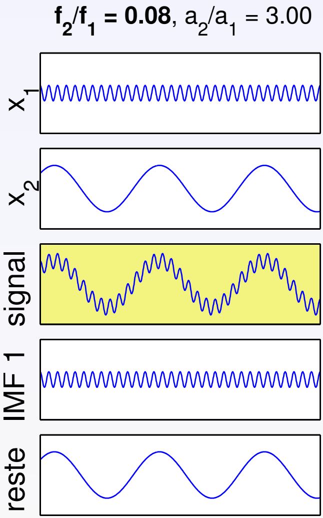

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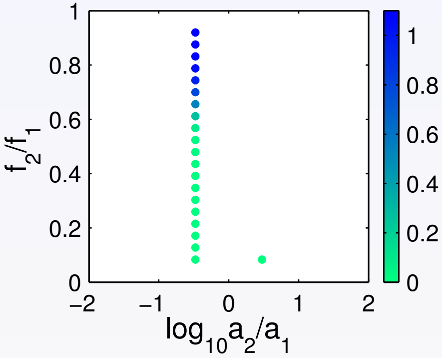

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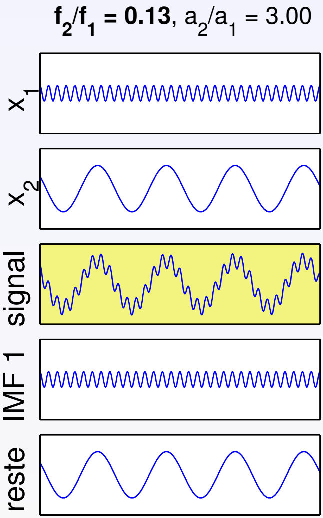

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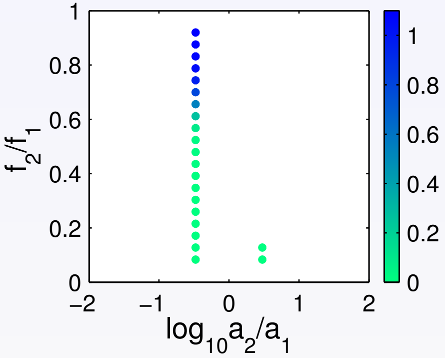

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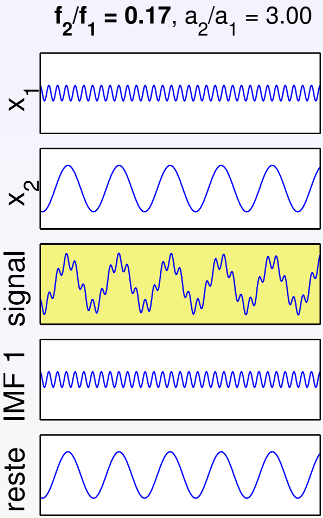

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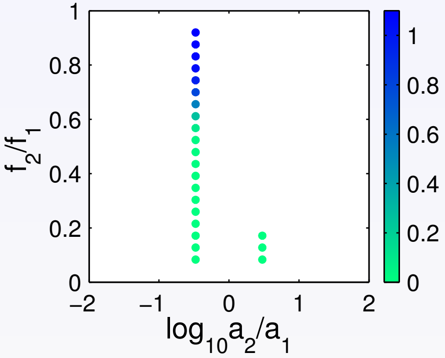

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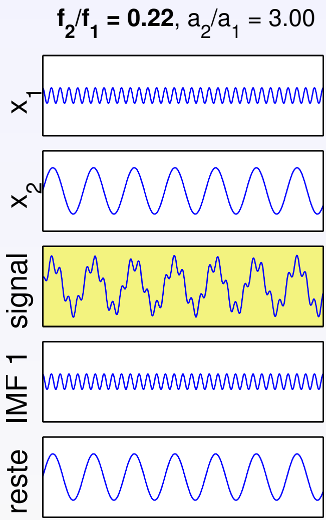

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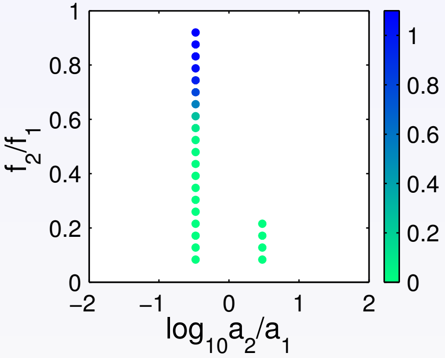

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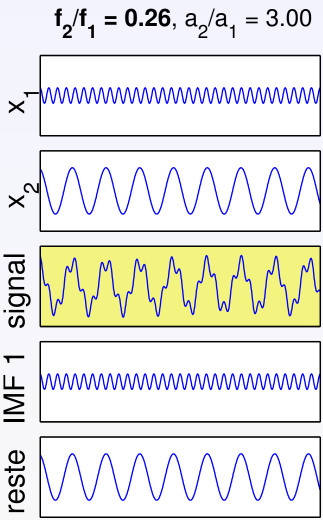

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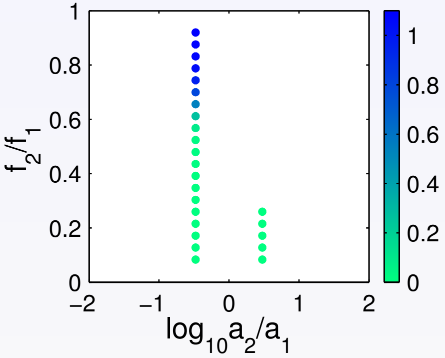

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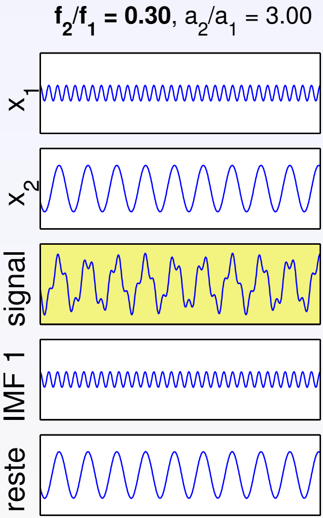

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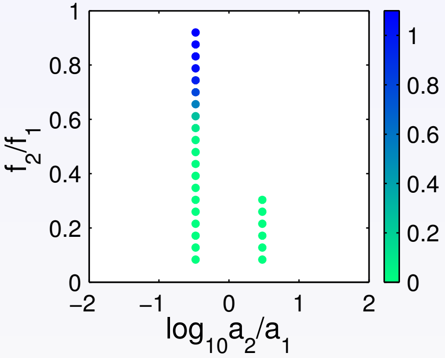

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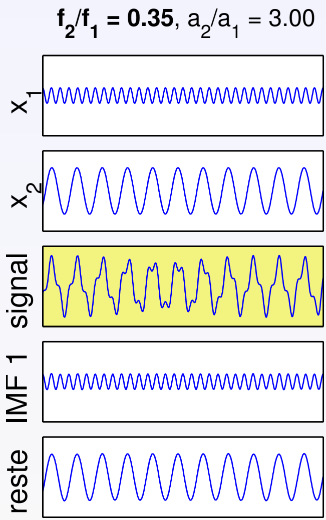

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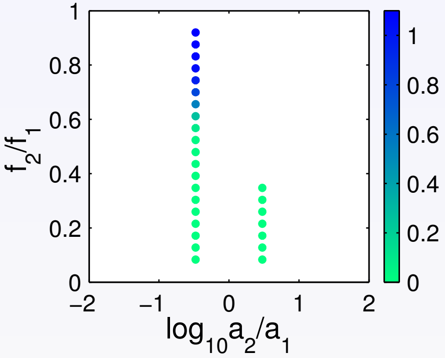

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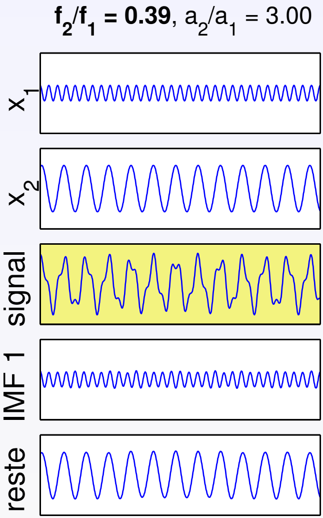

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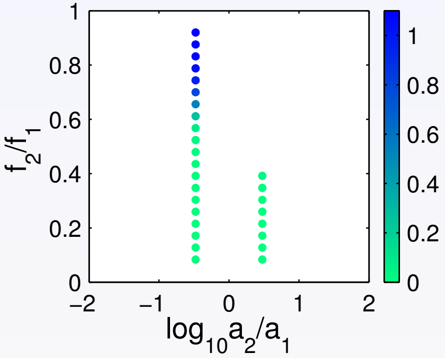

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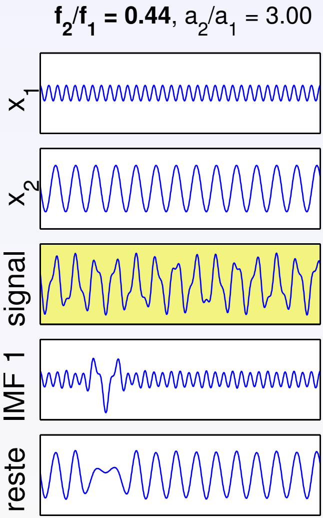

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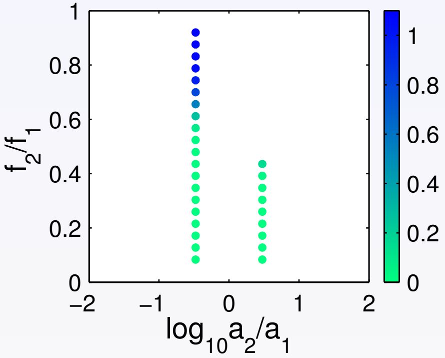

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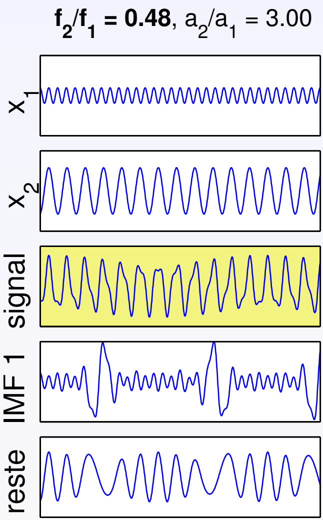

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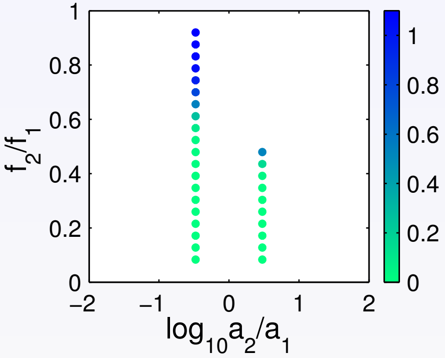

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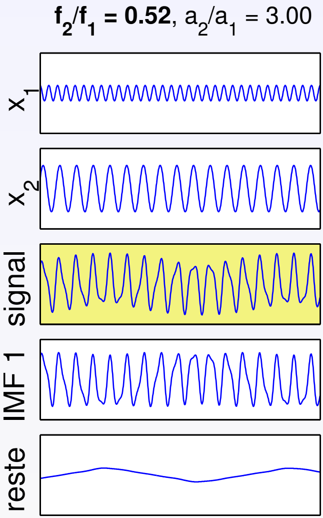

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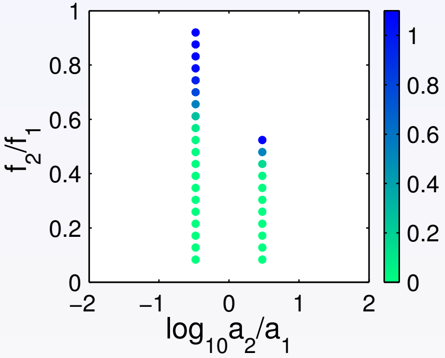

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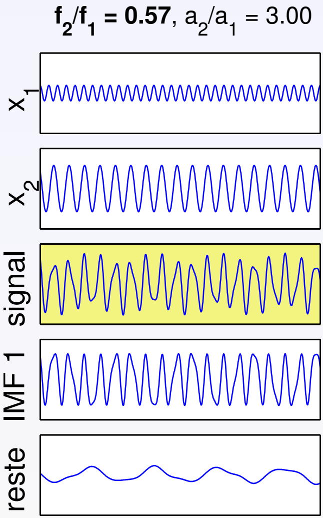

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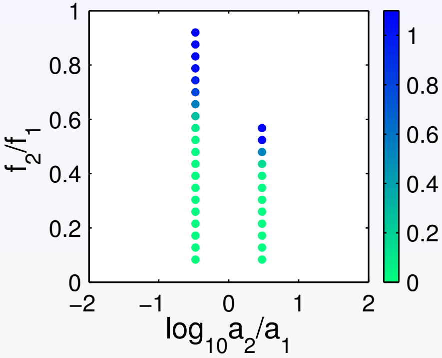

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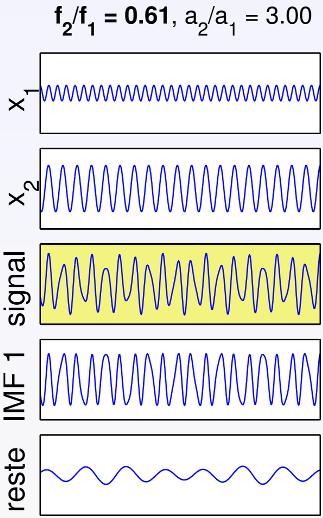

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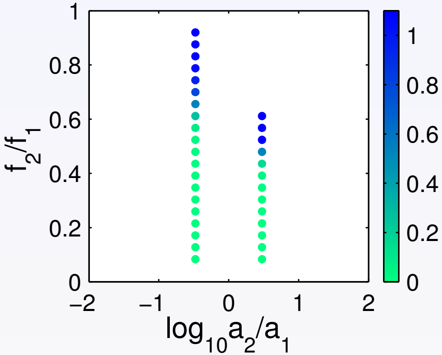

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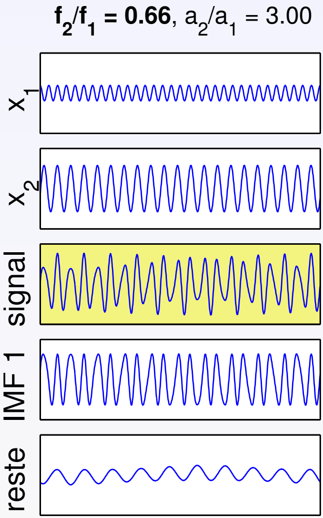

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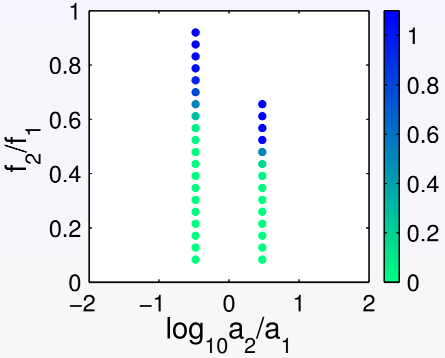

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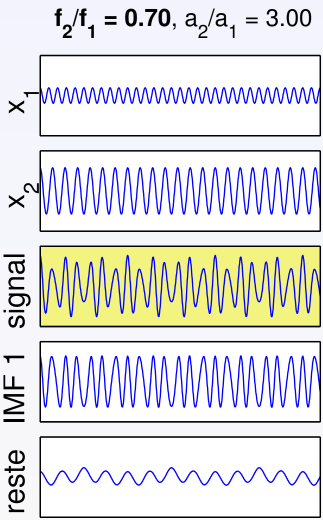

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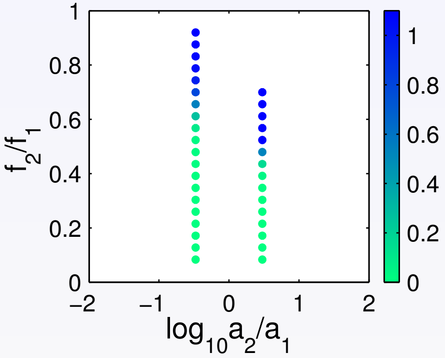

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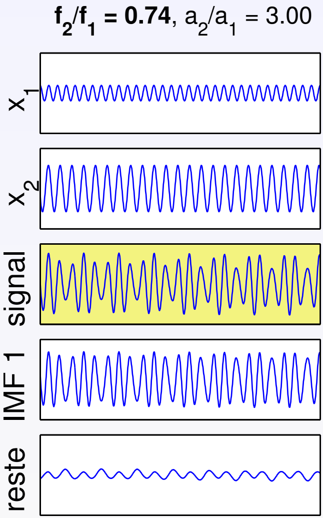

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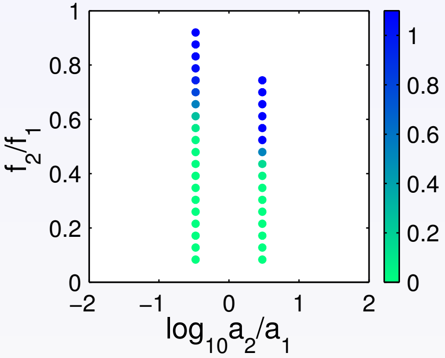

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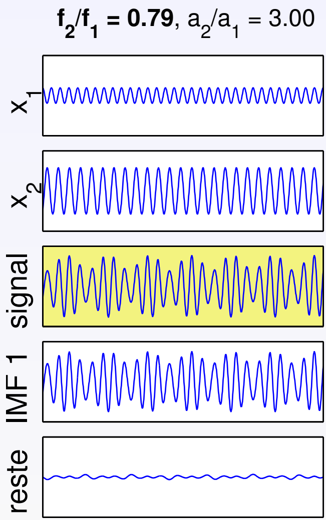

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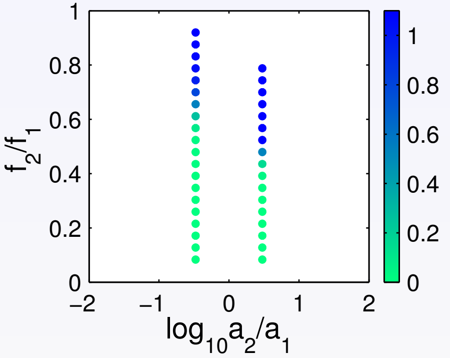
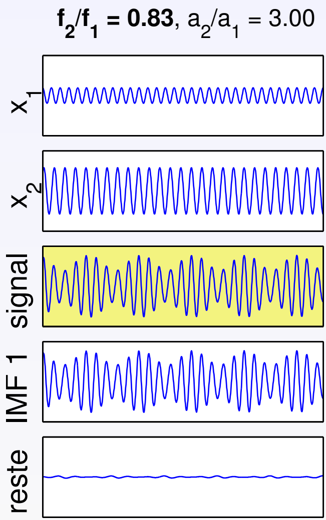


sum of two tones



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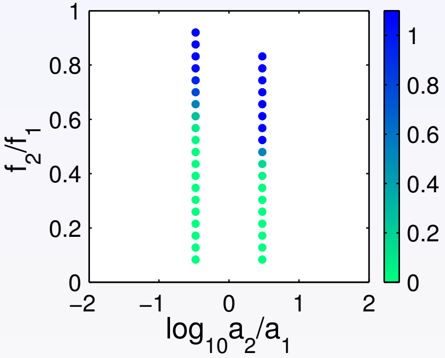
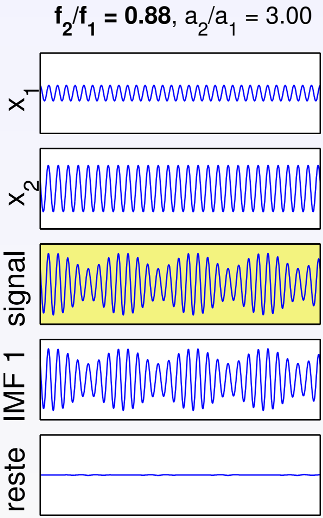


sum of two tones



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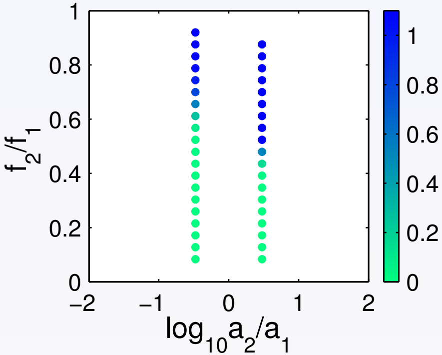

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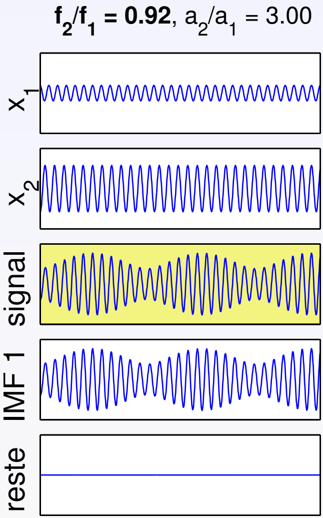

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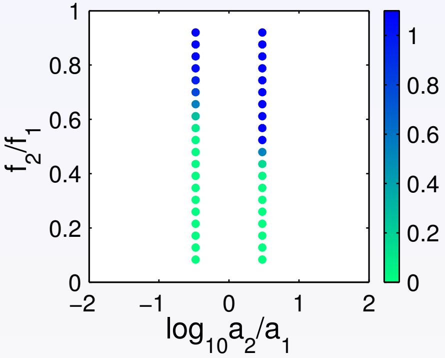

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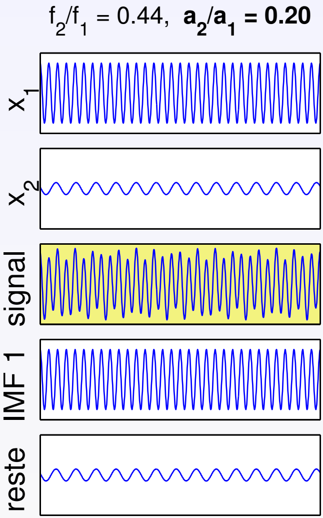

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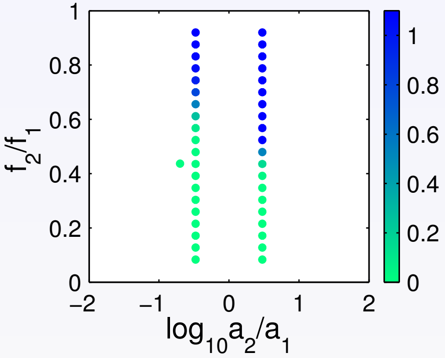

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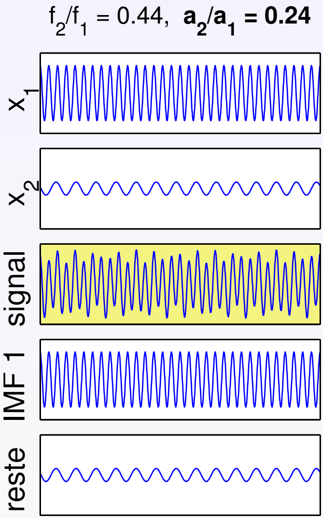

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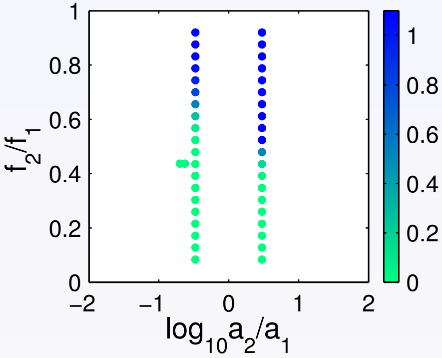

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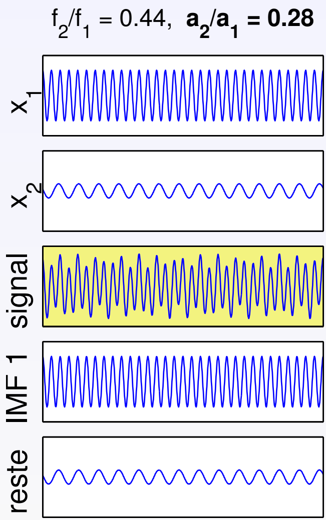

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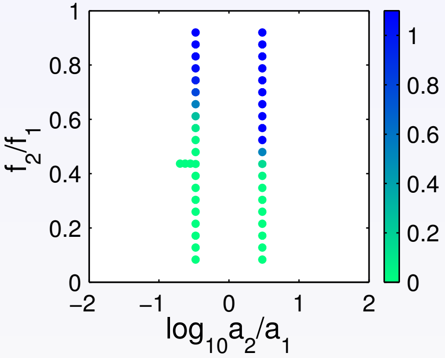

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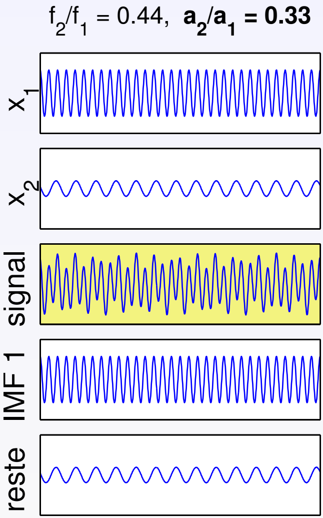

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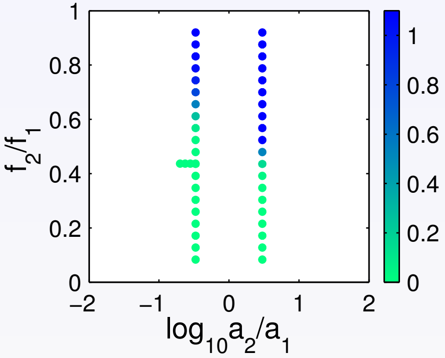
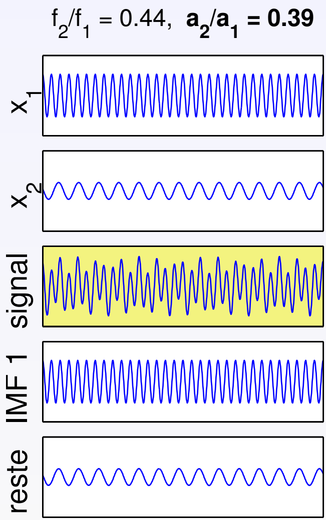


sum of two tones



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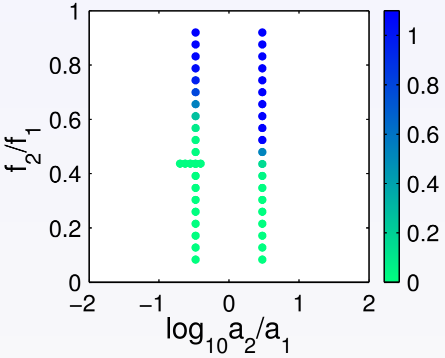
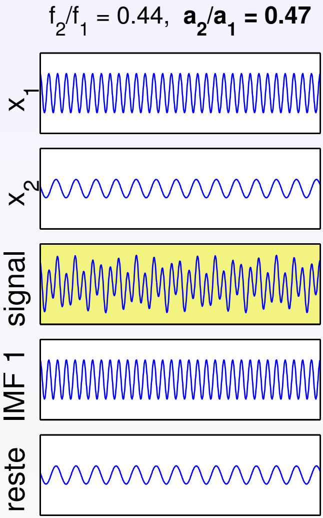


sum of two tones



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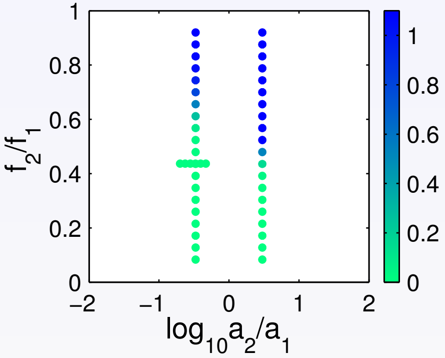
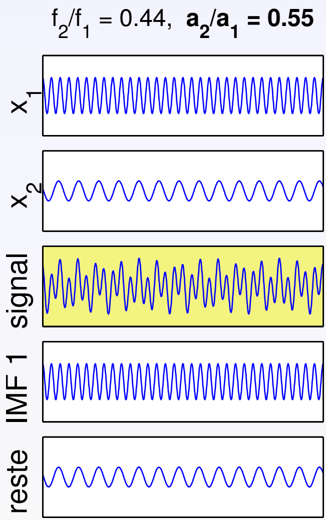


sum of two tones



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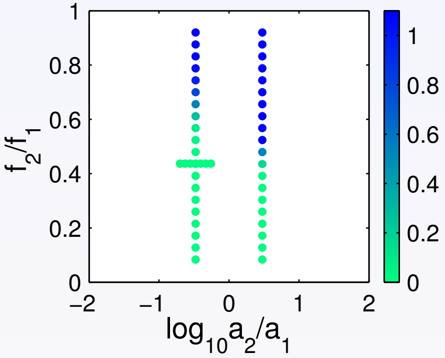
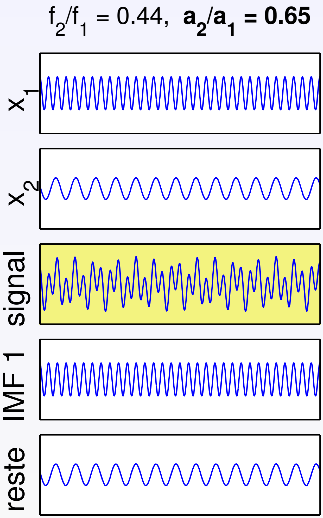


sum of two tones



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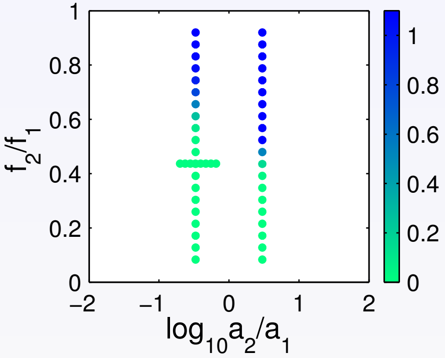
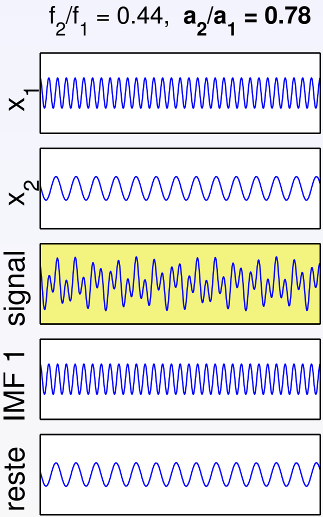


sum of two tones



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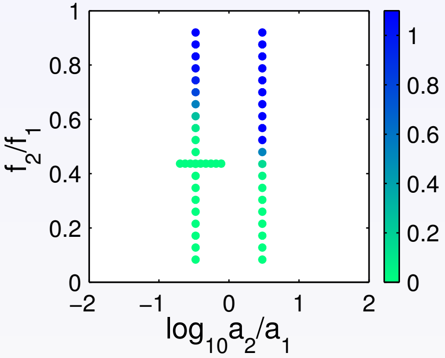

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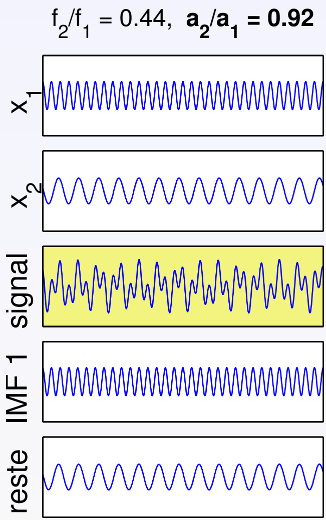

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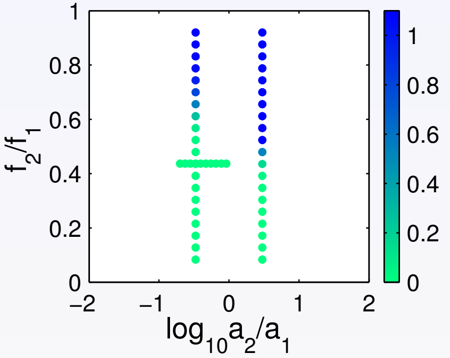
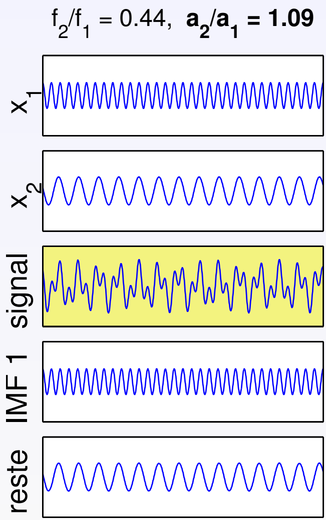


sum of two tones



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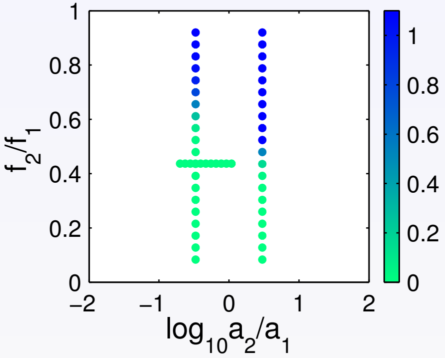
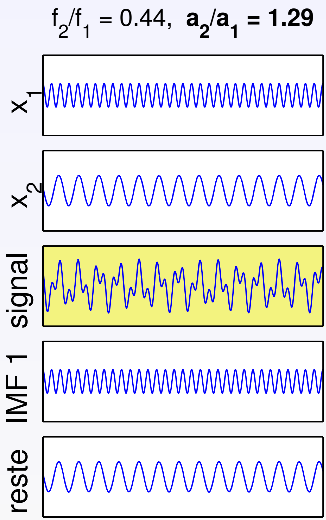


sum of two tones



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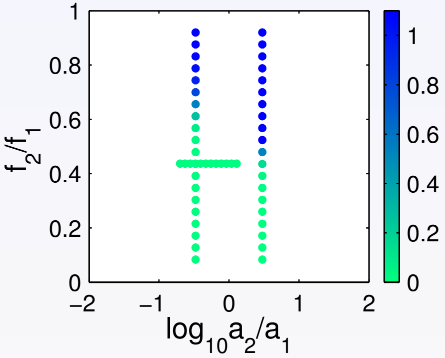
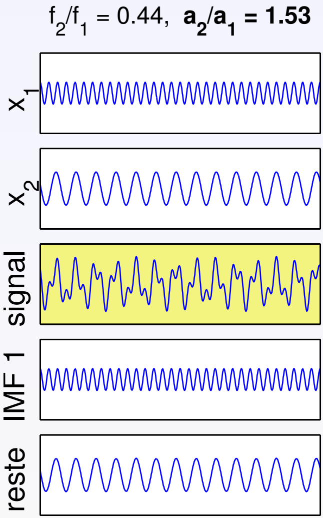


sum of two tones



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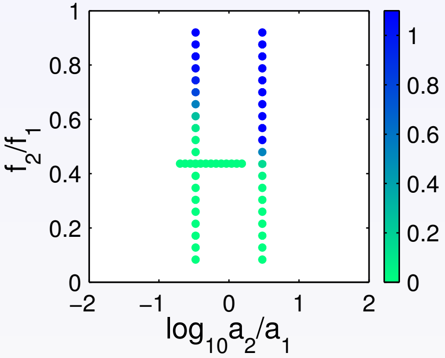
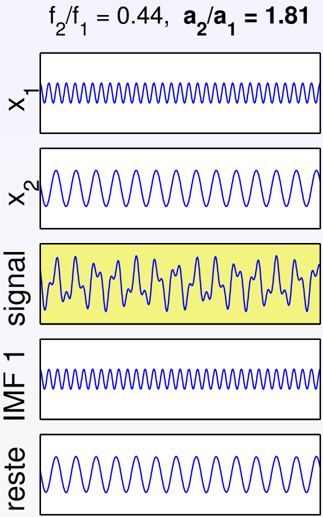


sum of two tones



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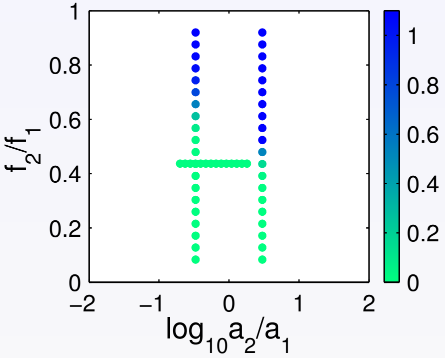

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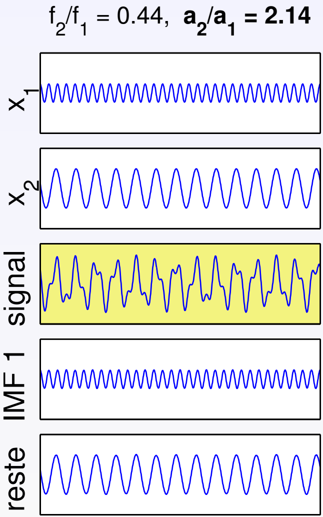

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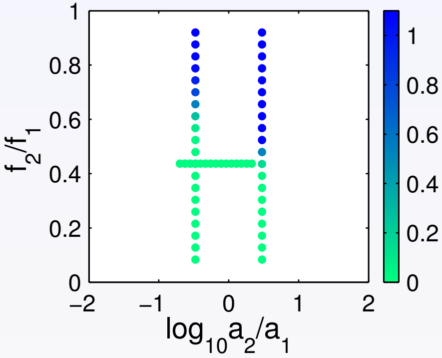

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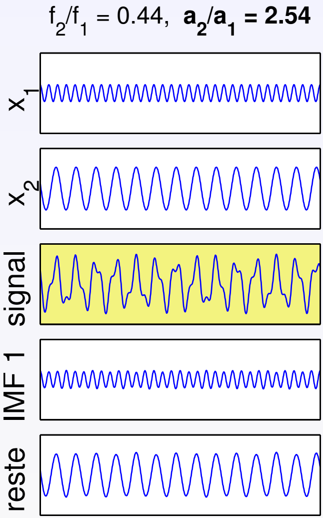

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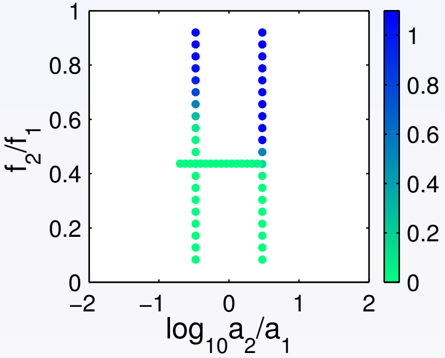
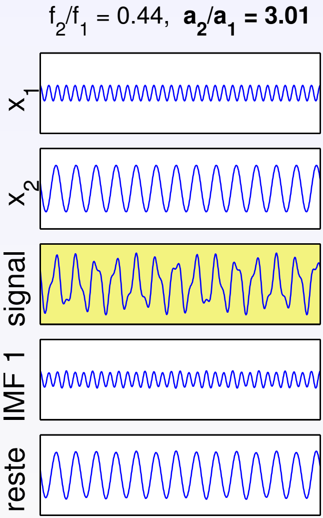


sum of two tones



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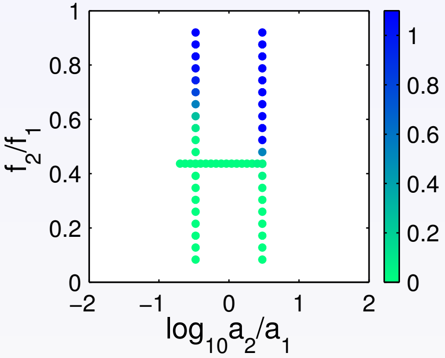

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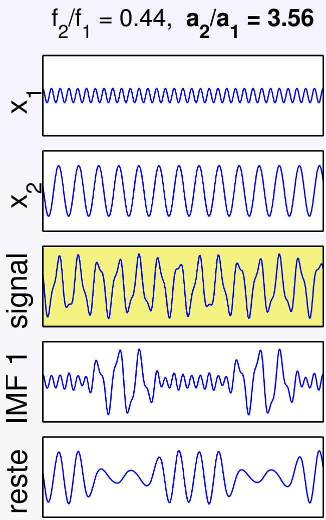

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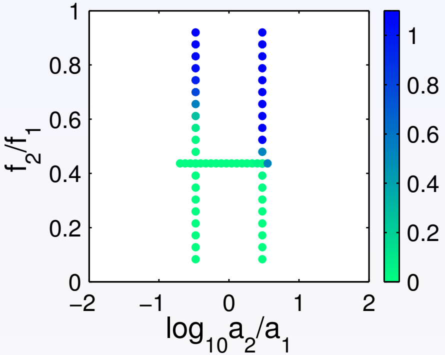
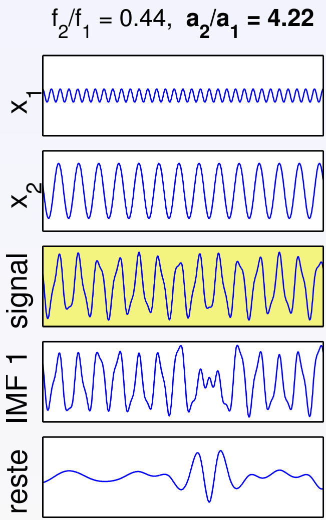


sum of two tones



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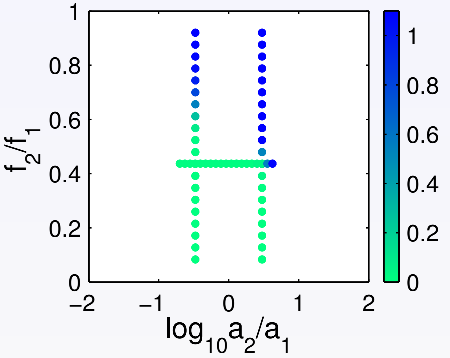
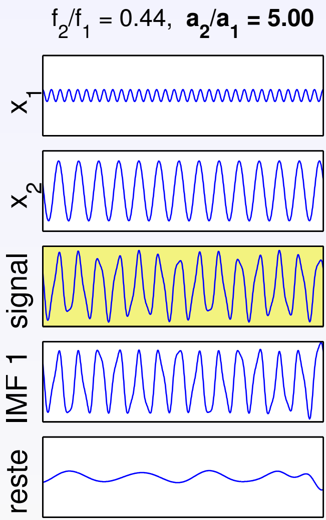


sum of two tones



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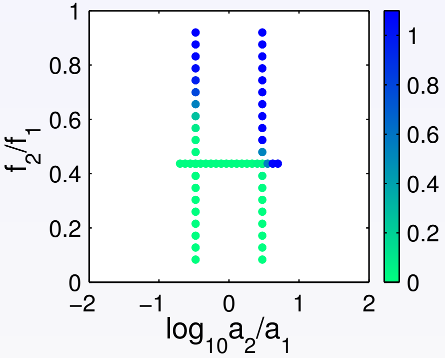

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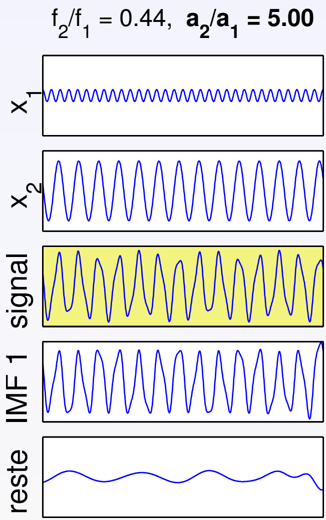

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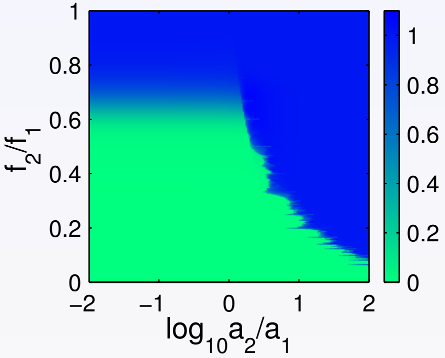

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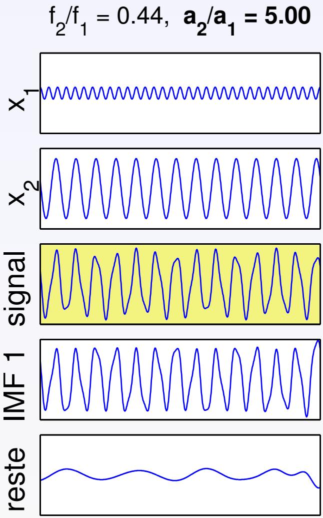

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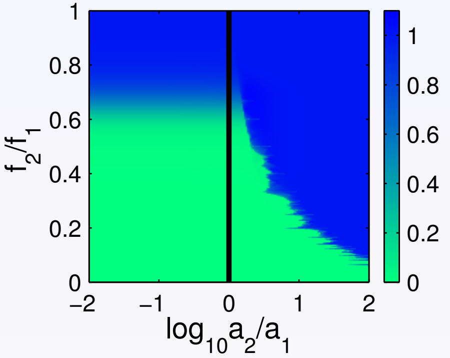
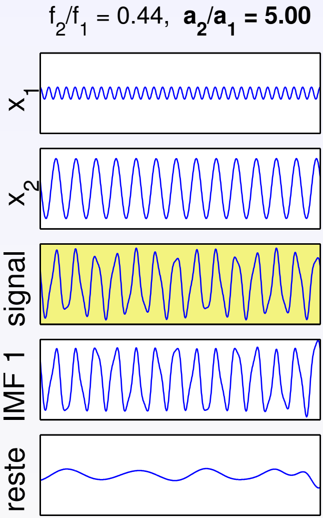


sum of two tones



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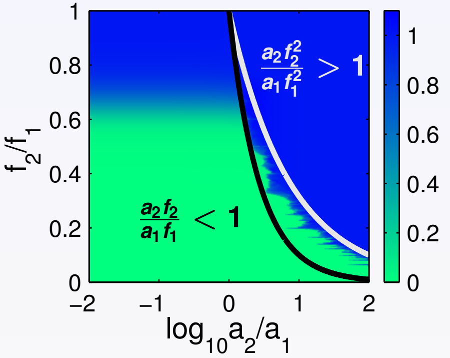
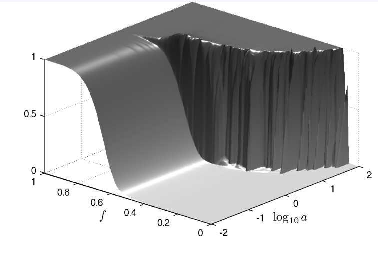


sum of two tones

- **Nonlinear** behaviour ⇒ dissymmetry of tones disentanglement w.r.t. amplitude ratio $a = a_2/a_1$, via the sign of $a - 1$:
    - *smooth variation* when $a < 1$ (HF dominant) & no a-dependence
    - *abrupt phase transition* when $a > 1$ (LF dominant) & strong a-dependence



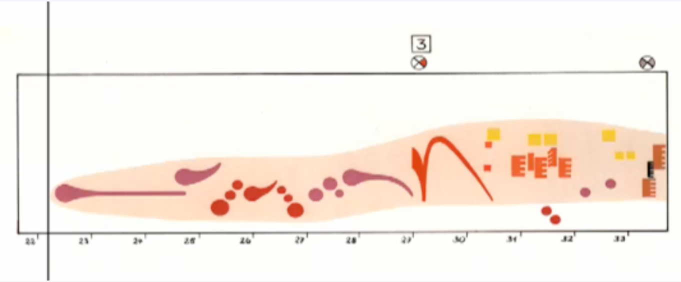
- **Data-driven** separation ⇒ good match to "beating effect" perception ⇒ connection with hearing?

concluding remarks

- Fourier: 200 years and still alive!
- basic ideas related to decompositions and frequency still central in “modern” approaches, whatever the variations (localized and/or evolutive tones, nonlinear techniques,. . .)
- time-frequency as a natural language

back to music notation

Rainer Wehinger' visual listening score created in the 70's to accompany Gyorgy Ligeti's *Artikulation*



http://www.youtube.com/watch?v=71hNl_skTZQ

(thanks to Laurent Chevillard & Gabriel Rilling)