An introduction to time-frequency distributions

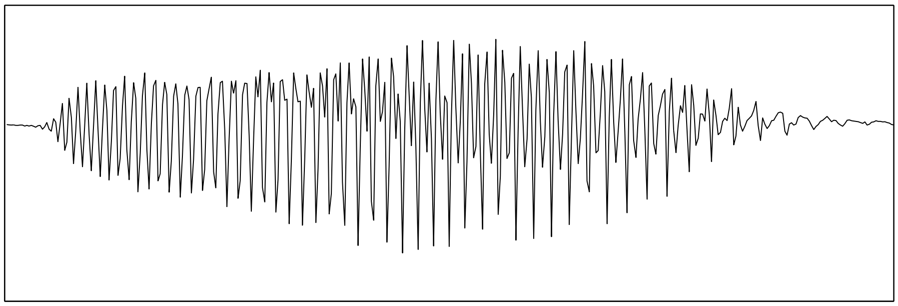
Patrick Flandrin*

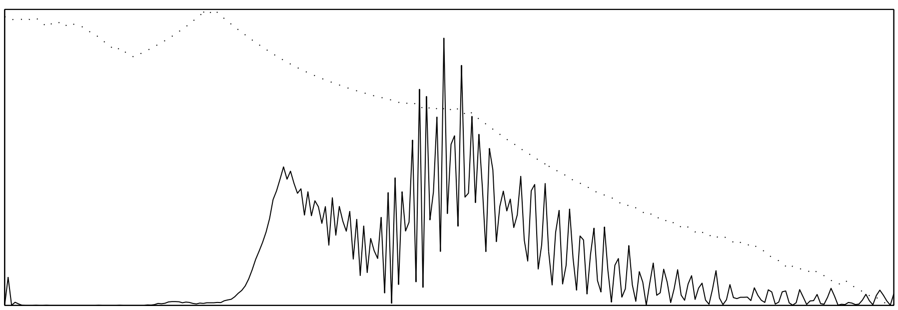
CNRS—Ecole Normale Supérieure de Lyon (France)

*thanks to E. Chassande-Mottin (Lyon/Nice), F. Auger (St-Nazaire), P. Gonçalvès (Grenoble) and I. Daubechies (Princeton)

Time or frequency ?

Example of a *bat echolocation call*:





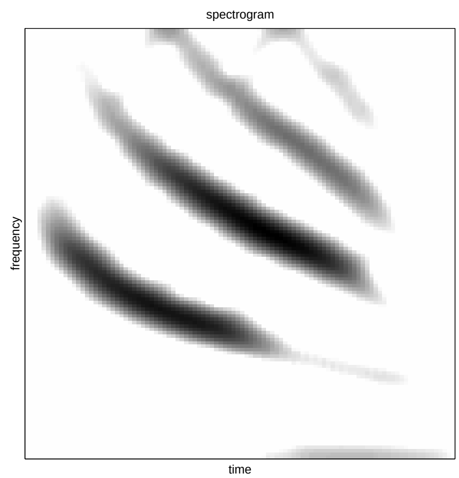
- *Same* information is displayed, but from two *mutually exclusive* perspectives

- Need for some *more meaningful* descriptions

"Mathematics vs. Physics"

Time and frequency

- Description is improved by using time and frequency *jointly*:



- Representation of a signal on a *"musical score"*

- "Time-frequency" appears as a natural *language* for nonstationary signals, but this calls for *mathematical definitions* supporting *physical interpretation*

Local Fourier analyses as a starting point

Making the Fourier transform *local* in time and/or frequency defines the same quantity $F_x^{(h)}(t,\omega)$, but with three possible interpretations:

1. *Short-time Fourier transform*:

$$F_x^{(h)}(t,\omega) = \int_{-\infty}^{\infty} \underbrace{x(s)\,\overline{h(s-t)}}_{\text{windowed signal}}\; e^{-i\omega(s-t/2)}\,ds$$

2. *Band-pass filtering*:

$$F_x^{(h)}(t,\omega) = \int_{-\infty}^{\infty} \underbrace{X(\xi)\,\overline{H(\xi-\omega)}}_{\text{filtered spectrum}}\; e^{it(\xi-\omega/2)}\,\frac{d\xi}{2\pi}$$

3. *Gabor's "logons"*:

$$F_x^{(h)}(t,\omega) = \langle x, h_{t,\omega}\rangle$$

$$h_{t,\omega}(s) := \underbrace{h(s-t)\,e^{i\omega(s-t/2)}}_{\text{time−frequency atom}}$$

Local methods and time-frequency localization

- *Signal* $x(t)$ is recovered in the limit of *infinitely narrow* windows $h(t) \to \delta(t)$

- *Spectrum* $X(\omega)$ is recovered in the limit of *infinitely narrow* filters $H(\omega) \to \delta(\omega)$

- *Joint* localization is limited by *Heisenberg's inequality*

$$\inf_{t_0,\omega_0} \|(t-t_0)\,x\|_2 \; \|(\omega-\omega_0)\,X\|_2 \geq \frac{1}{2},$$

  with equality for Gaussians

- Perfect localization on *chirps* would call for *locally adapted* (signal-dependent) windows/filters/atoms

Self-adaptation in local methods

- Taking for the window the *time-reversed signal* itself (*matched filter* principle) leads to
$$F_x^{(x-)}(t,\omega) = \frac{1}{2} W_x\left(\frac{t}{2},\frac{\omega}{2}\right),$$
with
$$W_x(t,\omega) = \int x\left(t+\frac{\tau}{2}\right)\overline{x\left(t-\frac{\tau}{2}\right)}\, e^{-i\omega\tau}\, d\tau$$
the *Wigner(-Ville) distribution* (Wigner, '32; Ville, '48)

- Perfect localization of the Wigner distribution on *lines* of the time-frequency plane:
$$x(t) = \exp\{i\alpha t^2/2\} \Rightarrow W_x(t,\omega) = \delta\left(\omega - \alpha t\right)$$

- *Quadratic* transform (energy distribution)

A geometrical interpretation

- For a phase signal $x(t) = \exp\{i\varphi(t)\}$, whose "instantaneous frequency" is $\omega_x(t) = d\varphi/dt$, the Wigner distribution is, at each time t, the Fourier transform of the phase signal $\exp\{i\Phi_t(\tau)\}$, with

$$\Phi_t(\tau) := \varphi\left(t + \frac{\tau}{2}\right) - \varphi\left(t - \frac{\tau}{2}\right)$$

- This new signal has for "instantaneous frequency"

$$\frac{\partial}{\partial\tau}\Phi_t(\tau) = \frac{1}{2}\left[\omega_x\left(t + \frac{\tau}{2}\right) + \omega_x\left(t - \frac{\tau}{2}\right)\right],$$

  a quantity which *exactly* coincides with $\omega_x(t)$ if and only if $\varphi(t)$ is at most quadratic (*linear chirps*)

Two consequences

- Localization from *quadratic phase compensation*

- *Quadratic superposition principle*:

$$W_{ax+by} = |a|^2 W_x + |b|^2 W_y + I,$$

  with I an *oscillating term* which lies *midway* between the interacting components

Janssen's *interference formula* (Janssen, '82):

$$|W_x(t,\omega)|^2 =$$

$$\iint W_x\left(t+\frac{\tau}{2},\omega+\frac{\xi}{2}\right) W_x\left(t-\frac{\tau}{2},\omega-\frac{\xi}{2}\right) \frac{dt\,d\omega}{2\pi}$$

Localization revisited: *a line is the only curve of the plane which is defined as the locus of all of its midpoints*

Localization on nonlinear curves

- Idea: localization is based on a *constructive interference* principle

- Application: *modified* "midpoint geometries" may lead to modified Wigner distributions with localization properties on nonlinear curves of the plane (F. & Gonçalvès, '96)

- Example: localization on *power-law* group delays:

$$t_X(\omega) = t_0 + c\,\omega^{k-1}, k \leq 0$$

  can be achieved in the class of *affine* Bertrand distributions (Bertrand & Bertrand, '92)

From Wigner to Bertrand

$$W_X(t,\omega) =$$

$$\int \underbrace{X\left(\omega + \frac{\xi}{2}\right)}_{\text{shift}\,+} \underbrace{\overline{X\left(\omega - \frac{\xi}{2}\right)}}_{\text{shift}\,-} \underbrace{e^{i\omega t}\,\frac{d\omega}{2\pi}}_{\text{Fourier}}$$

$$\downarrow$$

$$B_X^{(k)}(t,\omega) =$$

$$f \int \underbrace{X\left(\omega\lambda_k(u)\right)}_{\text{dilation}} \underbrace{\overline{X\left(\omega\lambda_k(-u)\right)}}_{\text{compression}} \underbrace{\mu_k(u)\, e^{i\omega t u}\, du}_{\text{weigthed Fourier}}$$

Bertrand distributions are defined for *analytic signals* and $B_X^{(2)} = W_X$

In the limit of *narrowband signals*, $B_X^{(k)} = W_X$ for all k's

Beyond Wigner and Bertrand — 1.

- *Covariance* principles applied to *quadratic* transforms lead to *classes* of distributions

- Shifts in time and frequency $\rightarrow$ Cohen's class (Cohen, '66)

$$C_x(t,\omega) = \int\int W_x(s,\xi)\,\Pi(s-t,\xi-\omega)\,\frac{ds\,d\xi}{2\pi}$$

- Shifts in time and dilations $\rightarrow$ affine classes (Bertrand and Bertand, '92, Rioul and F., '92)

$$\Omega_x(t,\omega) = \int\int W_x(s,\xi)\,\Pi(\omega(s-t),\xi/\omega)\,\frac{ds\,d\xi}{2\pi}$$

Beyond Wigner and Bertrand — 2.

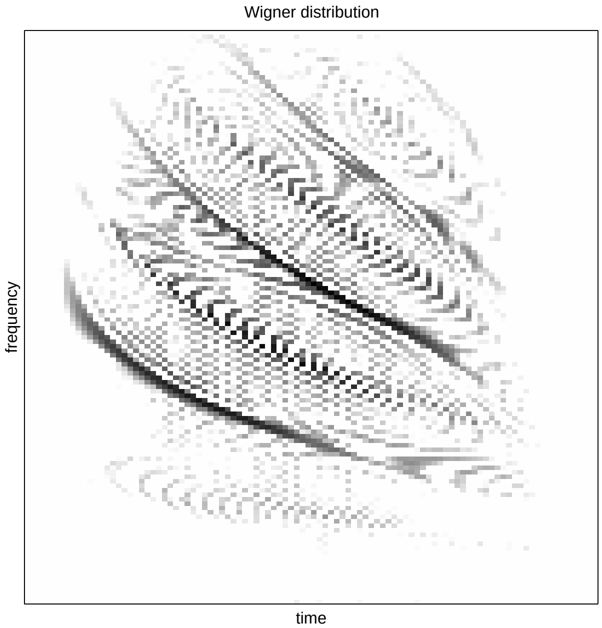
- Distributions are “parameterized” by an “arbitrary” function Π

- Specific distributions may be tailored to specific required properties

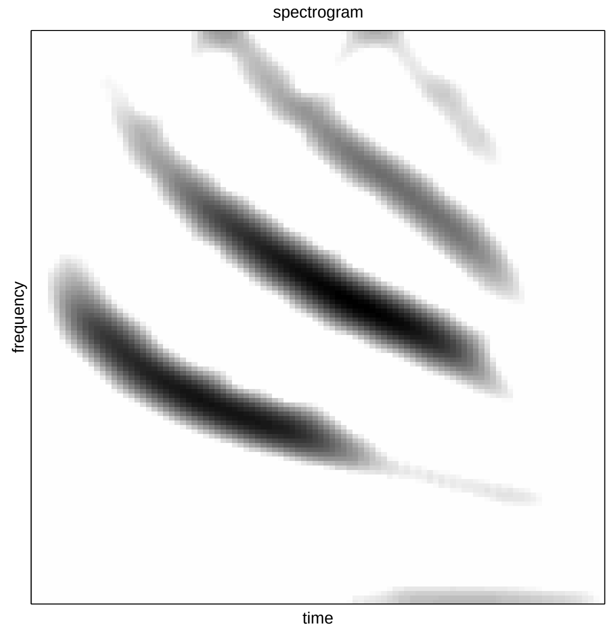
- In most cases, generalized distributions are *smoothed* versions of (localizable) mother distributions $\Rightarrow$ lower time-frequency resolution

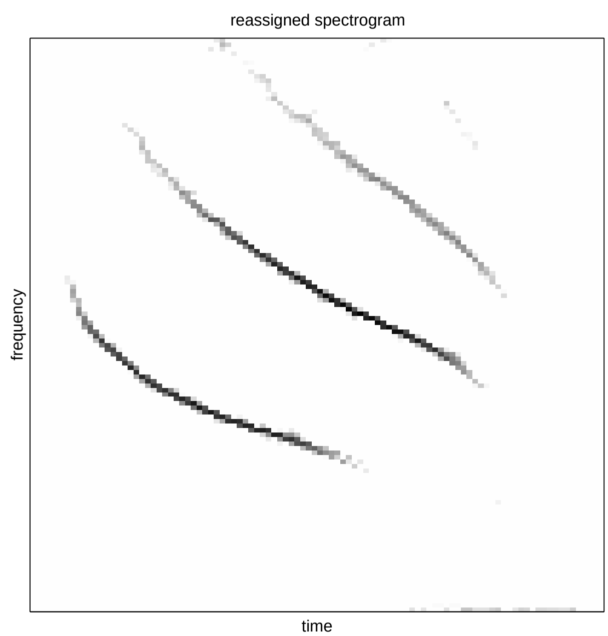
Back to the bat chirp

Wigner



spectrogram





reassigned spectrogram

Spectrograms and Wigner

- A *spectrogram* is the squared magnitude of a short-time Fourier transform

  $$\left|F_x^{(h)}(t,\omega)\right|^2 = \left|\int x(s)\,\overline{h(s-t)}\,e^{-i\omega(s-t/2)}\,ds\right|^2,$$

  with $h(t)$ a *low-pass* analyzing window

- An equivalent definition can be given as

  $$\left|F_x^{(h)}(t,\omega)\right|^2 = \iint W_x(s,\xi)\,W_h(s-t,\xi-\omega)\,\frac{ds\,d\xi}{2\pi}$$

  with W_X the Wigner distribution

The interpretation

is that spectrograms are *smoothed* Wigner distributions

Summary

- *linear* short-time Fourier transforms, and therefore *squared* linear transforms (spectrograms) cannot be sharply localized

- truely *quadratic* (Wigner-type) transforms can be localized, but create cross-terms between different components

- *trade-off* between localization and cross-terms

The objective

is to get, *simultaneously*, the sharp localization of truely quadratic transforms and the low level of cross-terms of squared linear transforms

A solution

is to make use of the nonlinear technique of *reassignment*, introduced by Kodera *et al.* in the mid-70's

A mechanical analogy

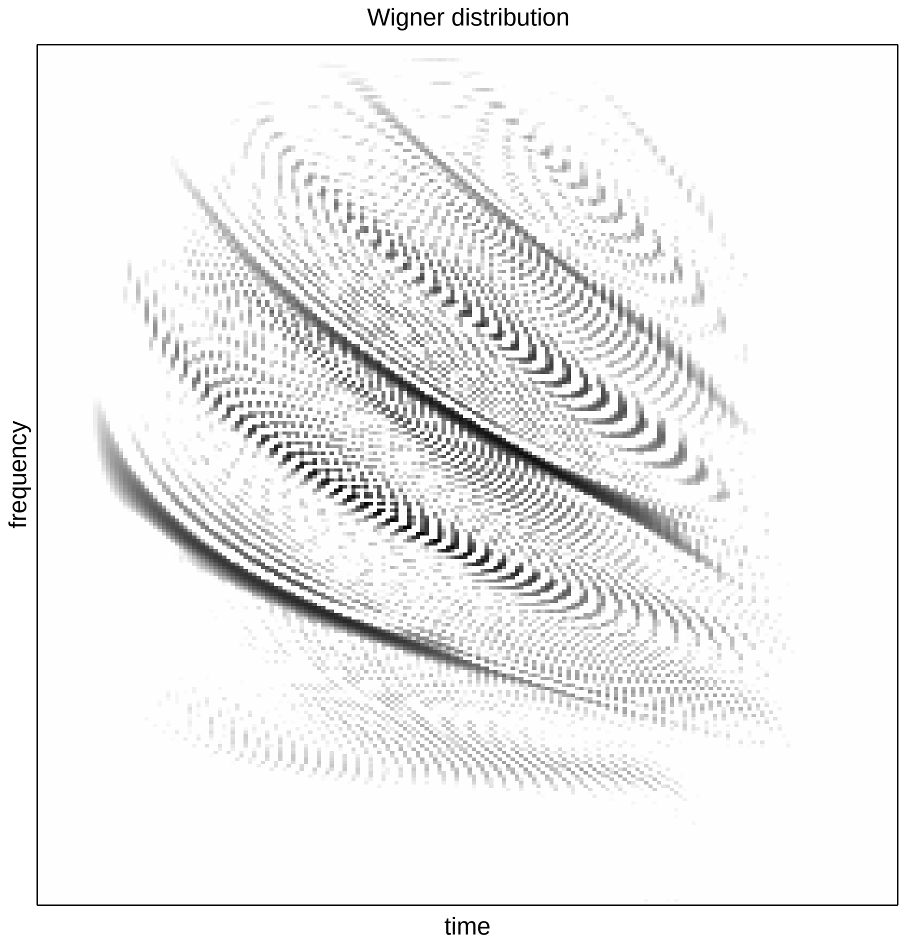
The spectrogram smoothing operator acts locally over a small domain of the time-frequency plane, namely the essential support of the Wigner distribution of the window at the considered location

Thinking of the Wigner distribution of the signal within this domain as a distribution of mass, evaluating a spectrogram at a given point amounts to

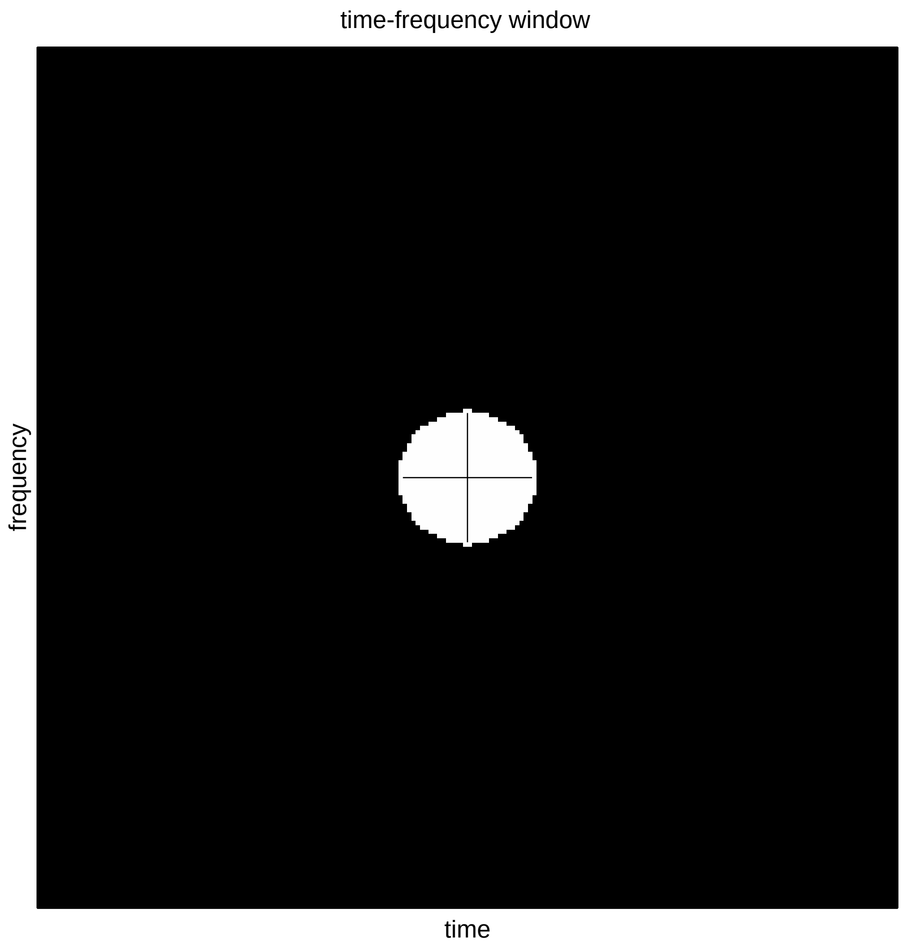
1. reducing the local mass distribution to *one* number (the total mass) by summing up all contributions in the domain

2. assigning this number to the *geometrical* center of the domain

Smoothing the Wigner distribution



Smoothing the Wigner distribution



The idea of reassignment

is to replace the *geometrical* center of the domain by the *center of gravity* of the distribution within the domain and, therefore, to *reassign* computed values of the smoothed distribution to local centroïds

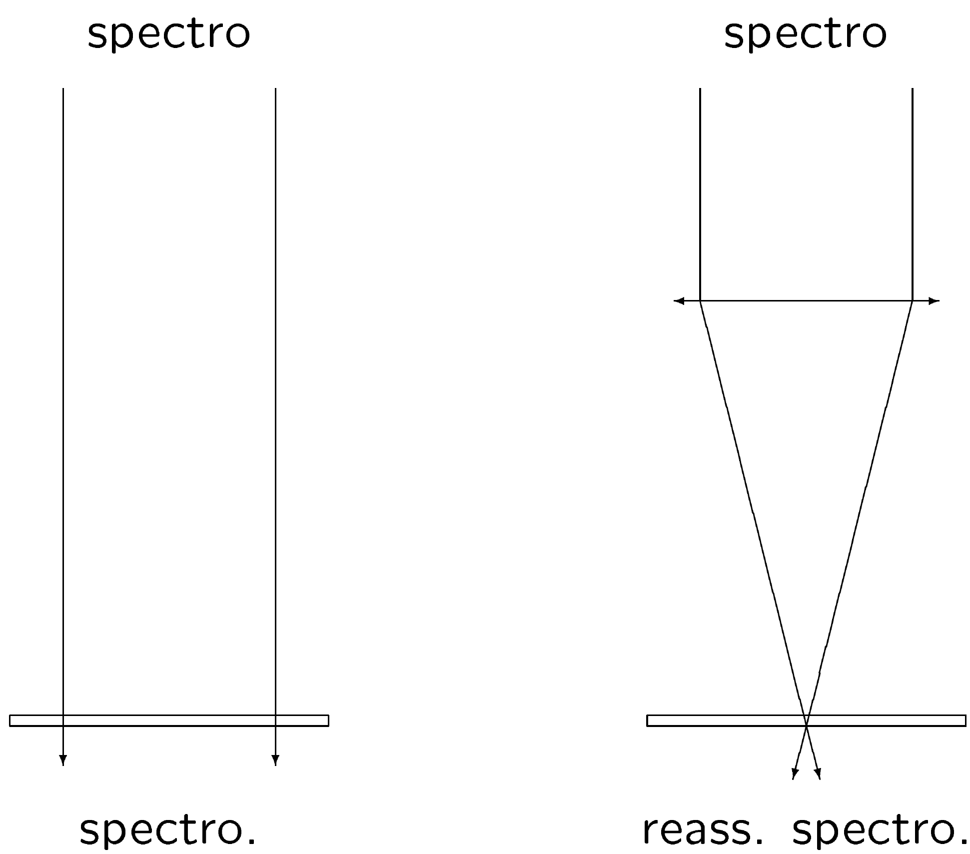
$$S_x^{(h)}(t, \omega)$$

$$\downarrow$$

$$\iint S_x^{(h)}(s, \xi)\, \delta\left(t - \hat{t}_x(s, \xi), \omega - \hat{\omega}_x(s, \xi)\right) \frac{ds\, d\xi}{2\pi}$$

Remark — Reassignment has been originally introduced (by Kodera *et al.*) for spectrograms only, but it applies in fact to *any* distribution which results from a smoothing of some localizable mother distribution: Cohen's class based on Wigner, affine class based on Bertrand, hyperbolic class based on Altes, ... (Auger & F., '95)

An optical analogy



"Lens" is *local* and *signal-dependent*: *adaptive optics*

Focus on *lines* of the image plane: *caustics* for nonlinear chirps

Reassignment in practice

For each time-frequency point (t, ω), local centroïds $\hat{t}_x(t, \omega)$ and $\hat{\omega}_x(t, \omega)$ have to be evaluated

In the case of spectrograms, we have (Auger & F., '95)

$$\hat{t}_x(t, \omega) = t + \text{Re}\left\{\frac{F_x^{(\mathcal{T}h)}}{F_x^{(h)}}\right\}(t, \omega)$$

and

$$\hat{\omega}_x(t, \omega) = \omega - \text{Im}\left\{\frac{F_x^{(\mathcal{D}h)}}{F_x^{(h)}}\right\}(t, \omega),$$

with $(\mathcal{T}h)(t) = t\, h(t)$ and $(\mathcal{D}h)(t) = (dh/dt)(t)$.

Similar relations hold for scalograms.

As compared to a conventional spectrogram, a reassigned spectrogram amounts to computing three short-time Fourier transforms instead of one (and two only with Gaussian windows)

Perfect localization

Reassigned distributions localize as *perfectly* as the unsmoothed distribution on which they are based

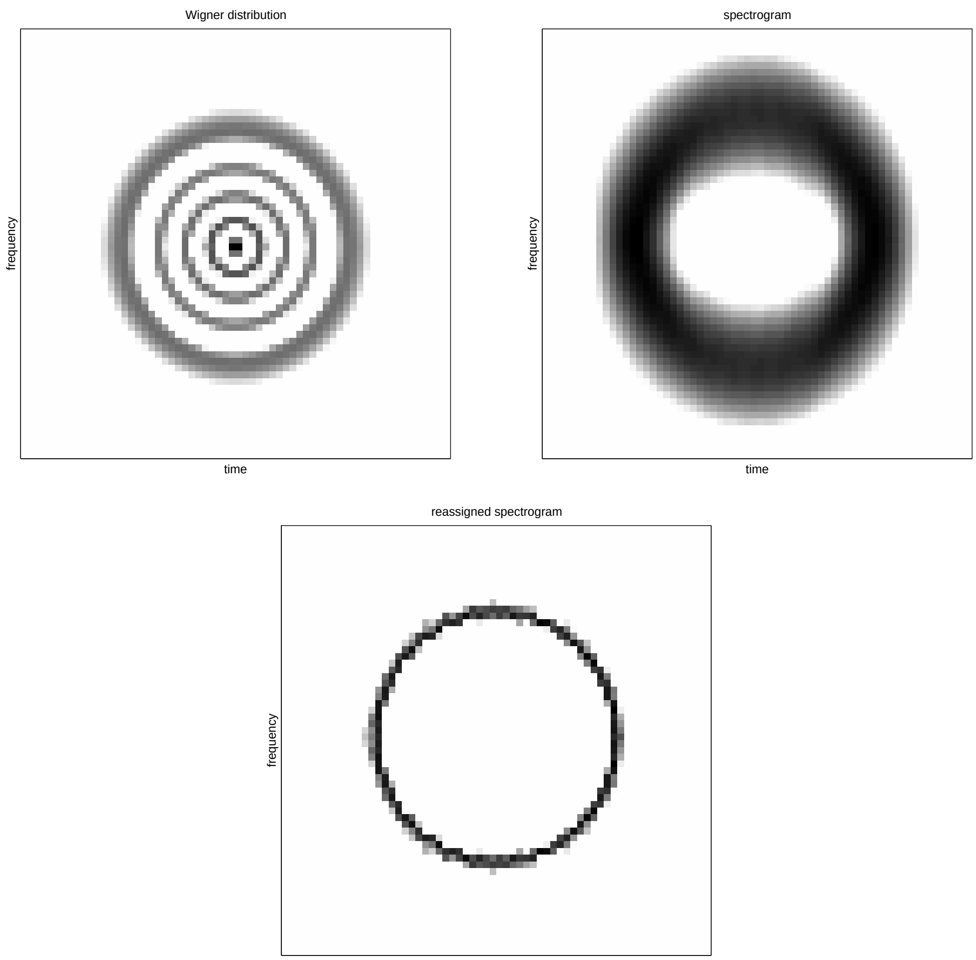
For reassigned smoothed Wigner distributions, localization is on *lines* of the time-frequency plane:

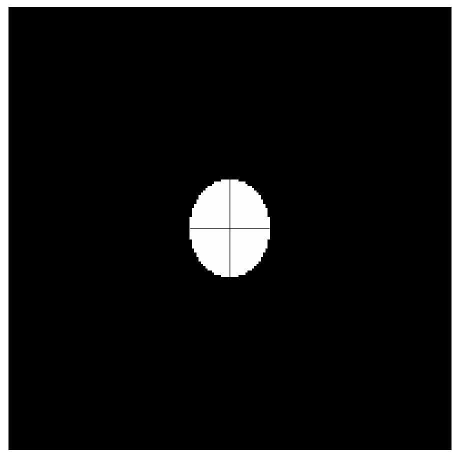
- *impulses* $\rightarrow \delta(t - t_0)$
- *pure tones* $\rightarrow \delta(\omega - \omega_0)$
- *linear chirps* $\rightarrow \delta(\omega - \alpha t)$

Approximate localization

Localization is still almost perfect as long as a chirp approximation is *locally* valid, within the time-frequency smoothing window

Example of a Hermite function





A new trade-off for noisy signals

- In the vicinity of a signal component, reassignment is *good*, since it reinforces localization

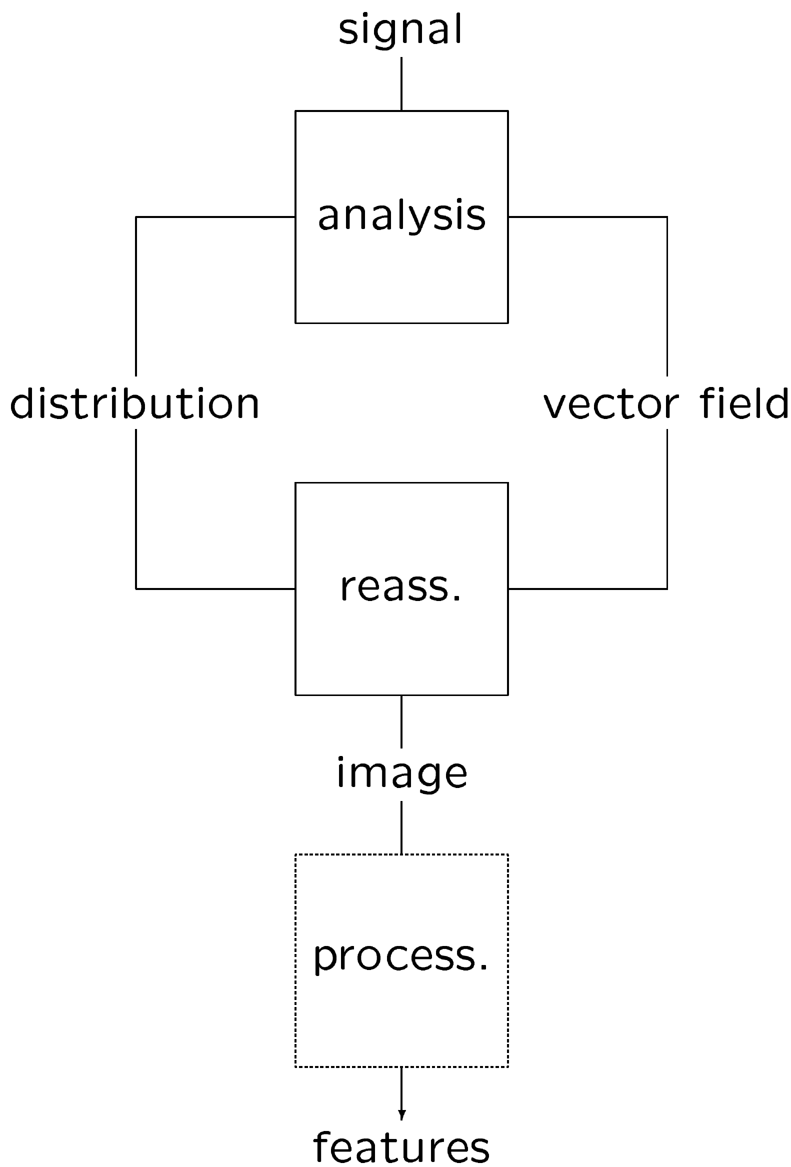
- In (broadband) noise-only regions, reassignment is *bad*, since it reinforces local peaks which depend on the realization

Idea

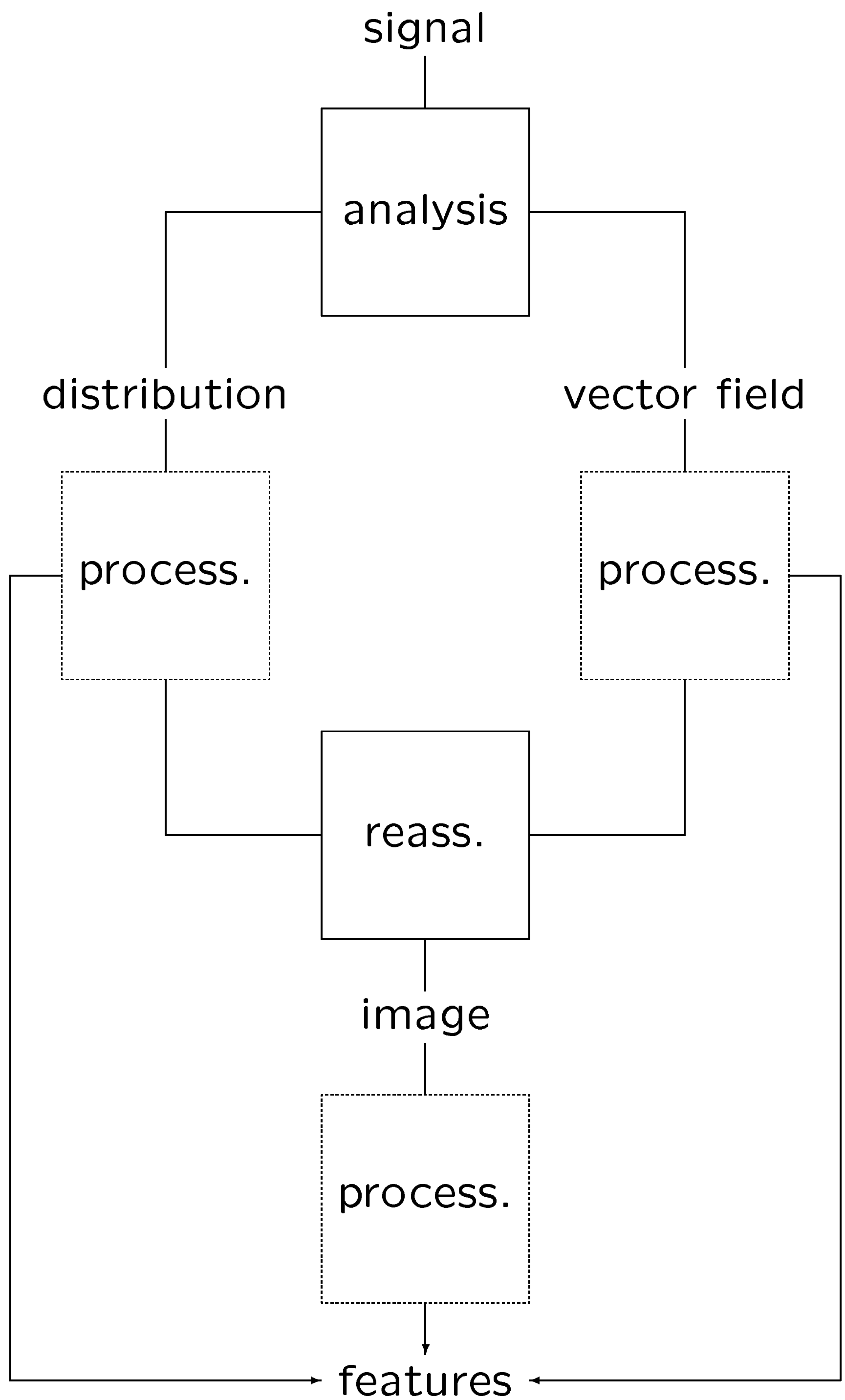
1. Identify noise-only regions

2. Inhibate reassignment in those regions

Reassignment for analysis and processing
— 1. —



Reassignment for analysis and processing — 2. —



Examples

- supervised reassignment
- differential reassignment
- time-frequency partitioning
- chirp detection

Concluding remarks

Time-frequency localization can be given simple *geometric* interpretations

Time-frequency localization is faced with *trade-offs* related to "uncertainty principles"

Reassignment is an *effective* and *easy* way to improve localization and readability of time-frequency distributions

A freeware Matlab toolbox is available at the URL

`http://crttsn.univ-nantes.fr/ãuger/tftb.html`