Fourier + 200

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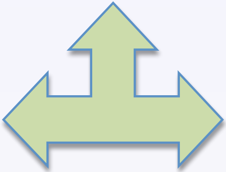
signal processing as a “3-body system”

« physics »

(laws of Nature, real world applications)

« mathematics »

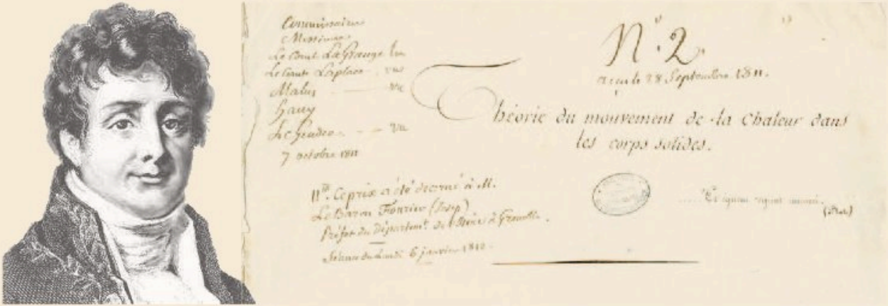
(models, proofs)



« computer science »

(algorithms)

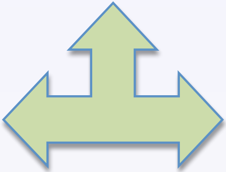
Fourier (1811)



the Fourier example

« physics »

(heat equation)



« mathematics »

(harmonic analysis)

« computer science »

(Fast Fourier Transform)

analysis/synthesis

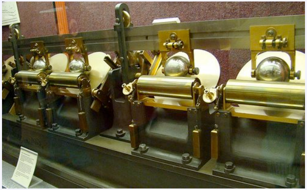
Fourier decomposition based on $e_f(t) := \exp\{i2\pi ft\}$

$$x(t) \to X(f) = \langle x, e_f \rangle, \ s.t. \ x(t) = \int \langle x, e_f \rangle \, e_f(t) \, df$$

- **mathematics**: "any" waveform is made of the superimposition of a (possibly infinite) number of harmonic modes which are *everlasting, undamped* and with a *fixed frequency*
- **physics**: keyrole played by the concept of *frequency* in relation with vibrations and waves
- **computer science**: further development of efficient algorithms (FFT = 1965) which favoured its *practical use*

cycles

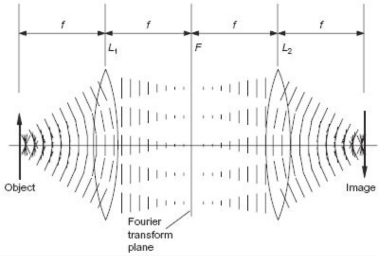
- **physics "of Nature"**, from *macrophysics* (celestial mechanics, tides, . . .) to *microphysics* (Quantum Mechanics)
- **physics "of engineers"** (rotating machines, modal analysis, surveillance of vibrating structures, . . .)

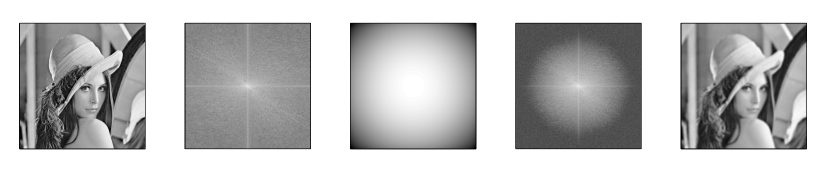


W. Thomson (Lord Kelvin), 1876-1878

lenses

- diffracted field
- Fourier image in the focal plane
- spatial filtering



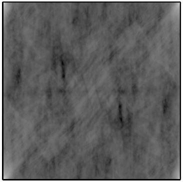


magnitude and phase

Fourier reconstruction with. . .

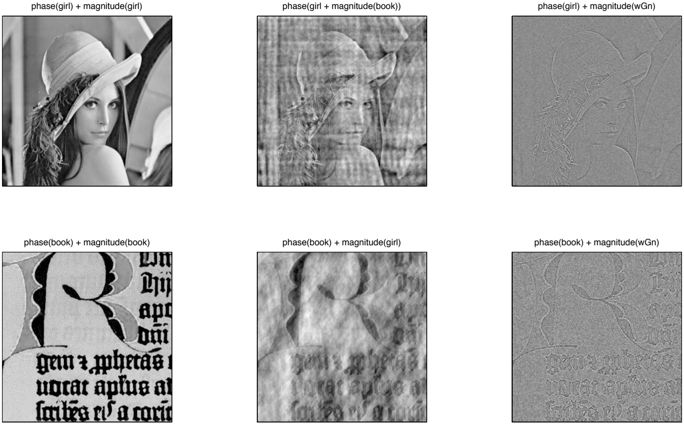
magnitude + phase

magnitude only



phase only

magnitude and phase



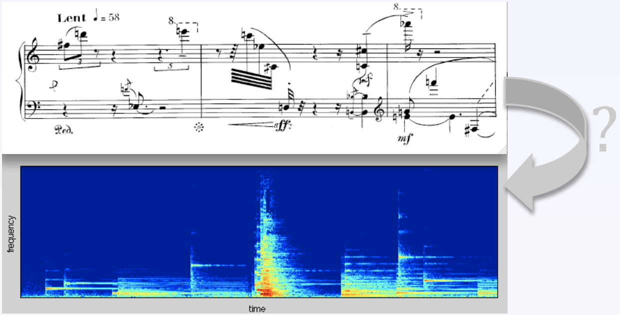
tones

- eigenmodes of cavities
- Helmholtz resonators
- inner ear (cochlea)

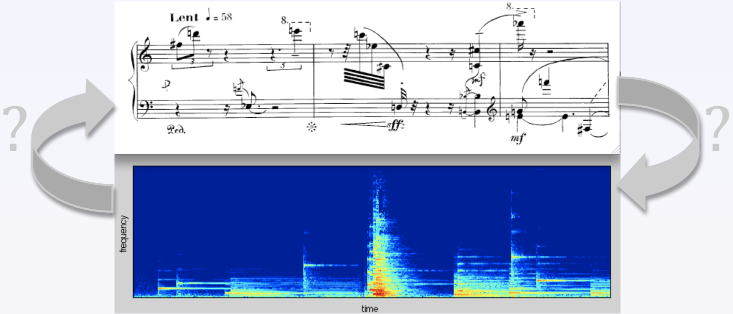
Appareil de Kœnig pour l'analyse du timbre des sons. Document Laboratoire de Mécanique et d'Acoustique, CNRS, Marseille.

music notation

music notation



music notation



from waves to wavelets

Issue

"localized modes" $\Rightarrow$ *switch to a 2-parameter transformation group that includes time*

$$x(t) \rightarrow T(t, \lambda) = \langle x, h_{t,\lambda} \rangle, \; s.t. \; x(t) = \iint \langle x, h_{s,\lambda} \rangle \, h_{s,\lambda}(t) \, d\mu(s, \lambda)$$

1. time-frequency: $\lambda = f$ and $h_{s,f}(t) = h(t - s)\, e_f(t)$

   $\rightarrow$ **short-time Fourier transform**

2. time-scale: $\lambda = a$ and $h_{s,a}(t) = |a|^{-1/2}\, h((s - t)/a)$

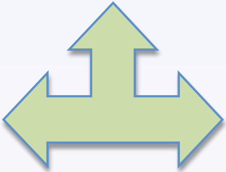
   $\rightarrow$ **wavelet transform**

the wavelet connection ($\sim$ 1980-90)

« physics »

vibroseismics for oil exploration

(Morlet)



« mathematics »

CWT, MRA, bases, etc.

(Grossmann, Meyer, Daubechies)

« computer science »

filter banks, fast algorithms

(Mallat, Cohen, Vetterli)

exclusion principles

« physics »

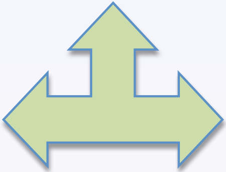
joint measurement of position and momentum

(Heisenberg, 1925)

« mathematics »

any Fourier pair of variables

(Weyl, 1927)



« computer science »

time and frequency

(Gabor, 1946 + ...)

classical formulation

Localization trade-off

based on a second order (variance-type) measure:
$\Delta t_x = (\int t^2 \, |x(t)|^2 \, dt)^{1/2}$ *and* $\Delta f_x = (\int f^2 \, |X(f)|^2 \, df)^{1/2} \Rightarrow$

$$\Delta t_x \, \Delta f_x \geq \frac{\|x\|}{4\pi} \; (> 0)$$

- no perfect pointwise localization
- variations: same limitation with other spreading measures, e.g., entropy (Hirschman, 1957)
- common denominator: minimum achieved with **Gaussians**

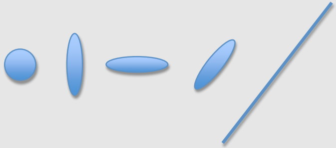
extension

no pointwise localization does not mean no localization

Stronger uncertainty relation (Schrödinger, 1935)

$$\Delta t_x \, \Delta f_x \geq \frac{\|x\|}{4\pi} \sqrt{1 + 16\pi^2 \left(\int t \; (\partial_t \arg x(t)) \; |x(t)|^2 \, dt \right)^2}$$

*bound achieved for “**squeezed states"*** $\{\exp(\alpha t^2 + \beta t + \gamma)\}$, *with **linear** “**chirps**” as a limit when* $\mathrm{Re}\{\beta\} = 0$ *and* $\mathrm{Re}\{\alpha\} \to 0_-$



time-frequency alternatives

From stationarity...

spectrum analysis "à la Wiener-Khintchine-Bochner": $\Gamma_x(f) = \mathcal{F}\{\gamma_x\}(f)$, *with* $\gamma_x(\tau) := \langle x, \mathbf{T}_\tau x \rangle$ *a* ***time-independent*** *correlation function*

...to nonstationarities

$\gamma_x \rightarrow$ ***time-frequency*** *correlation* $\langle x, \mathbf{T}_{\tau,\xi} x \rangle$ + *2D Fourier transform* $\Rightarrow$ ***Wigner-type transforms***

- *intrinsic definitions: no dependence on some measurement device (window, wavelet)*
- *perfect localization for linear chirps (with possible extensions to non linear cases)*

"distribution/correlation" duality

Definition

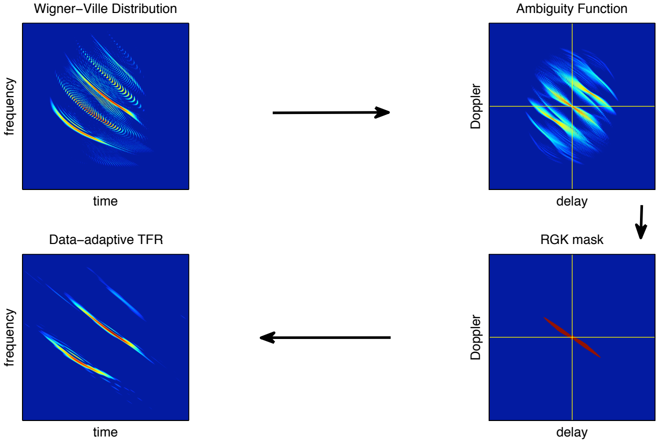
by definition, $W_x(t,f) \stackrel{TF-2D}{\longrightarrow} \mathcal{F}\{W_x\}(\xi,\tau) := A_x(\xi,\tau)$*:* ***ambiguity function*** *(AF)*

Interpretation

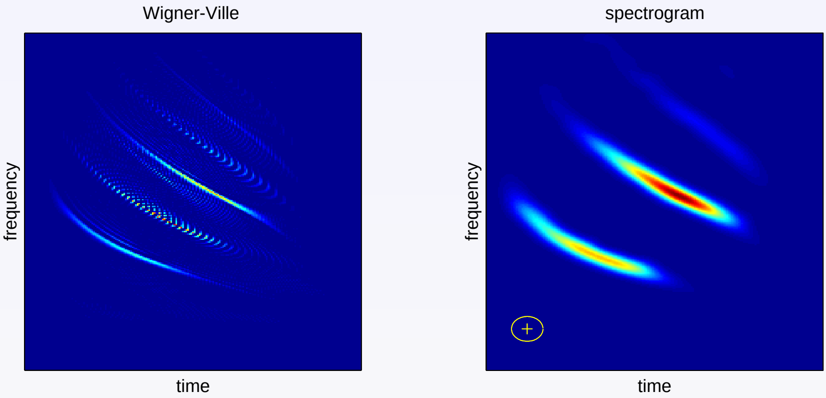
the TF-shift operator $(\mathbf{T}_{\xi,\tau}x)(t) := x(t-\tau)\, e^{-i2\pi\xi(t-\tau/2)}$ *is such that* $A_x(\xi,\tau) = \langle x, \mathbf{T}_{\xi,\tau}x\rangle \Rightarrow$ ***AF = TF correlation****, with*

- *"auto-terms" neighbouring the origin of the plane*
- *"cross-terms" at a distance from the origin that equals the TF distance between components*

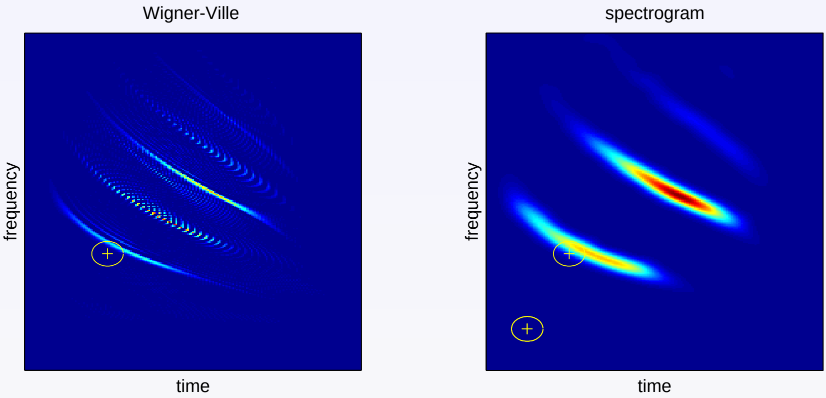
the other trade-off and its "classical" way out



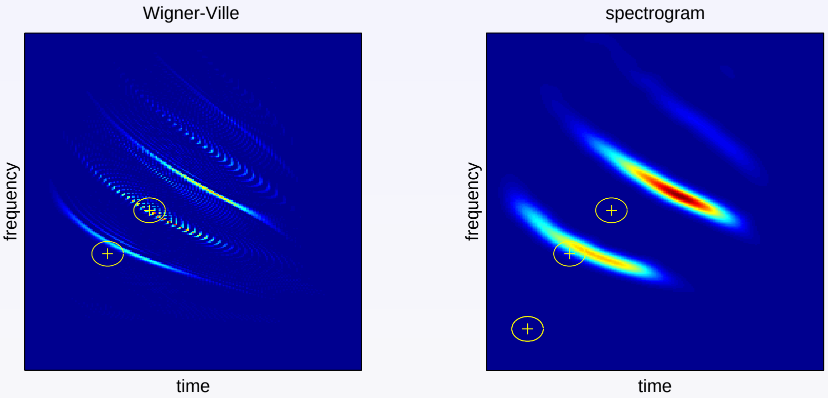
spectrogram = smoothed Wigner



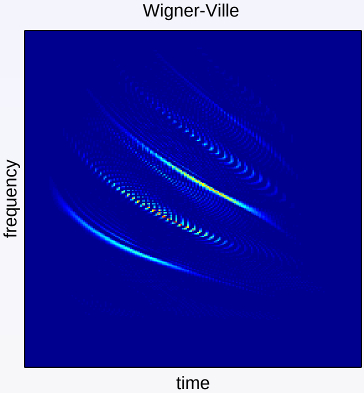
spreading of auto-terms

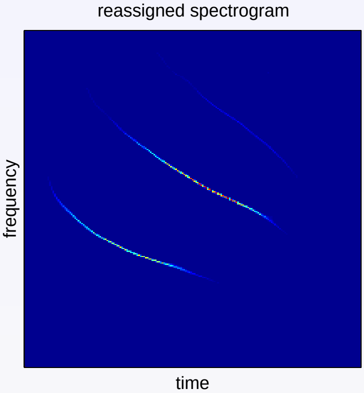


cancelling of cross-terms



reassignment (Kodera *et al.*, 1976, Auger & F., 1995)

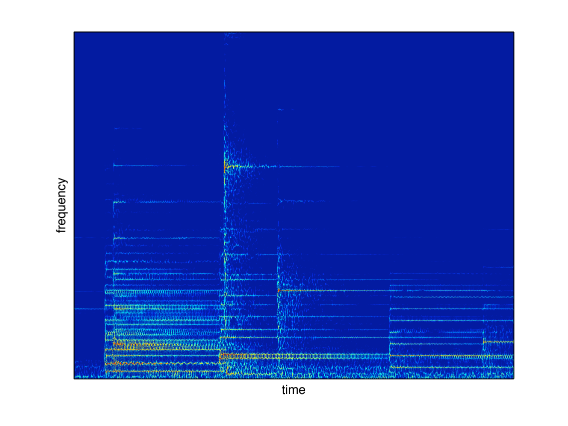




music

tones & transients

“visual score”

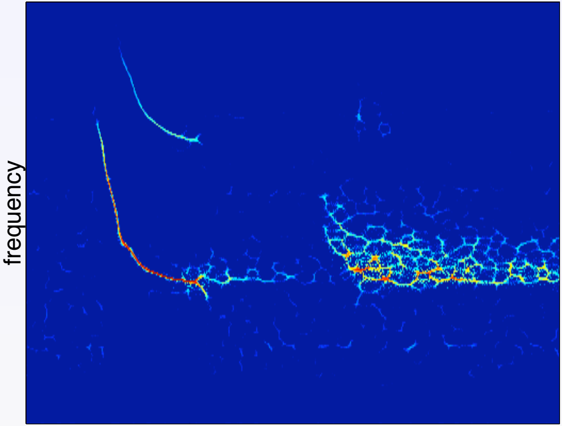


echolocation

bats

“animal sonar”

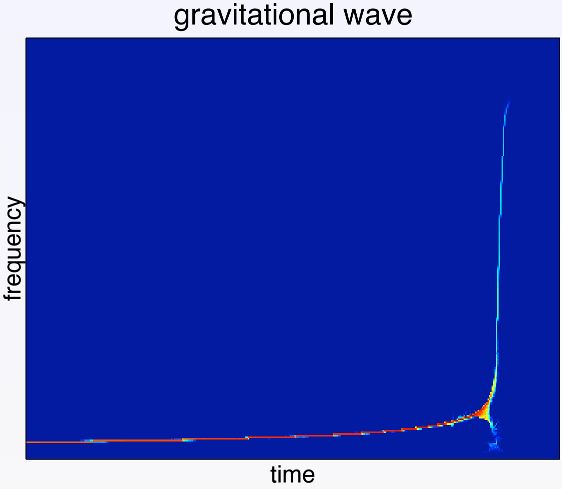
bat echolocation call + echo



gravitational waves

VIRGO

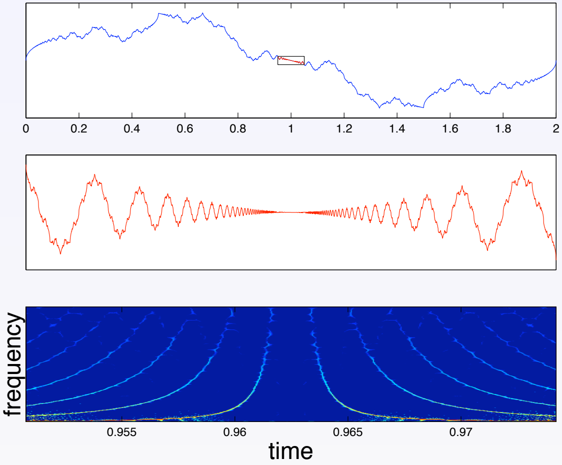
"coalescence of binaries"



Riemann’s function

B. Riemann

$\sigma(t) = \sum_{n=1}^{\infty} n^{-2} \sin \pi n^2 t$



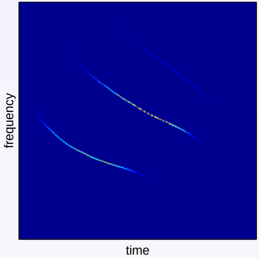
a "compressed sensing" approach

Discrete time

signal of dimension N $\Rightarrow$ TF distribution of dimension $\approx N^2$

Few components

$K \ll N \Rightarrow$ at most $KN \ll N^2$ non zero values in the TF plane



Sparsity

*minimizing the ℓ_0 quasi-norm not feasible, but almost optimal solution by **minimizing the ℓ_1 norm***

a "compressed sensing" approach"

Idea (F. & Borgnat, 2008-10)

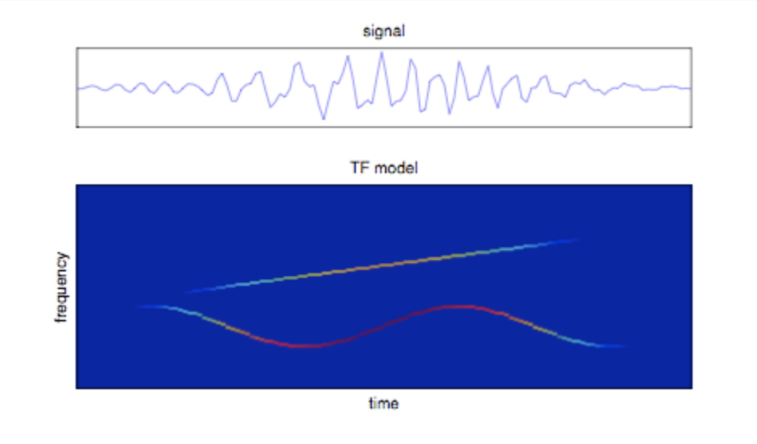
1. *choose a domain Ω neighbouring the origin of the AF plane*
2. *solve the program*

$$\min_{\rho} \|\rho\|_1 \, ; \, \mathcal{F}\{\rho\} - A_x = 0|_{(\xi,\tau)\in\Omega}$$

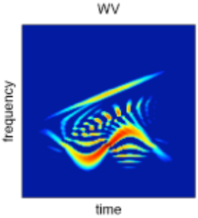
3. *the exact equality over Ω can be relaxed to*

$$\min_{\rho} \|\rho\|_1 \, ; \, \|\mathcal{F}\{\rho\} - A_x\|_2 \leq \epsilon|_{(\xi,\tau)\in\Omega}$$

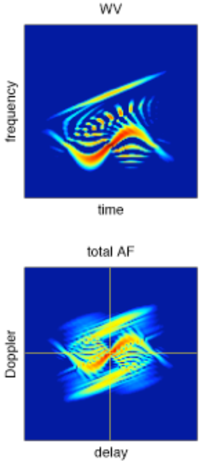
a toy example



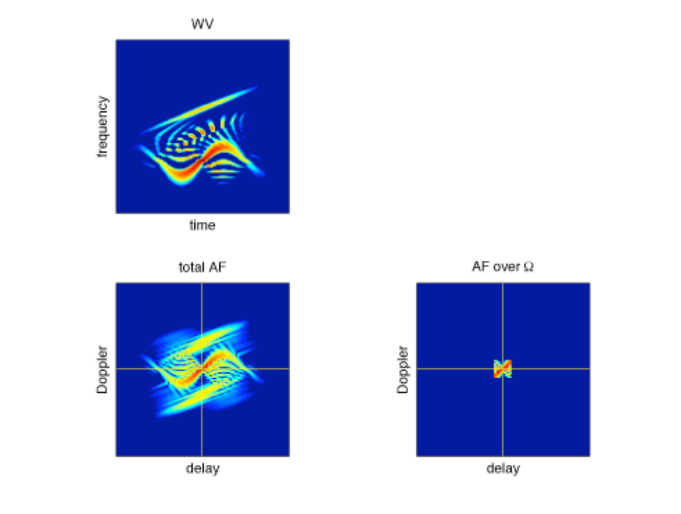
Wigner



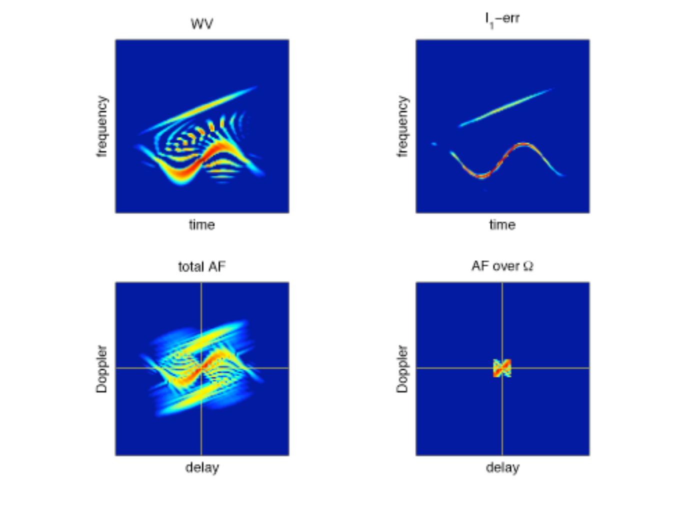
ambiguity



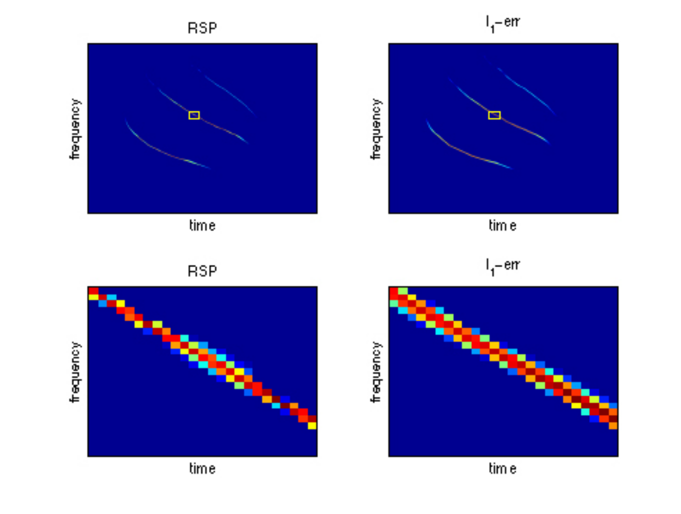
selection


WV
frequency
time
total AF
Doppler
delay
AF over Ω
Doppler
delay

sparse solution



comparison sparsity vs. reassignment



instantaneous frequency

Aim

model a signal $x(t) \in \mathbb{R}$ *as* $x(t) = a_x(t) \cos 2\pi \int^t f_x(s)\, ds$

- for a given t, "1 equation and 2 unkowns" $\Rightarrow$ **no unique representation**
- **multiplicity of solutions** under constraints
  - global
  - local
  - non harmonic

"global" approach (Gabor, 1946 ; Ville, 1948)

1. $e_f(t) = \cos 2\pi ft + i\,\mathcal{H}\{\cos 2\pi ft\}$, with $\mathcal{H}$ Hilbert

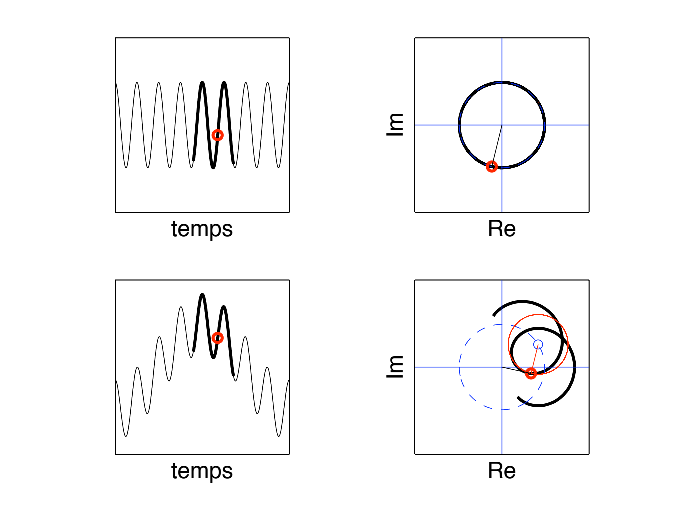
   monochromatic wave = **circle** in the complex plane + constant speed

2. $x(t) \rightarrow z_x(t) = x(t) + i\,\mathcal{H}\{x(t)\}$ (analytic signal)

   modulated "AM-FM" signal: circle $\rightarrow$ **"any" loop** around the origin of the complex plane + varying speed

3. **amplitude** : $a_x(t) = |z_x(t)|$
   instantaneous frequency : $f_x(t) = \frac{1}{2\pi}\partial_t \arg z_x(t)$

variation (Equis, Jacquot & F., 2011)

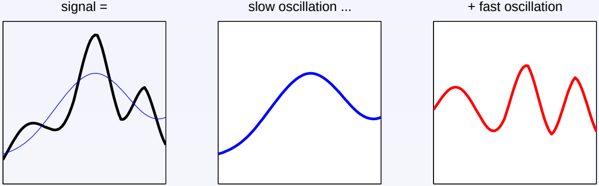


"local" approach (Teager, 1980 ; Kaiser, 1990)

1. $x(t) = a \cos 2\pi ft \Rightarrow \Psi(x) := (\partial_t x)^2 - x.\partial_t^2 x = 4\pi^2 a^2 f^2$

   $\Psi(x)$ **energy operator** taking the form
   $E(x) = x^2[n] - x[n-1]x[n+1]$ in discrete-time
2. similar local properties when $a \rightarrow a_x(t)$ and $f \rightarrow f_x(t)$
3. **instantaneous amplitude** : $a_x(t) = \Psi(x)/\sqrt{|\Psi(\partial_t x)|}$
   instantaneous frequency : $f_x(t) = \frac{1}{2\pi}\sqrt{|\Psi(\partial_t x)/\Psi(x)|}$

“non harmonic” approach (Huang *et al.*, 1998)



Idea of Empirical Mode Decomposition (EMD)

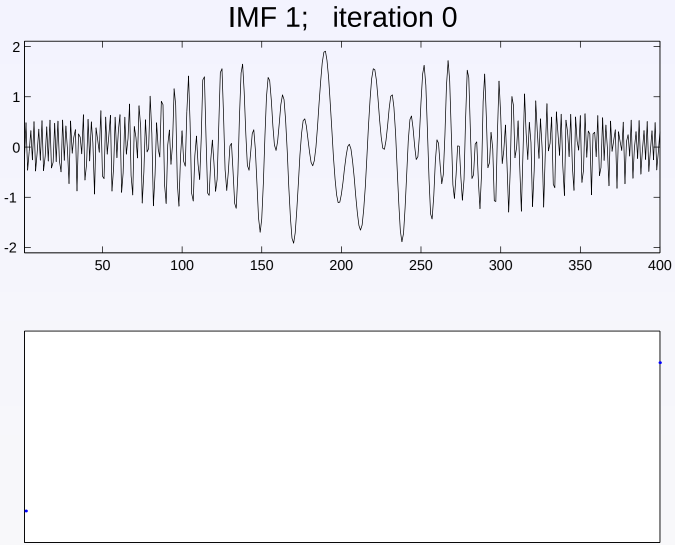
signal = fast oscillation + slow oscillation [& iteration]

- **data-driven** “fast vs. slow” disentanglement
- **“local"** analysis based on neighbouring extrema
- **oscillation** rather than frequency

algorithm

1. identify local maxima and minima
2. interpolate (cubic splines) to get an upper and a lower envelope
   1. subtract the mean of the envelopes from the signal
   2. iterate until “mean envelope $\approx$ 0" (*sifting*)
3. subtract the so-obtained mode from the signal
4. iterate on the residual

$$
\begin{aligned}
x(t) &= c_1(t) + r_1(t) \\
&= c_1(t) + c_2(t) + r_2(t) \\
&= \ldots\ldots\ldots\ldots\ldots\ldots \quad = \sum_{k=1}^{K} c_k(t) + r_K(t)
\end{aligned}
$$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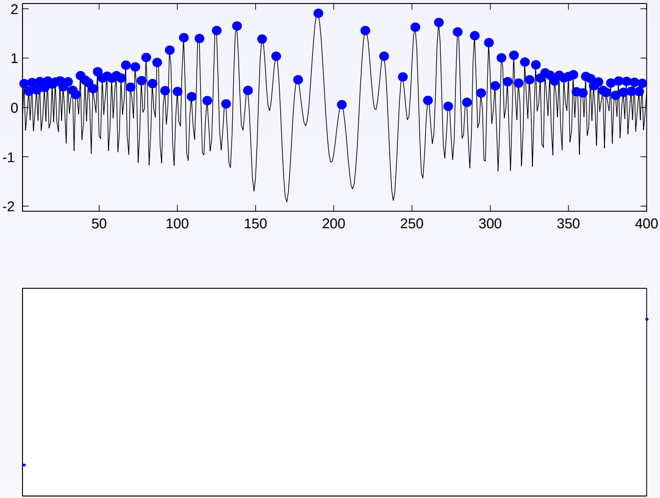
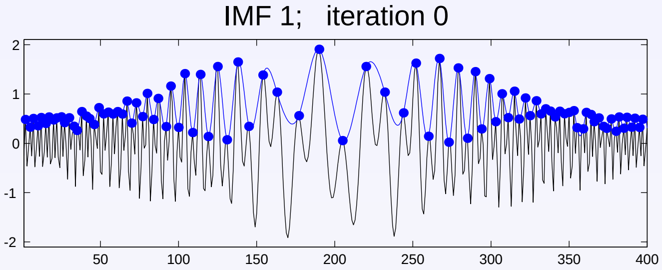
IMF 1; iteration 0

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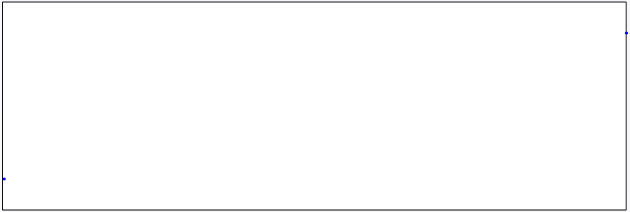

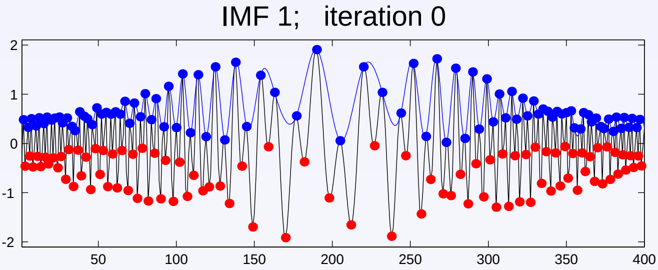
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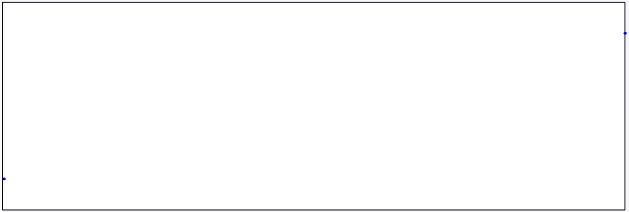

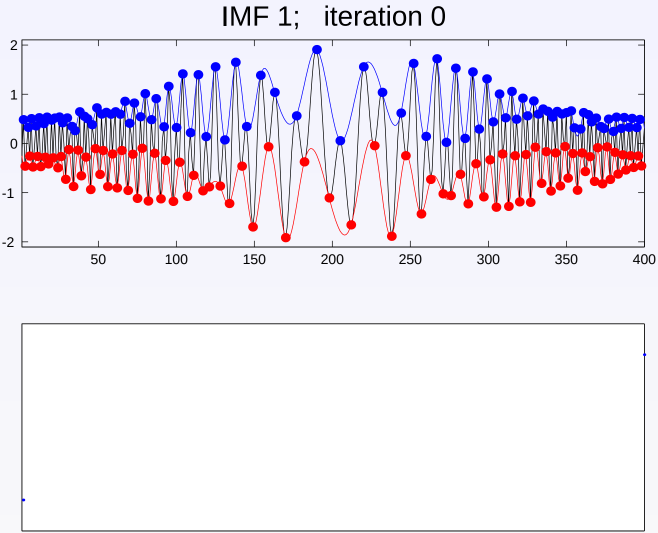
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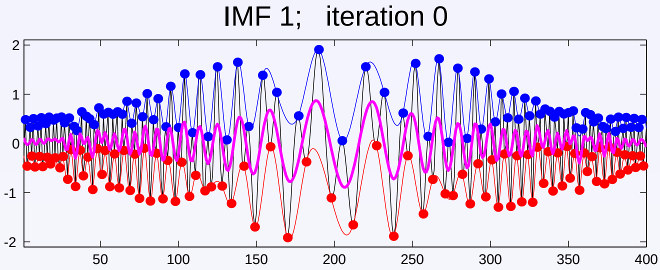
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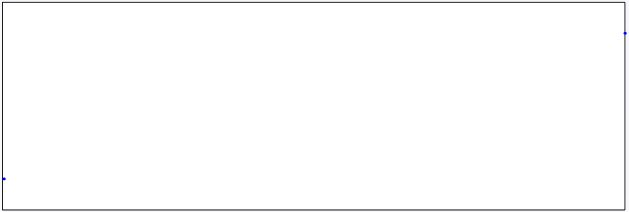

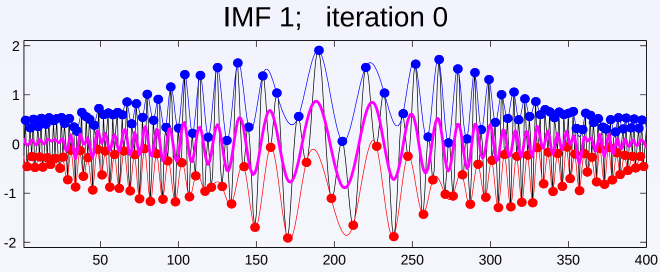
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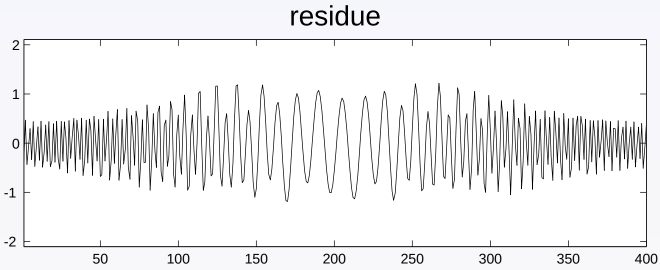
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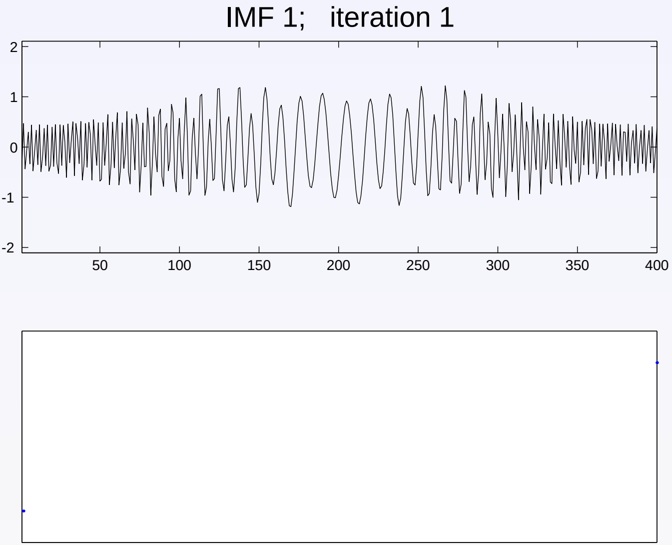
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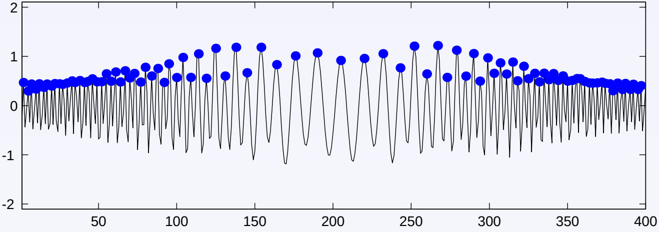
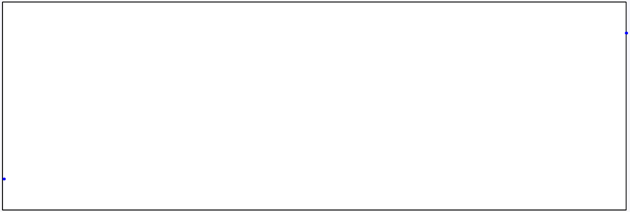
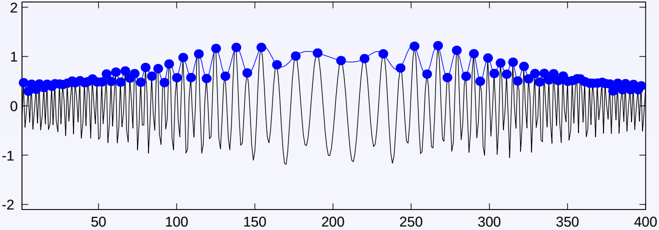
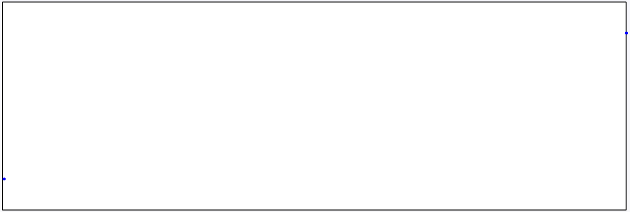
IMF 1; iteration 1



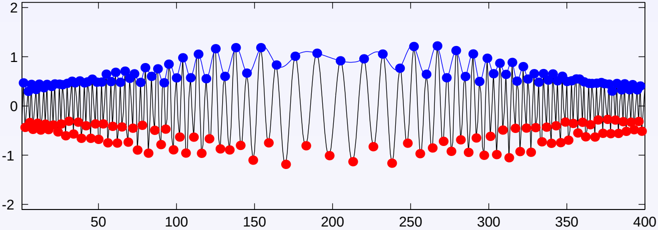


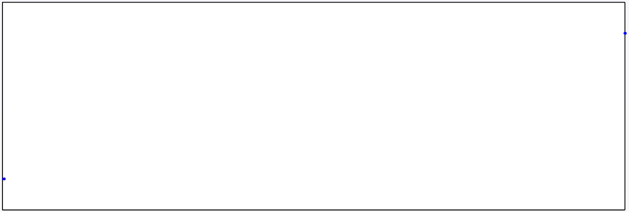
IMF 1; iteration 1



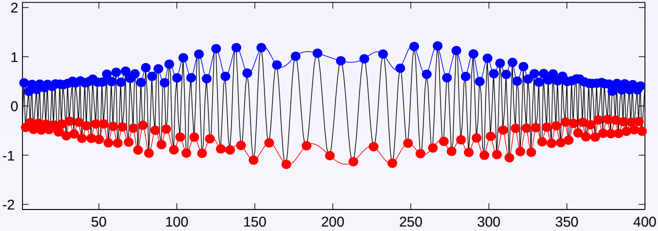


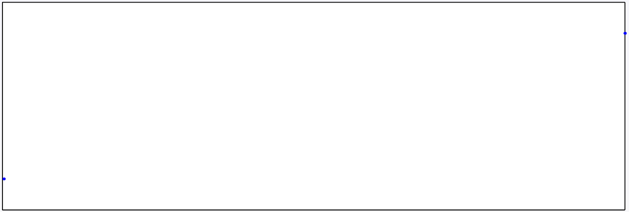
IMF 1; iteration 1



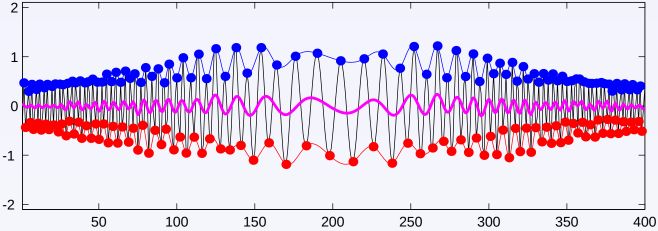


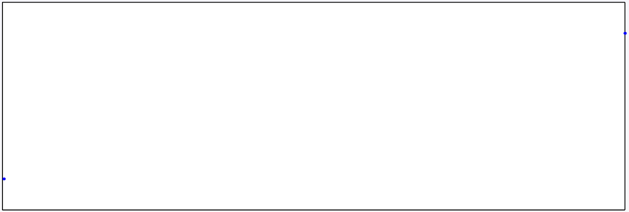
IMF 1; iteration 1

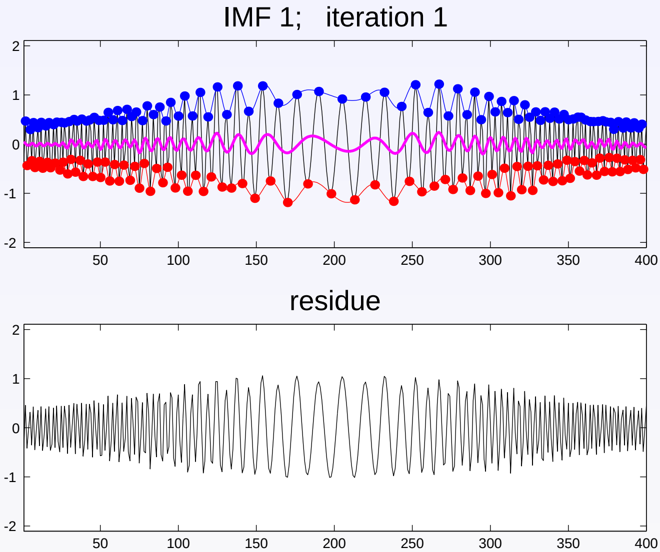


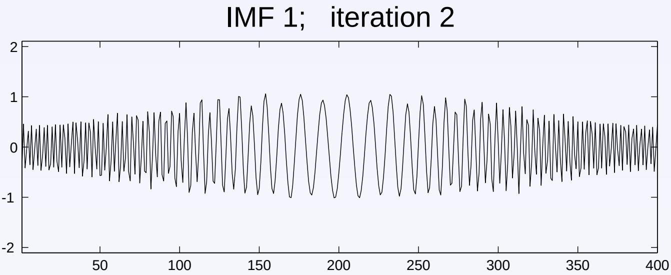


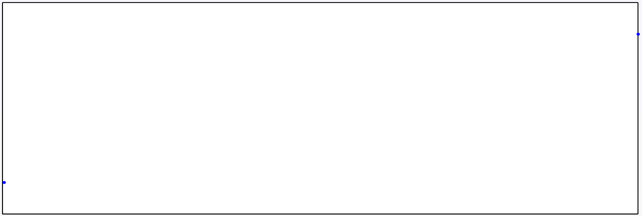
IMF 1; iteration 1

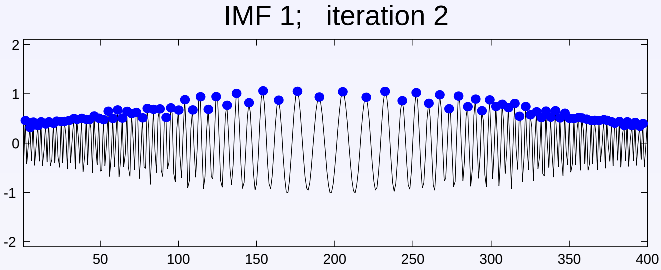


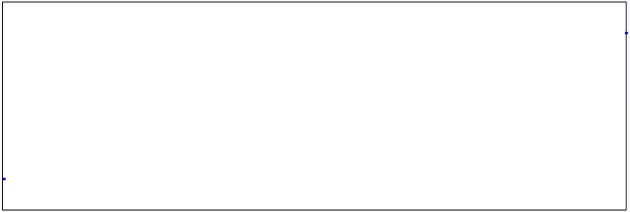



IMF 1; iteration 1
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

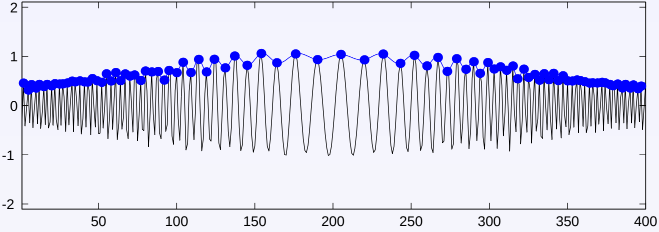

IMF 1; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400

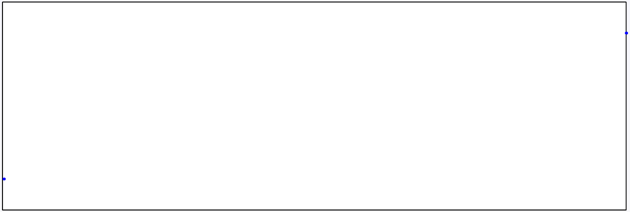



IMF 1; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400

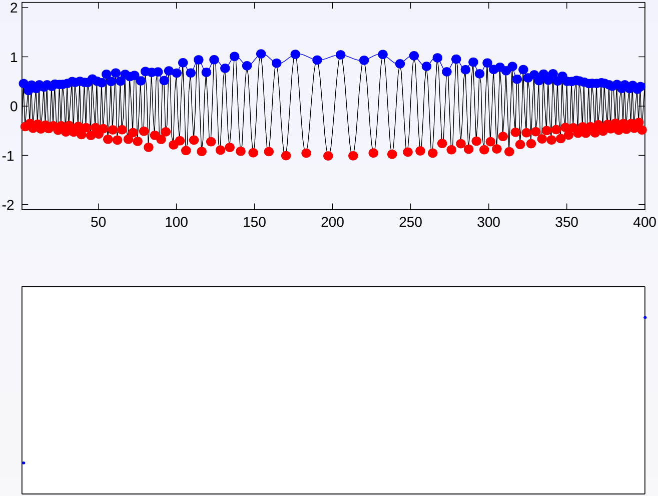


IMF 1; iteration 2

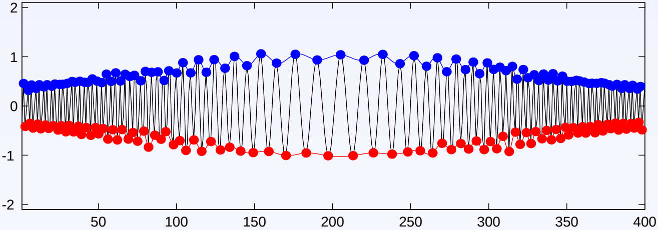


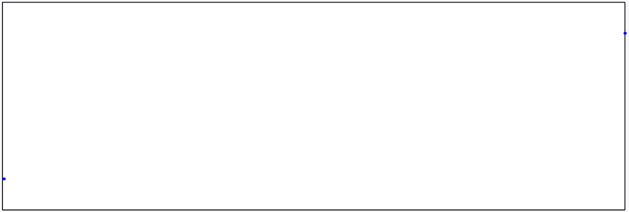


IMF 1; iteration 2

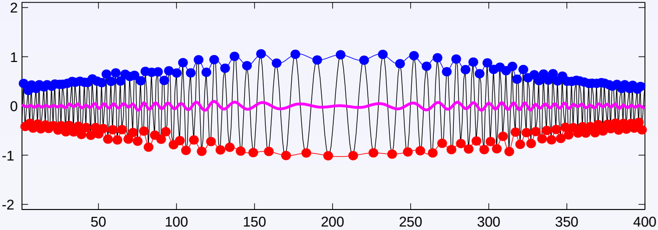


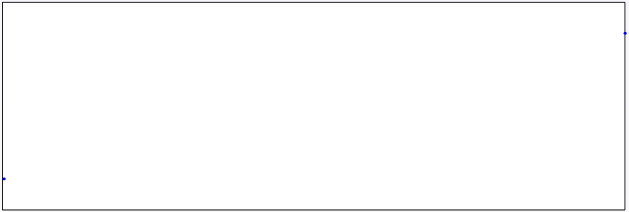
IMF 1; iteration 2

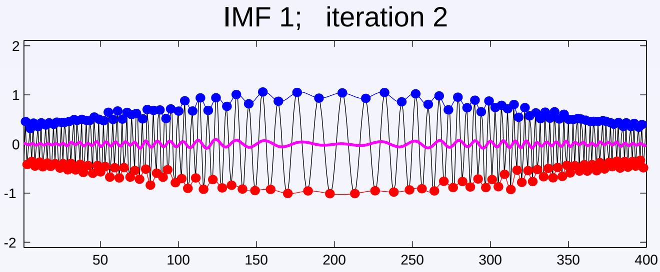


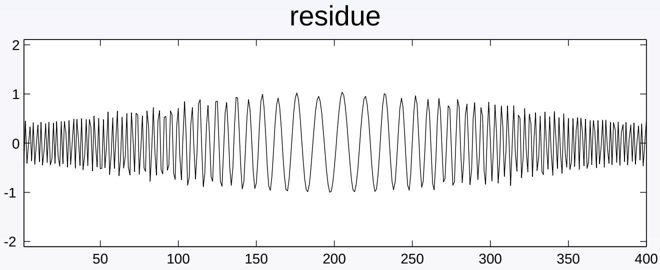


IMF 1; iteration 2

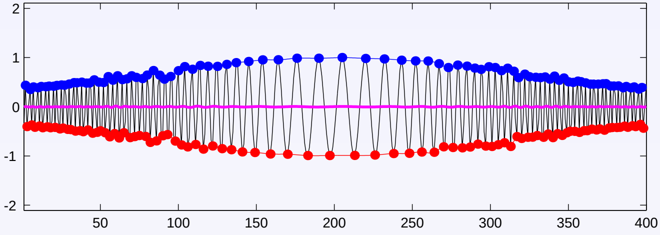




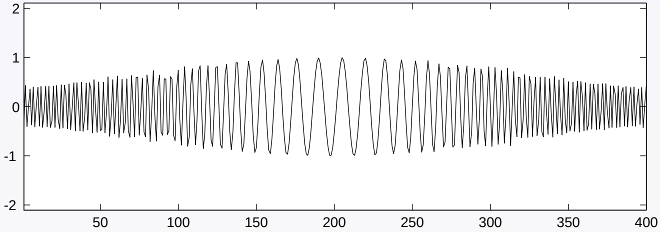

IMF 1; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400

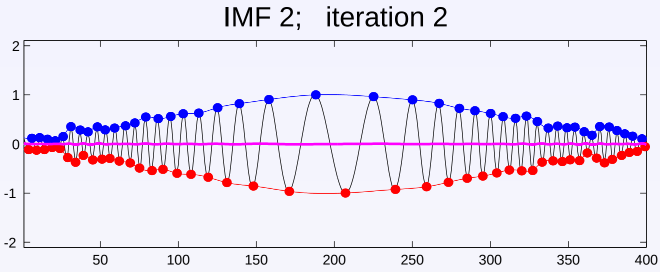

residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

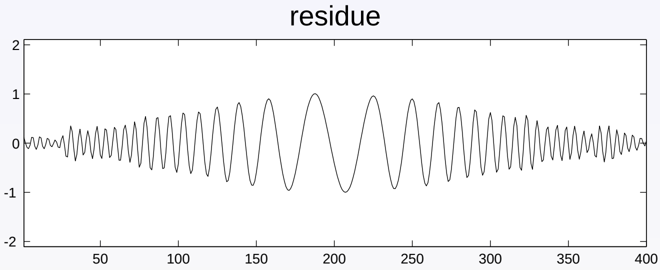
IMF 1; iteration 5

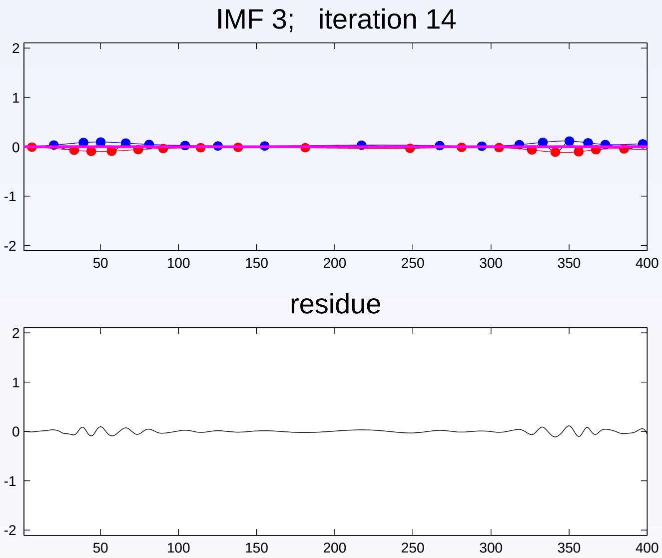


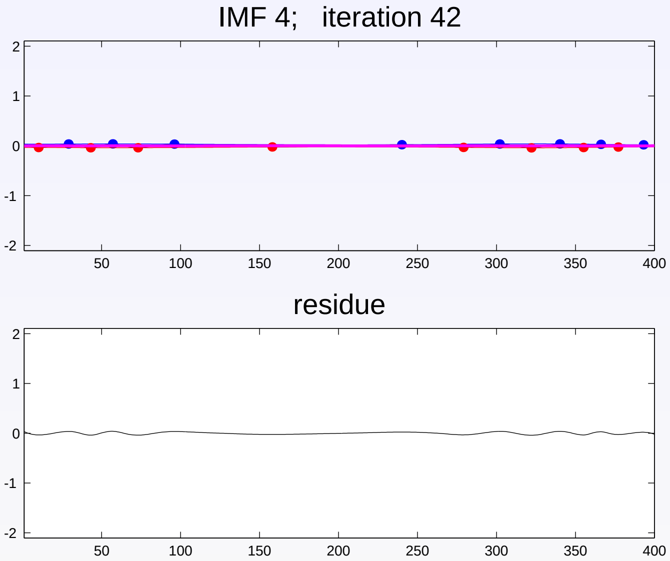
residue

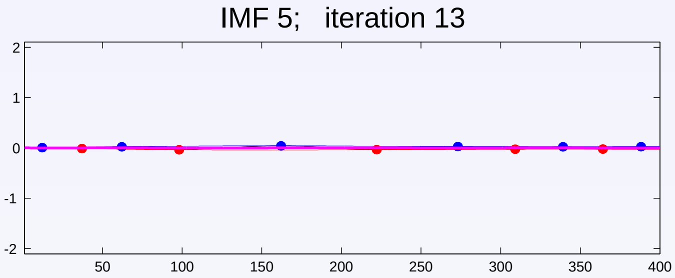


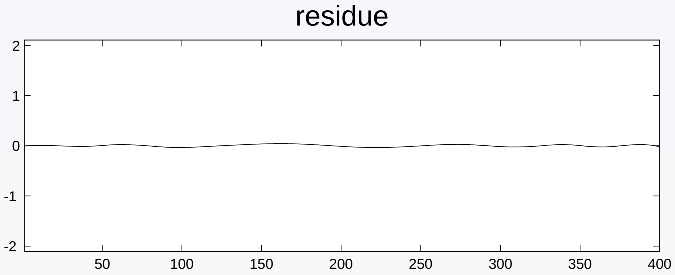

IMF 2; iteration 2
2
1
0
-1
-2
50 100 150 200 250 300 350 400

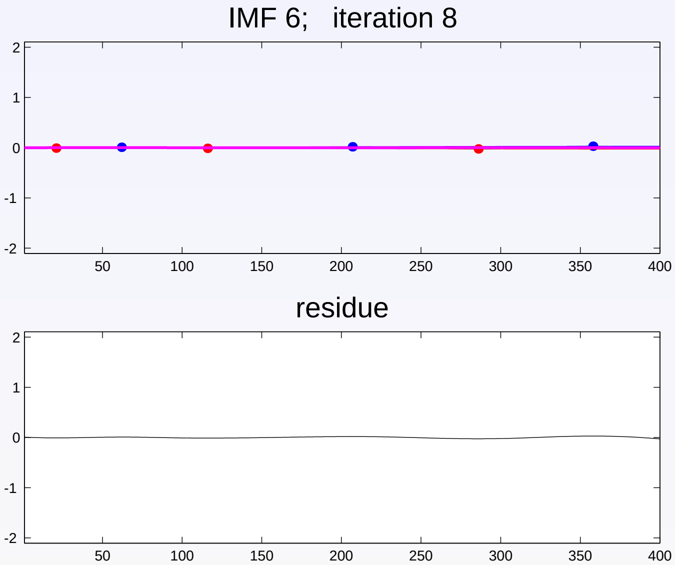

residue
2
1
0
-1
-2
50 100 150 200 250 300 350 400

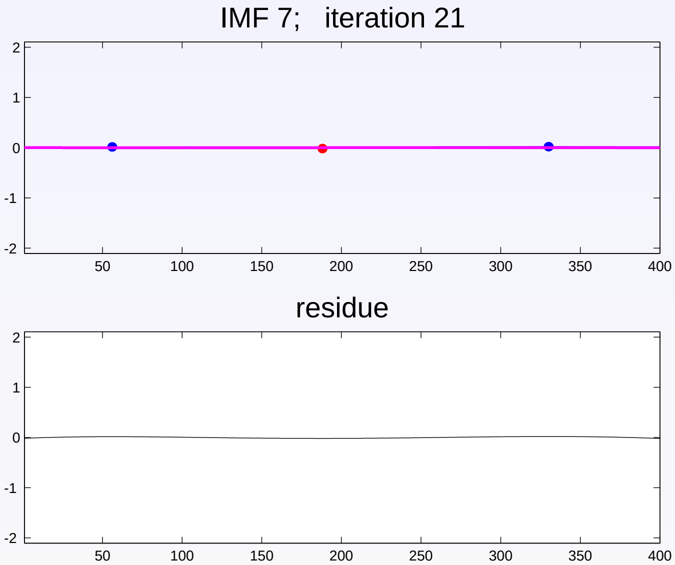

IMF 3; iteration 14
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

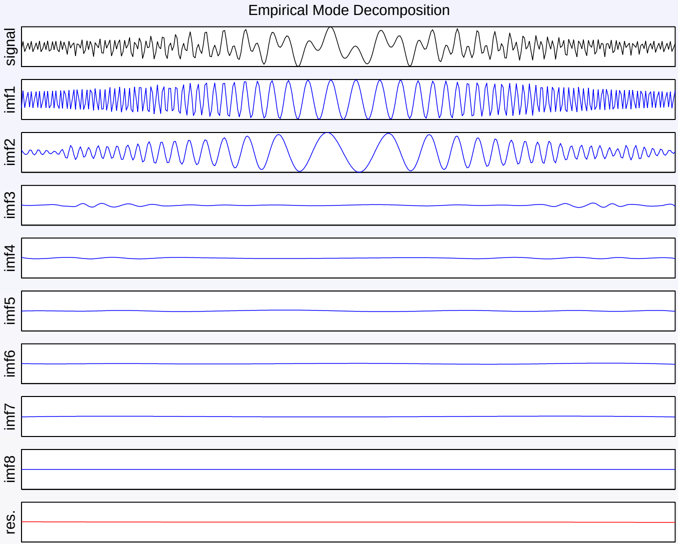

IMF 4; iteration 42
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

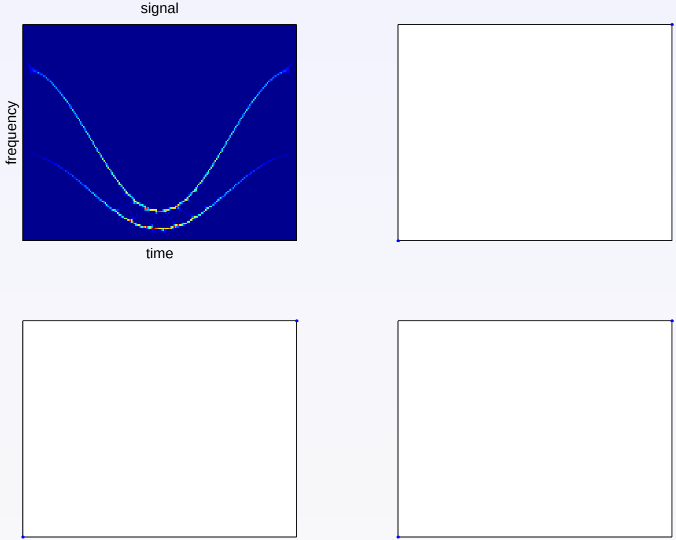

IMF 5; iteration 13
2
1
0
-1
-2
50
100
150
200
250
300
350
400

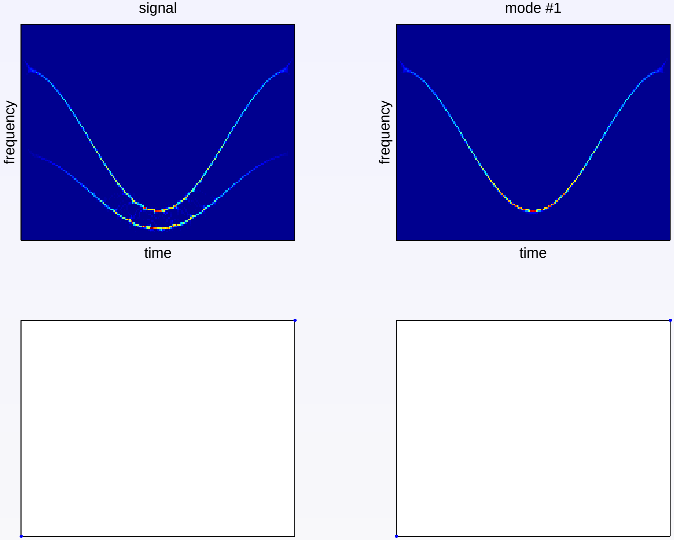

residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

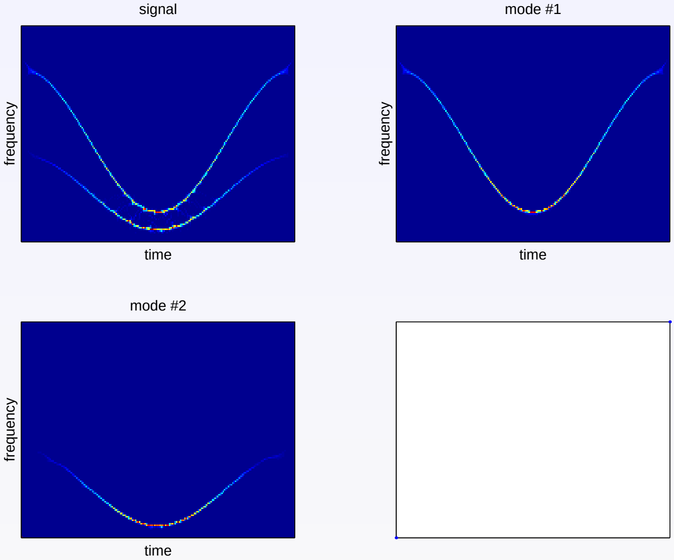

IMF 6; iteration 8
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

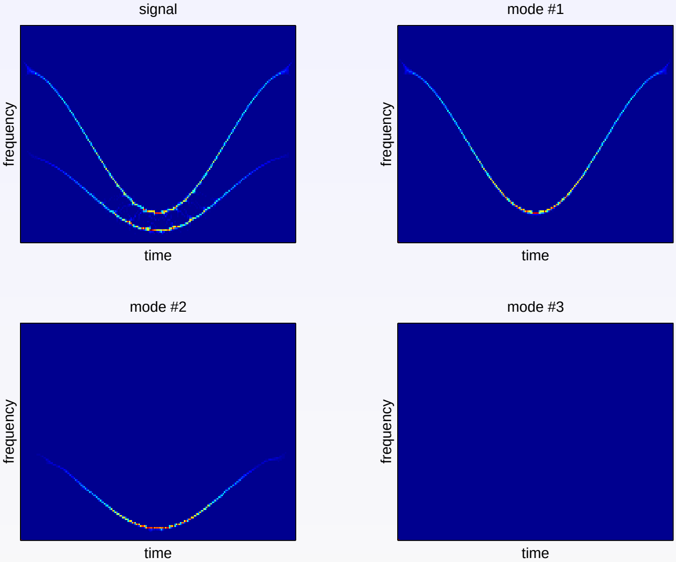

IMF 7; iteration 21
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400


Empirical Mode Decomposition
signal
imf1
imf2
imf3
imf4
imf5
imf6
imf7
imf8
res.


signal
frequency
time


signal
frequency
time
mode #1
frequency
time


signal
frequency
time
mode #1
frequency
time
mode #2
frequency
time

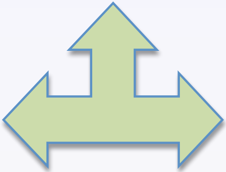

signal
frequency
time
mode #1
frequency
time
mode #2
frequency
time
mode #3
frequency
time

one or two components?

$$« \cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) »$$

« physics »

(production, perception)



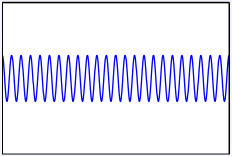
« mathematics »

(equivalent descriptions)

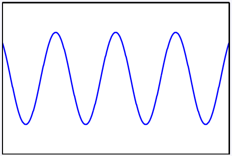
« computer science »

(model-based? data-driven?)

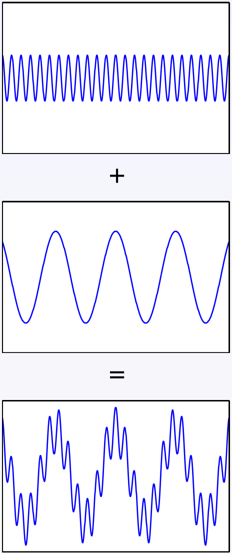
one or two components?



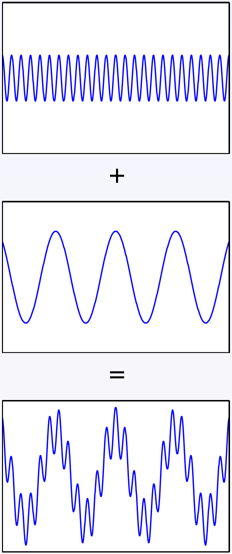
+

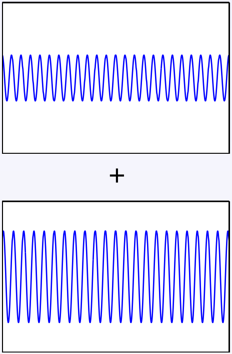


one or two components?

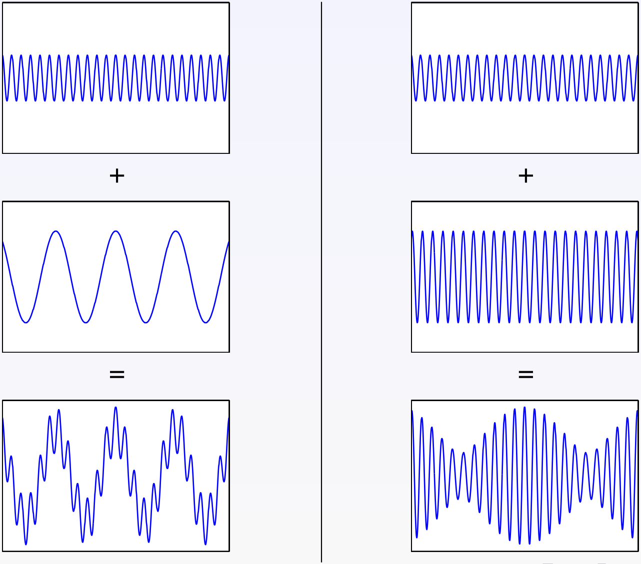


one or two components?

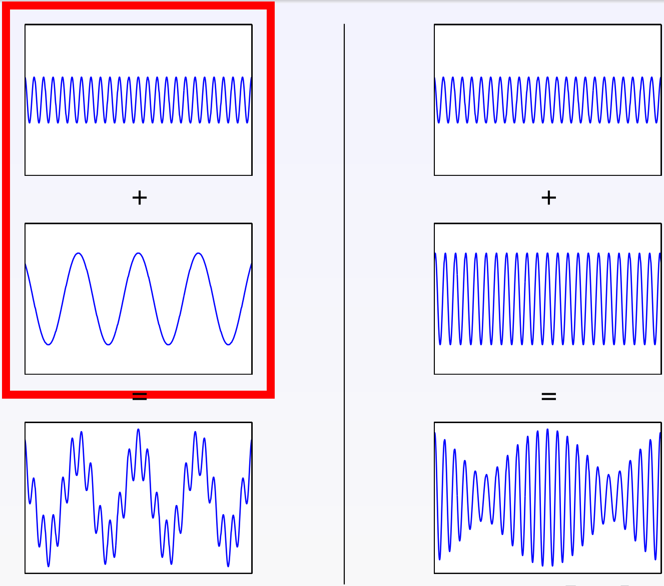




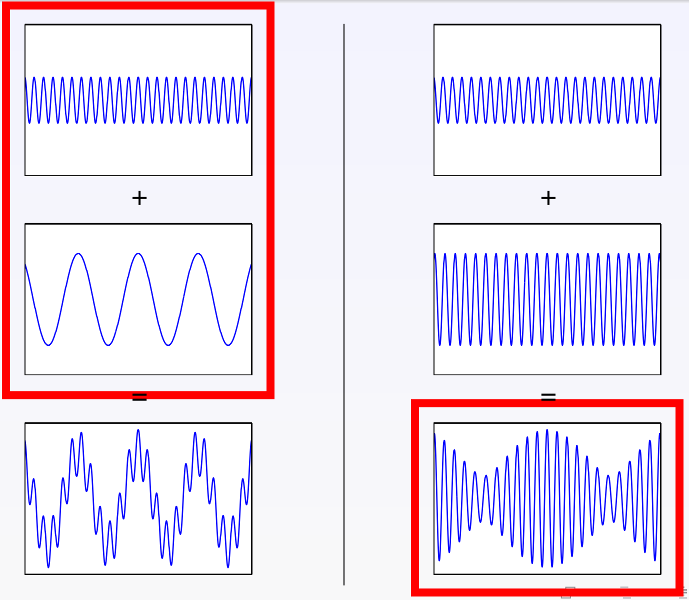
one or two components?



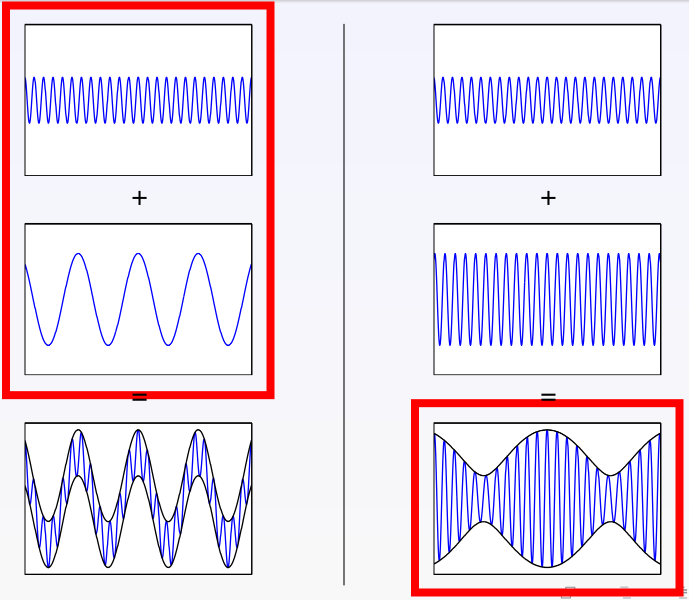
one or two components?



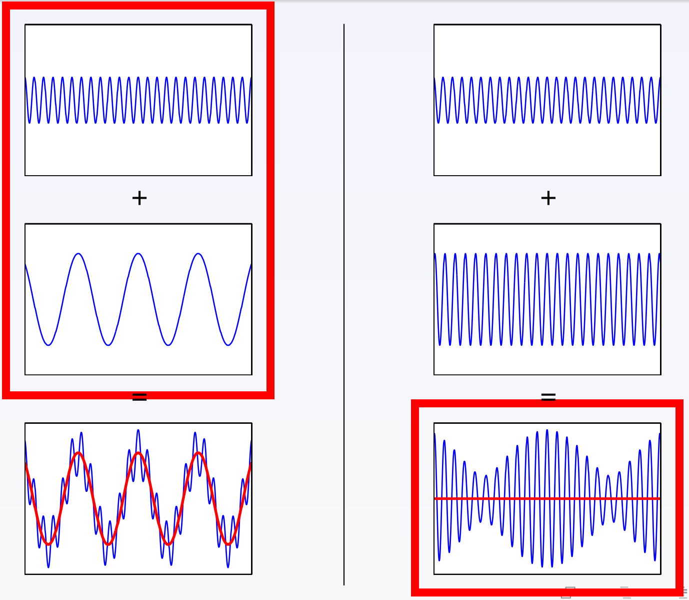
one or two components?



one or two components?



one or two components?



simulations

Signal

$$x(t) = \underbrace{a_1 \cos(2\pi f_1 t)}_{x_1(t)} + \underbrace{a_2 \cos(2\pi f_2 t + \varphi)}_{x_2(t)}, \quad f_1 > f_2$$

Analysis of its EMD

- only the **first IMF** is computed: if separation, it should be equal to the highest frequency component $x_1(t)$
- **criterion** (= 0 if separation) :

$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

- sampling effects are **neglected** : $f_1, f_2 \ll f_s$, with f_s the sampling frequency

simulations

Signal

$$x(t) = \underbrace{a_1 \cos(2\pi f_1 t)}_{x_1(t)} + \underbrace{a_2 \cos(2\pi f_2 t + \varphi)}_{x_2(t)}, \quad f_1 > f_2$$

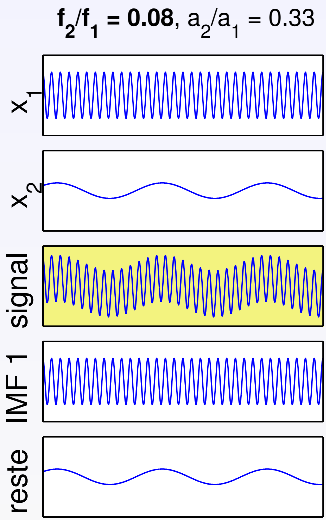
Analysis of its EMD

- only the **first IMF** is computed: if separation, it should be equal to the highest frequency component $x_1(t)$
- **criterion** ($= 0$ if separation) :

$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

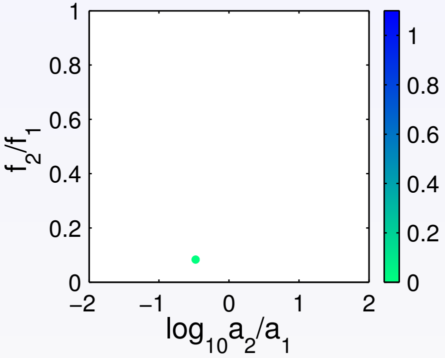
- sampling effects are **neglected** : $f_1, f_2 \ll f_s$, with f_s the sampling frequency

sum of two tones

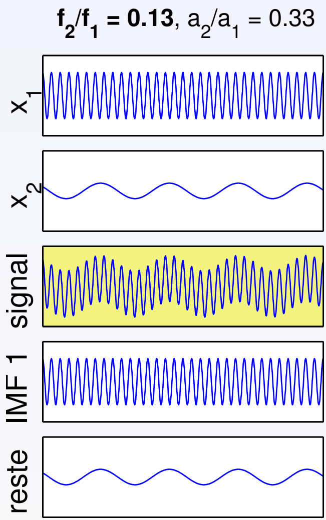


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\| IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

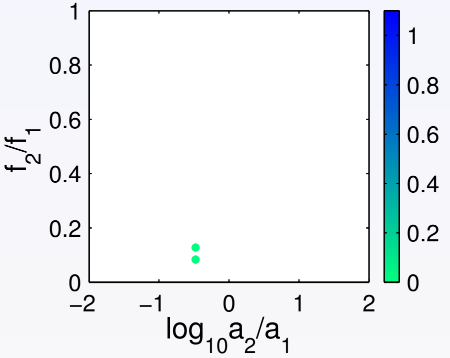


sum of two tones

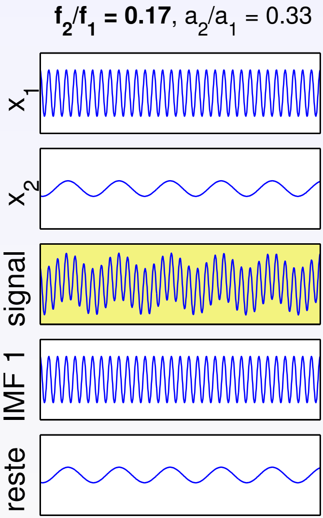


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

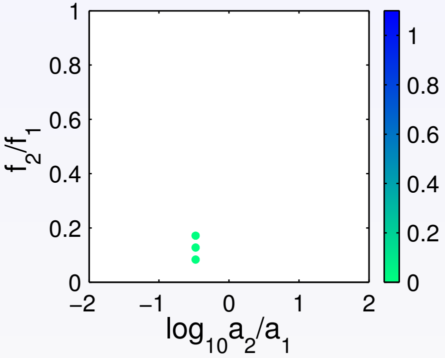


sum of two tones

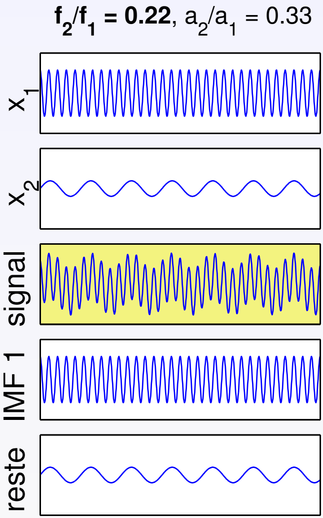


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

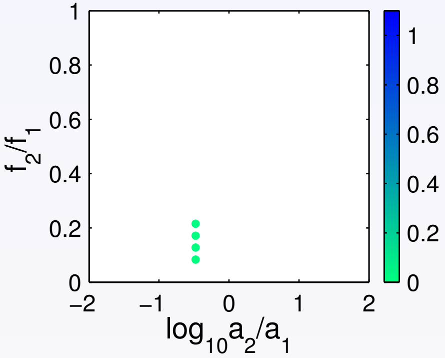


sum of two tones

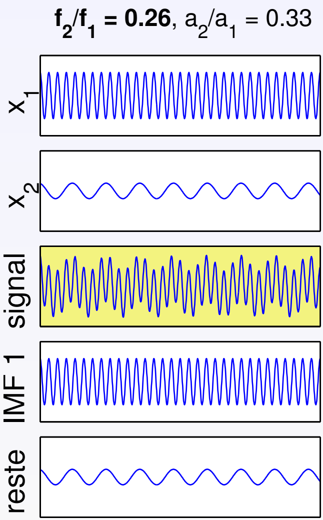


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

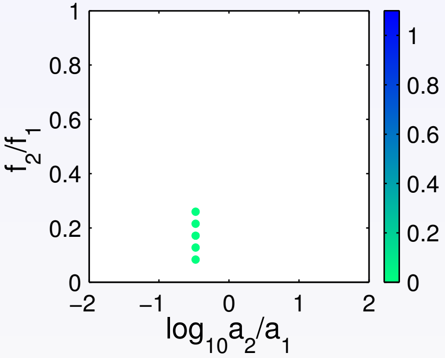


sum of two tones



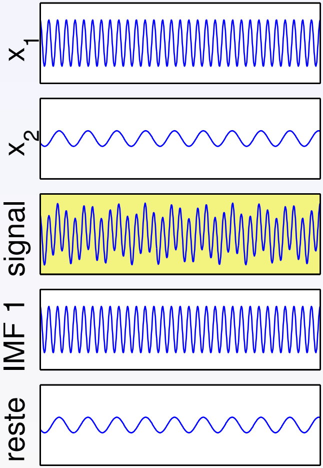
$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



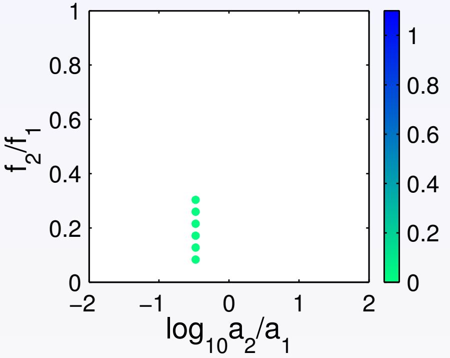
sum of two tones

f_2/f_1 = 0.30, a_2/a_1 = 0.33

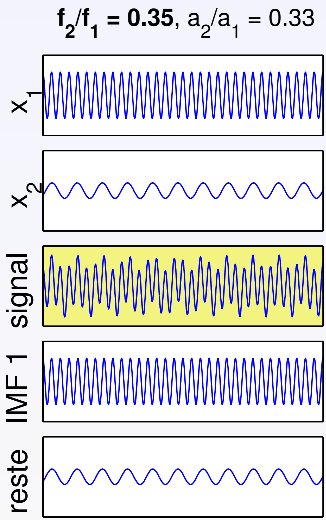


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

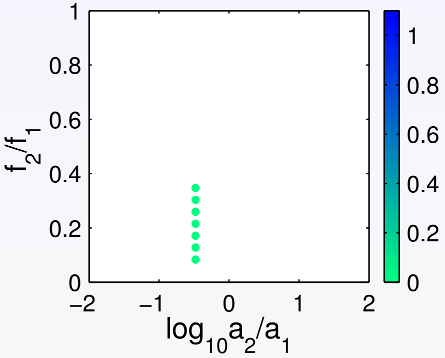


sum of two tones

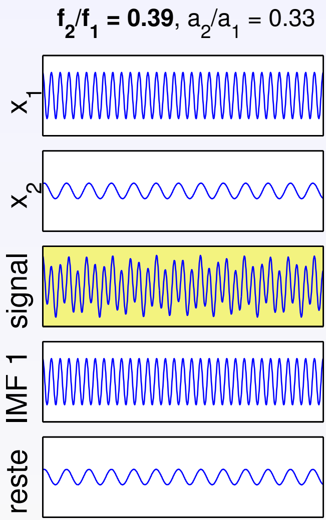


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

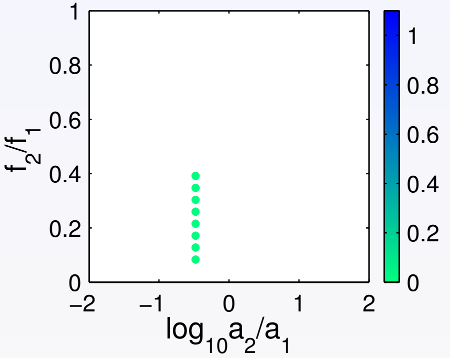


sum of two tones

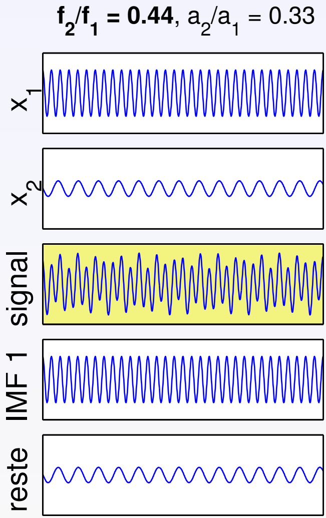


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

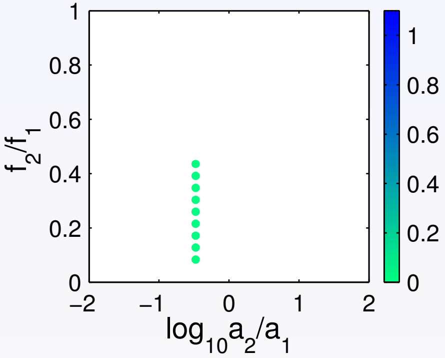


sum of two tones

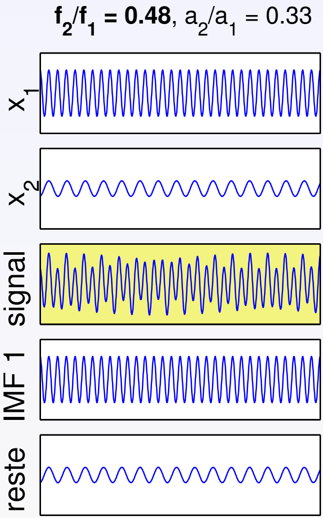


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

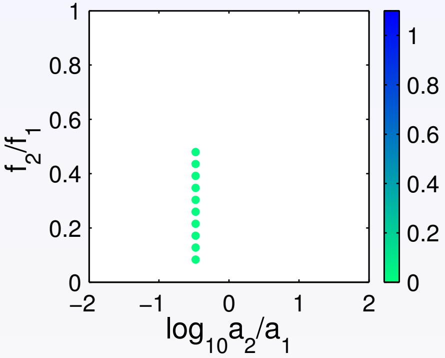


sum of two tones

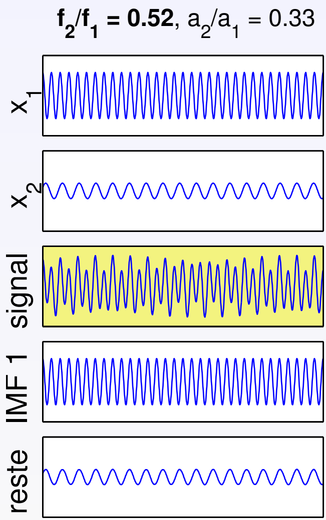


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

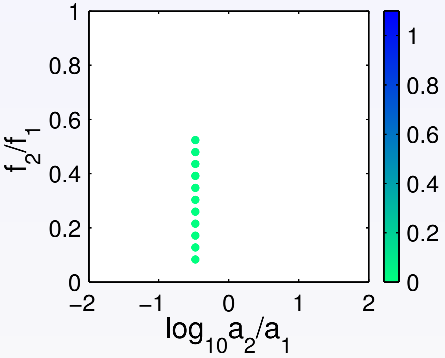


sum of two tones

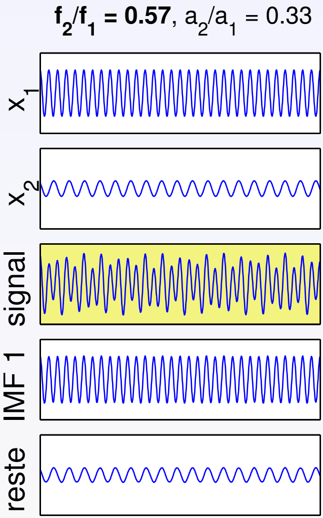


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

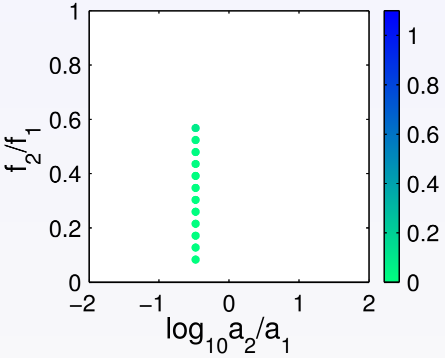


sum of two tones

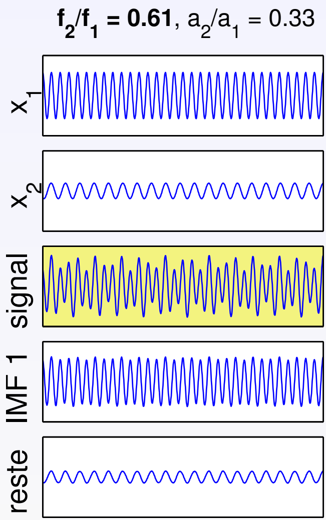


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

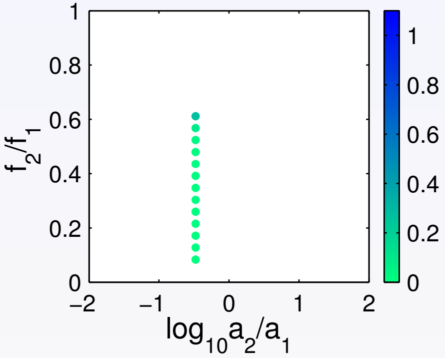


sum of two tones

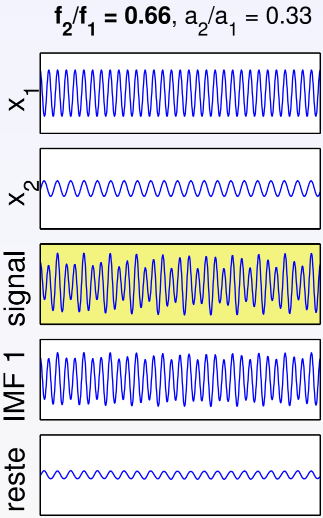


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

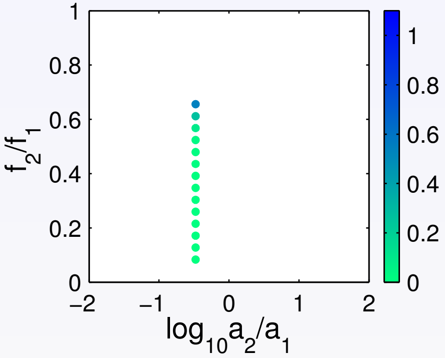


sum of two tones

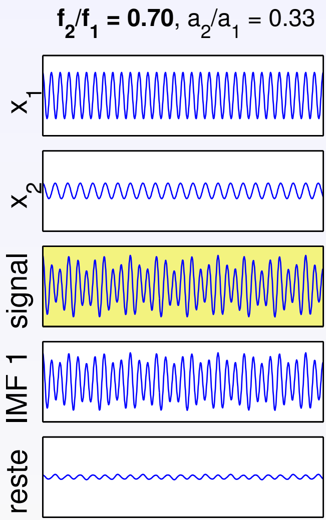


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

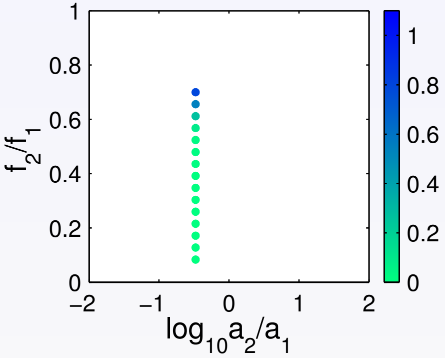


sum of two tones

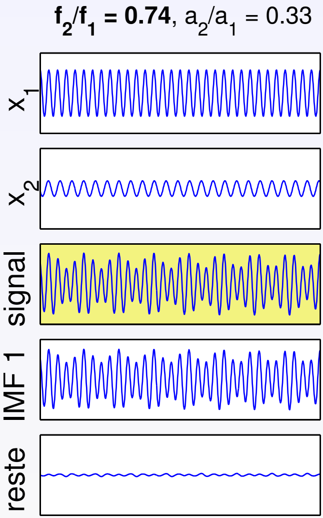


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

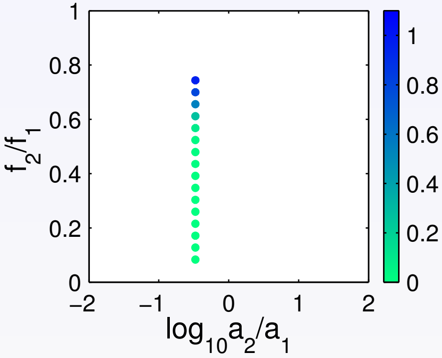


sum of two tones

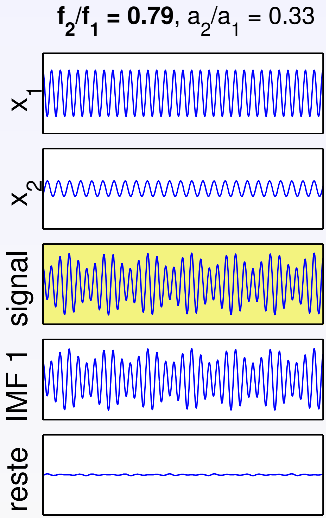


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

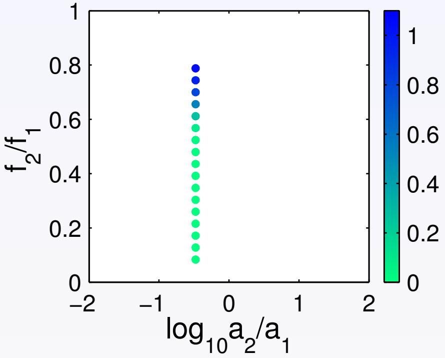


sum of two tones

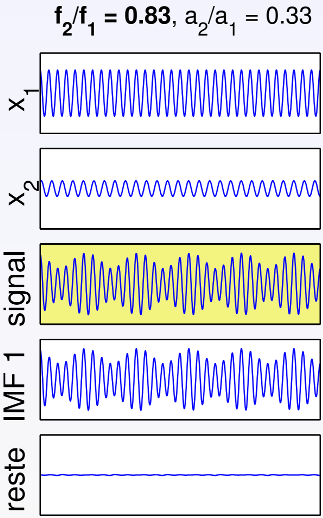


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

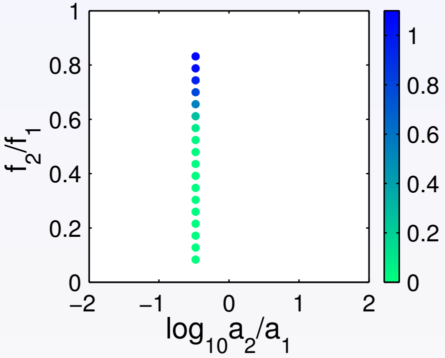


sum of two tones

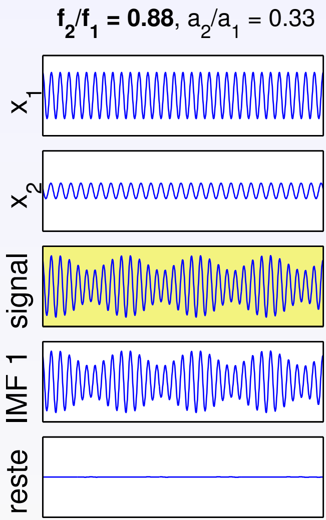


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

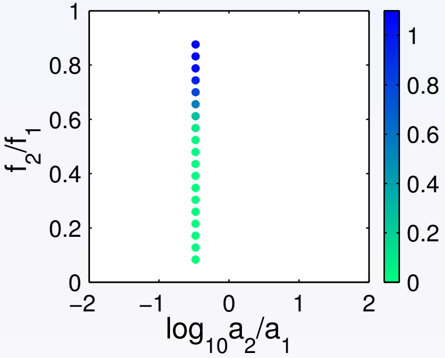


sum of two tones

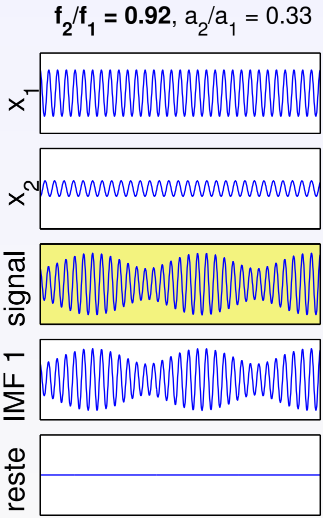


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation

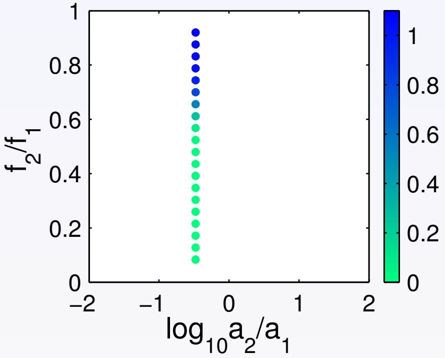


sum of two tones

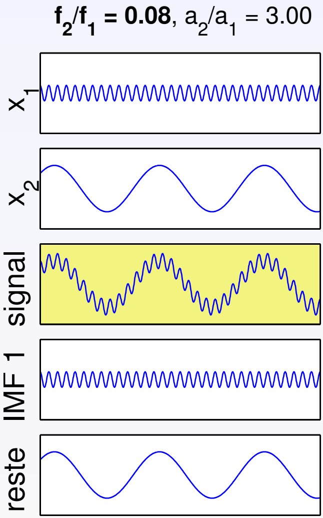


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

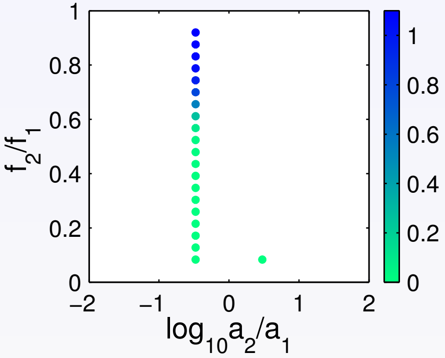


sum of two tones

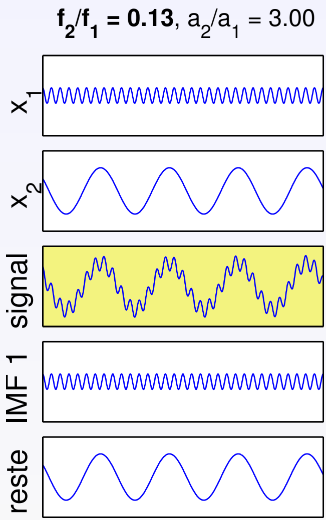


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

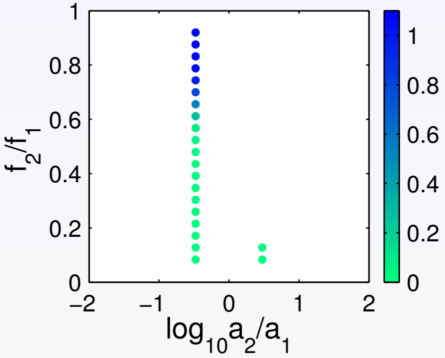


sum of two tones

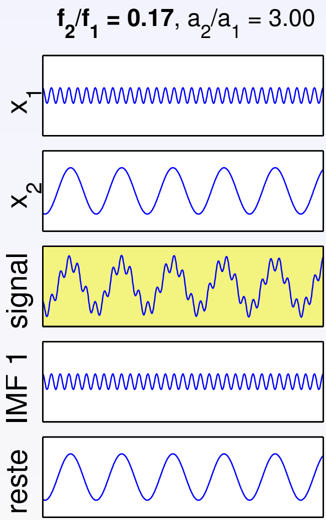


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

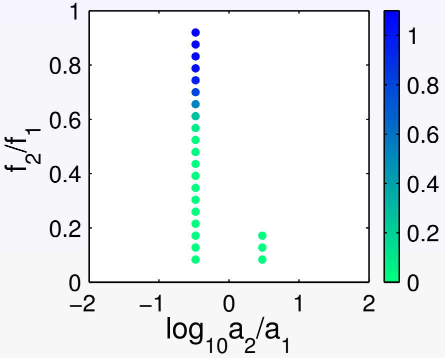


sum of two tones



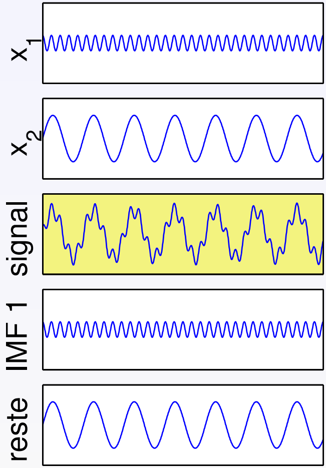
$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



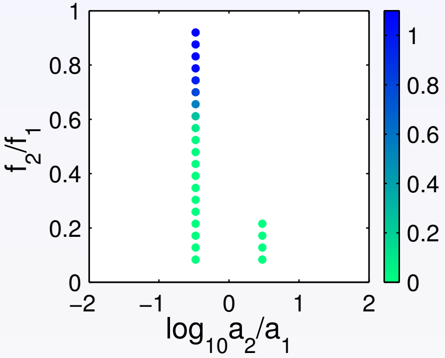
sum of two tones

f_2/f_1 = 0.22, a_2/a_1 = 3.00

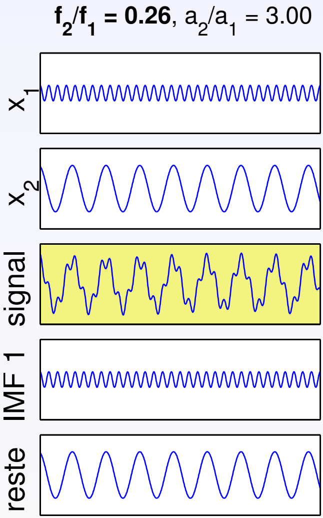


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

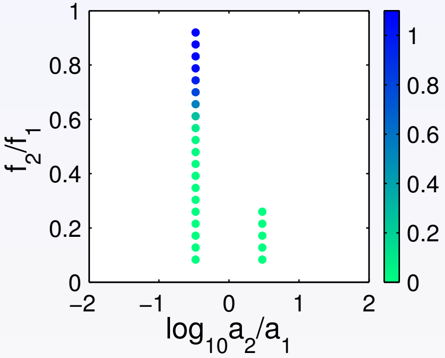


sum of two tones

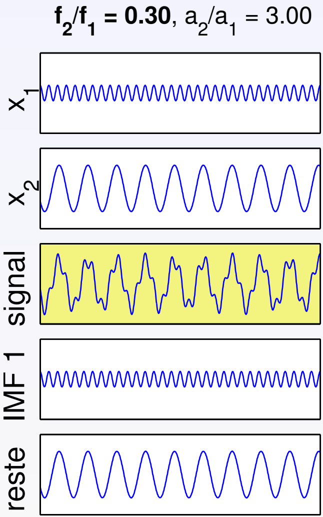


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

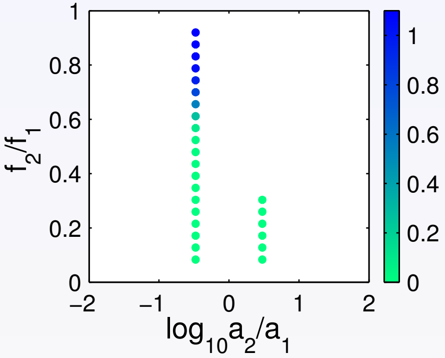


sum of two tones

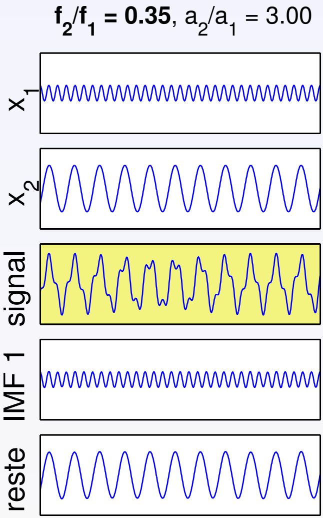


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

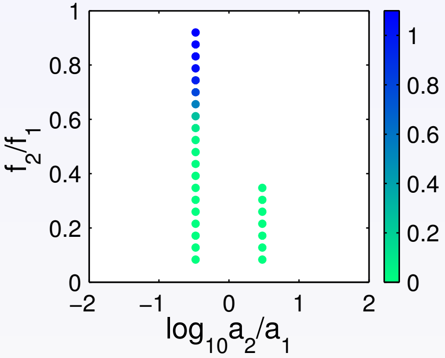


sum of two tones

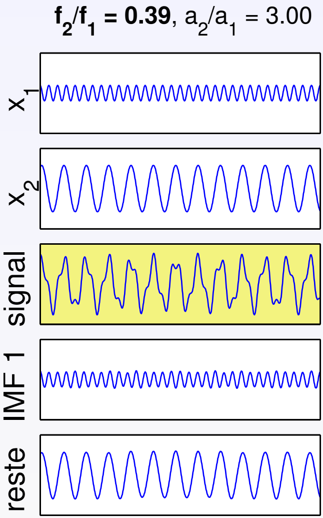


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

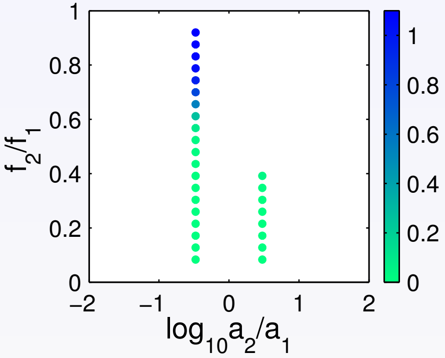


sum of two tones

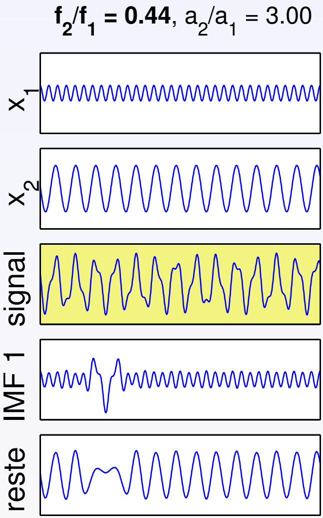


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

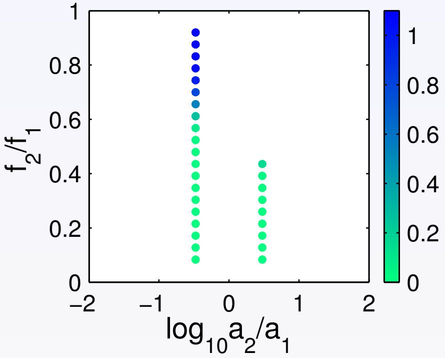


sum of two tones

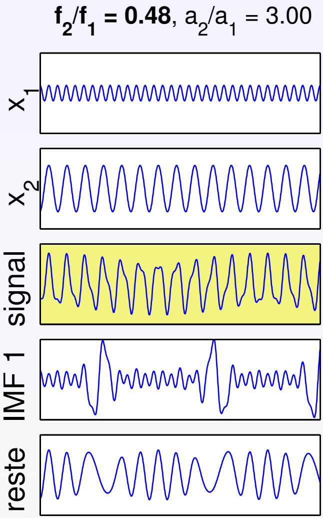


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

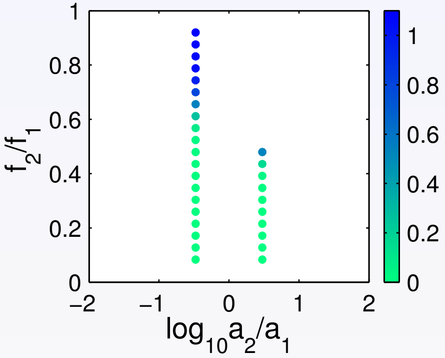


sum of two tones

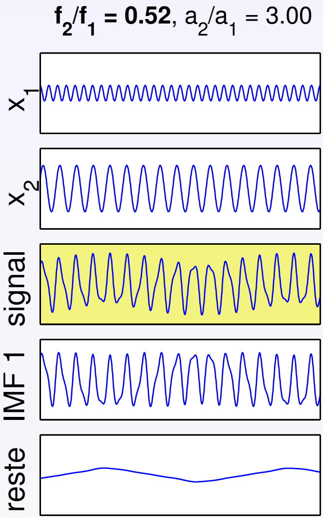


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

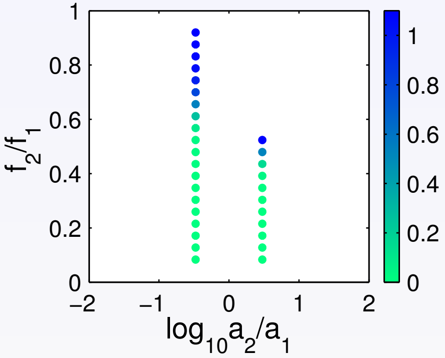


sum of two tones

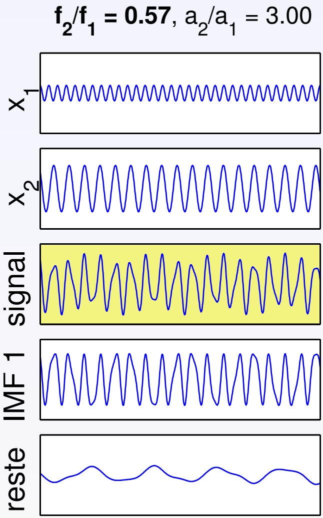


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

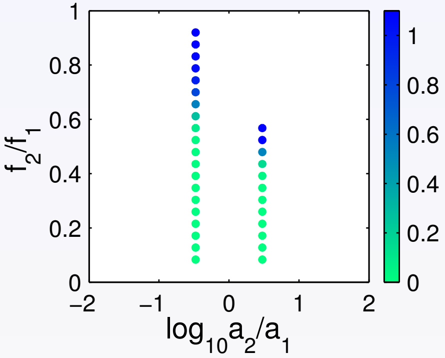


sum of two tones

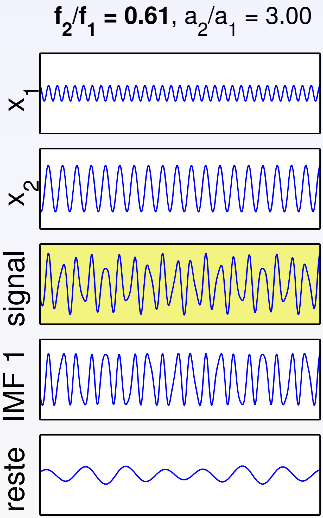


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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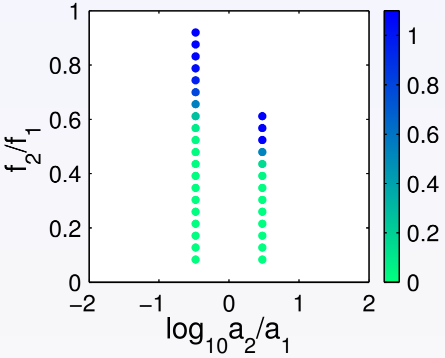
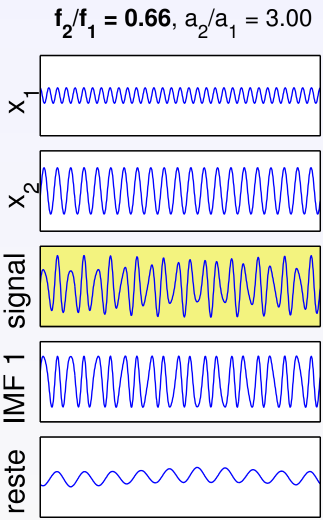


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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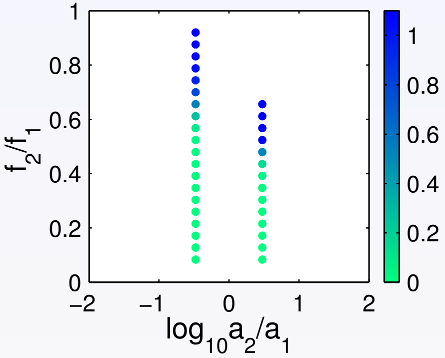
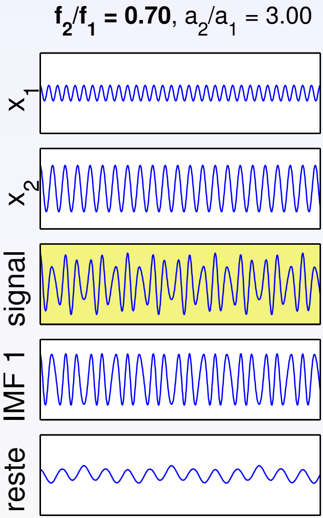


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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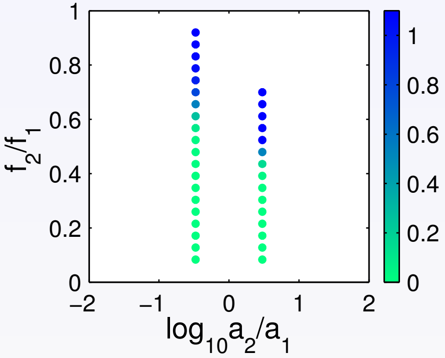
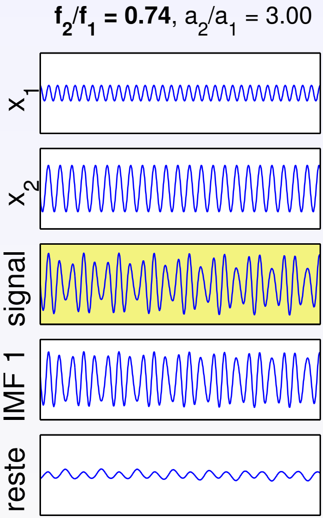


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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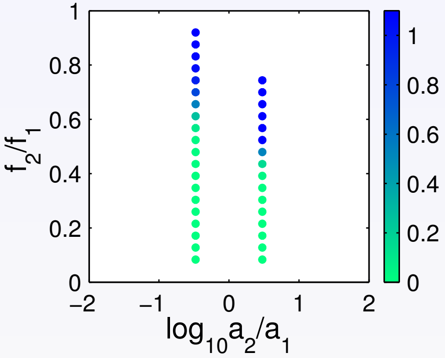
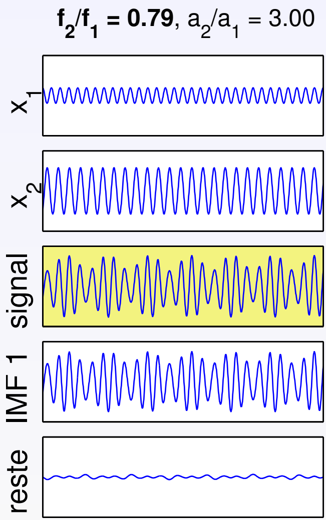


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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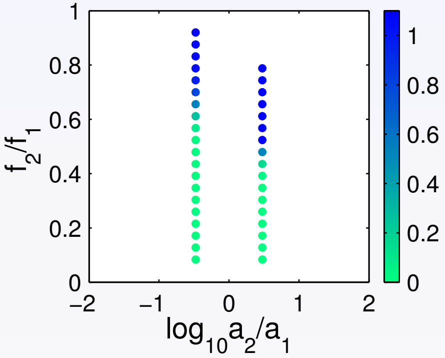
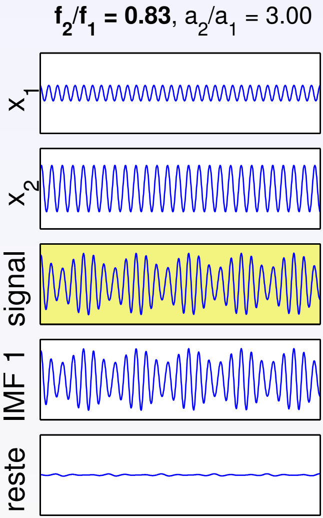


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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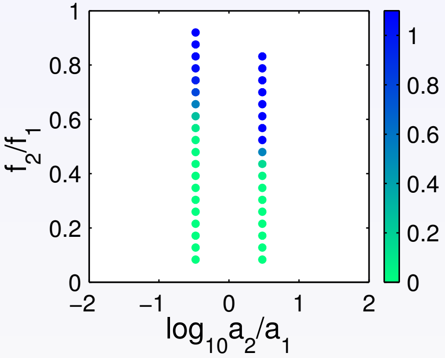
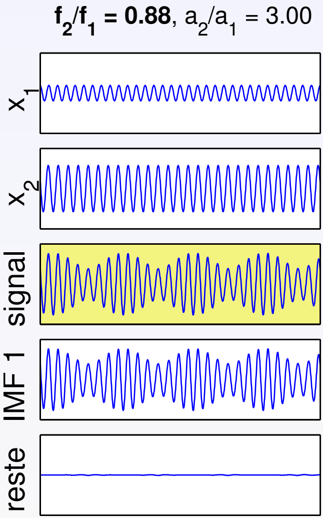


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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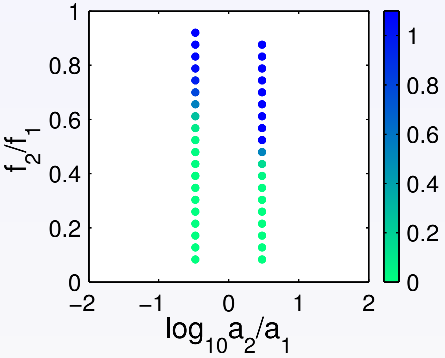
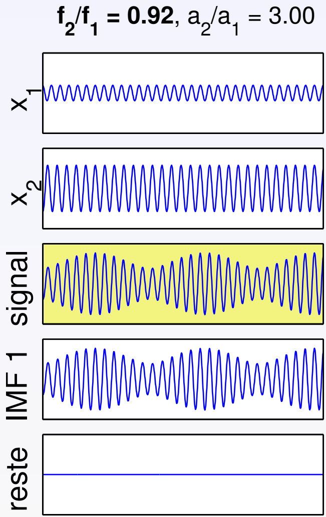


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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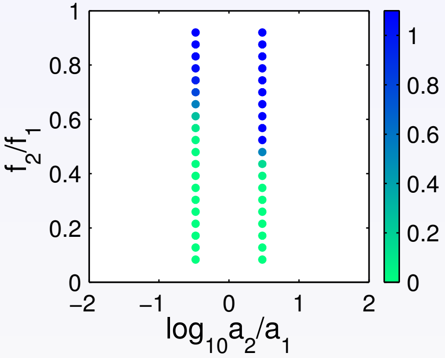
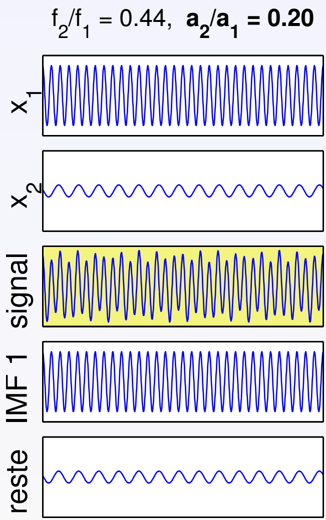


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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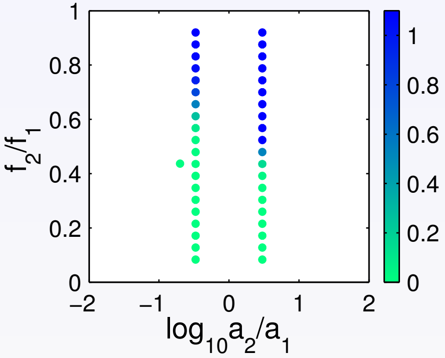
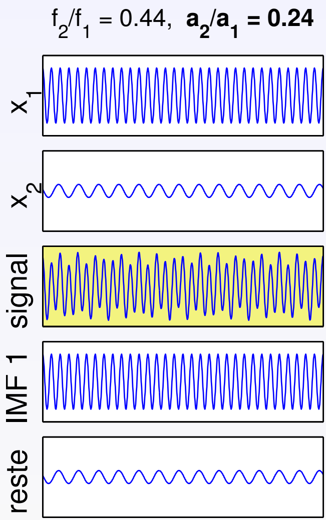


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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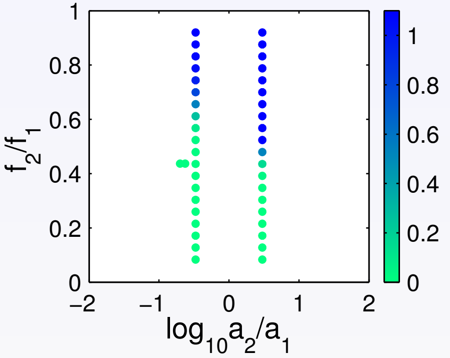
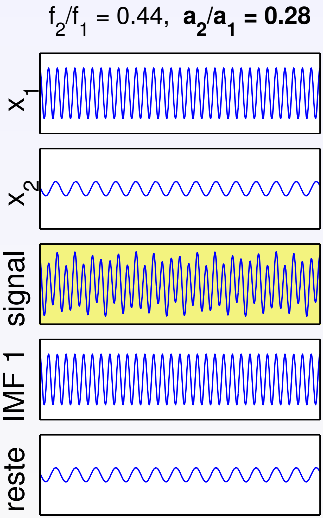


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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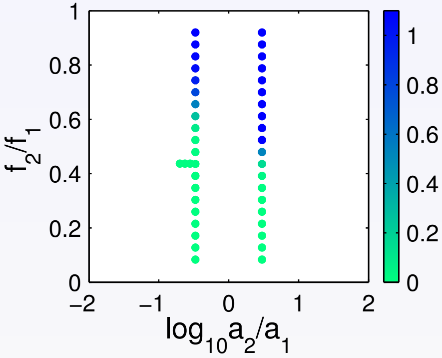
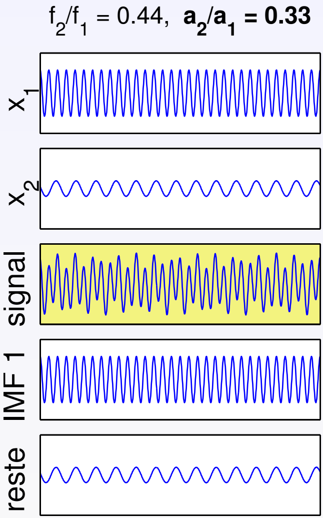


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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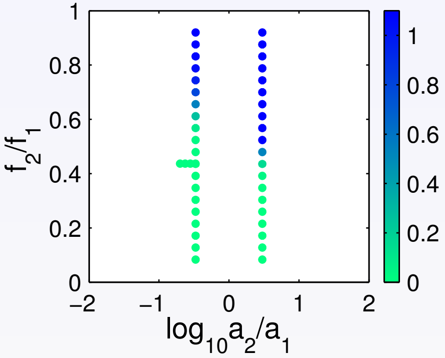
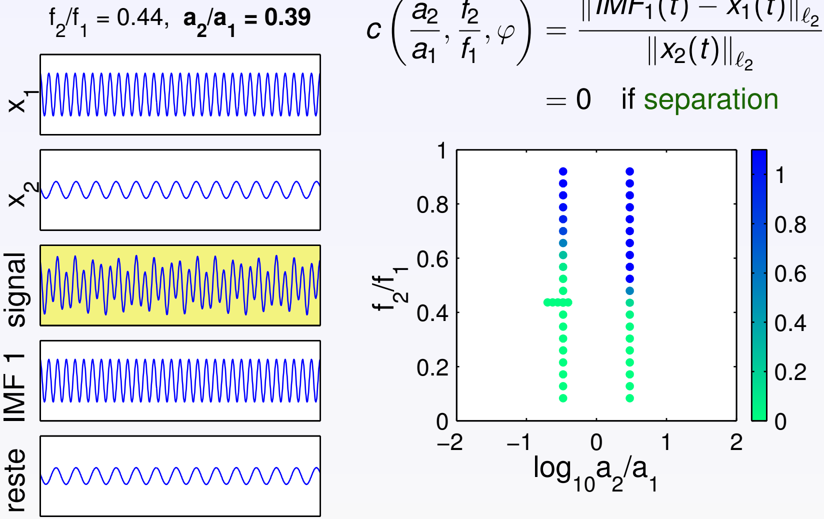


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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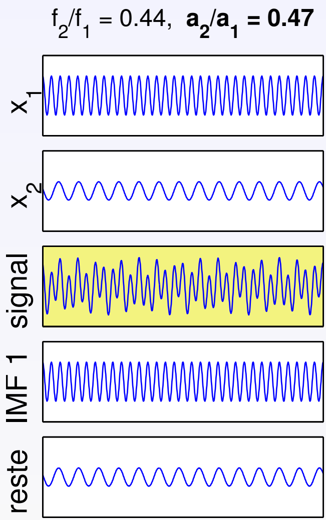
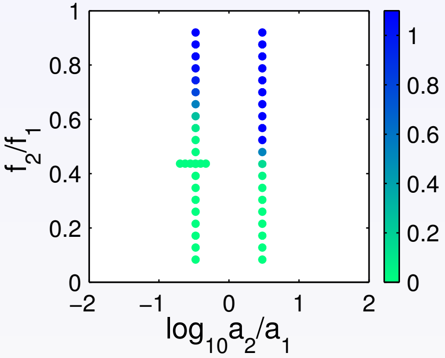
sum of two tones

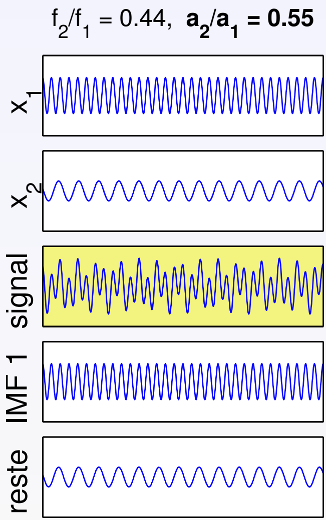


$$c\left(\frac{a_2}{a_1},\frac{f_2}{f_1},\varphi\right)=\frac{\|IMF_1(t)-x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

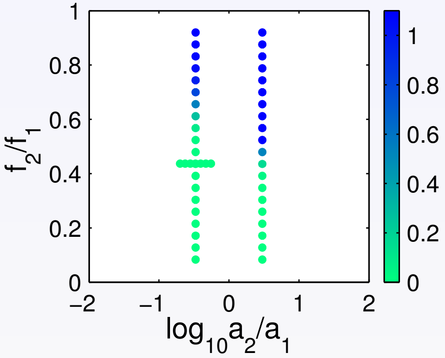


sum of two tones

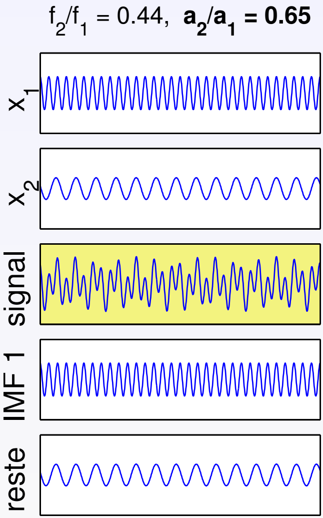


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation

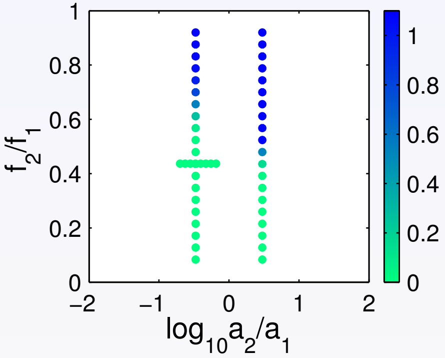


sum of two tones

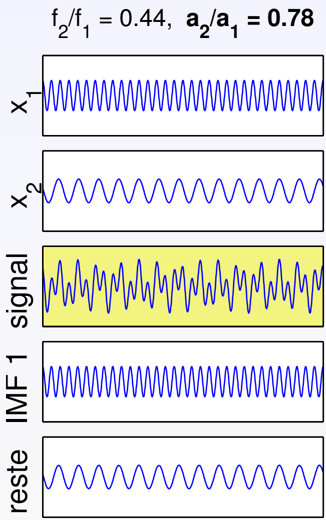


$$c\left(\frac{a_2}{a_1},\frac{f_2}{f_1},\varphi\right)=\frac{\|IMF_1(t)-x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

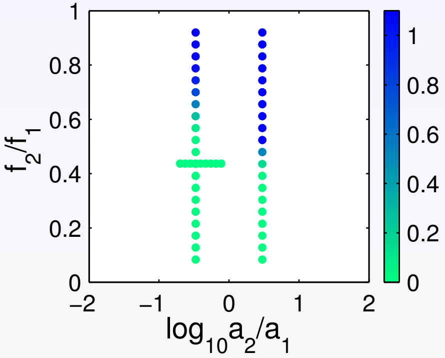


sum of two tones



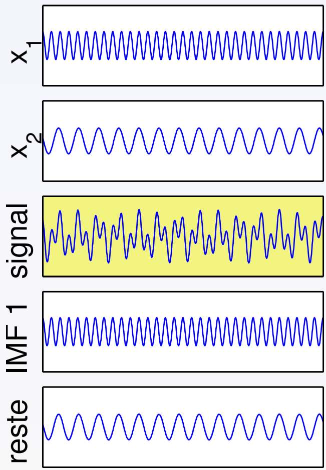
$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



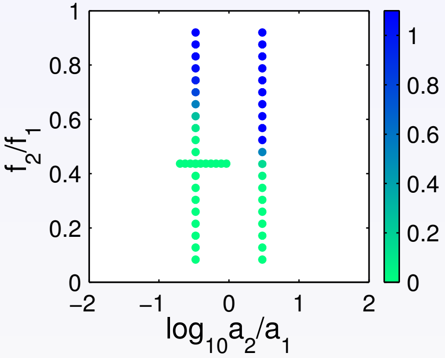
sum of two tones

$f_2/f_1 = 0.44$, $\mathbf{a_2/a_1 = 0.92}$

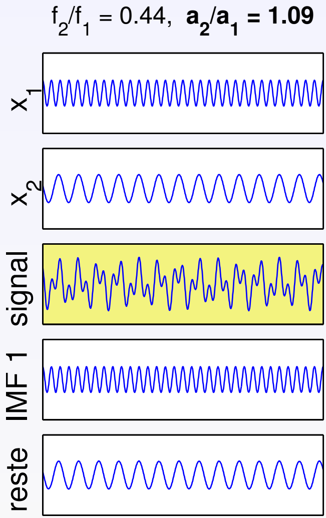


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

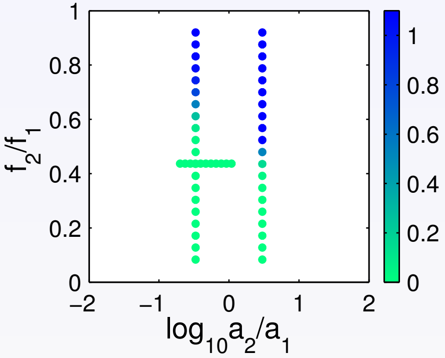


sum of two tones

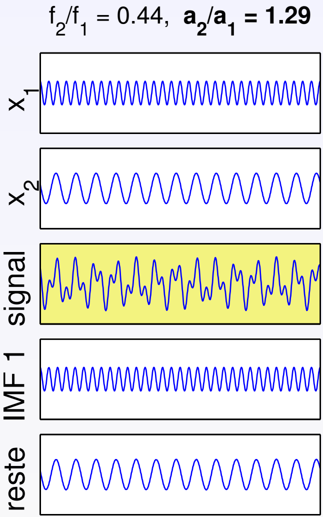


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation



sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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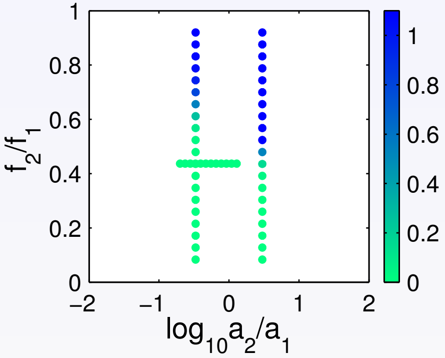
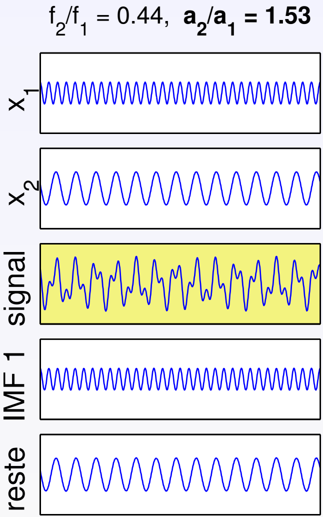


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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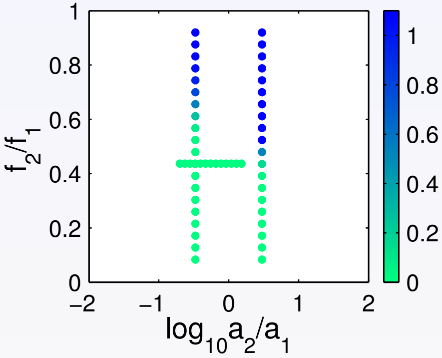
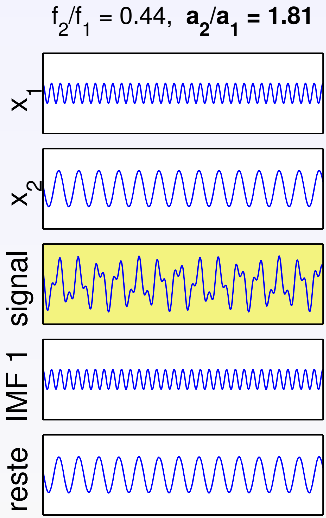


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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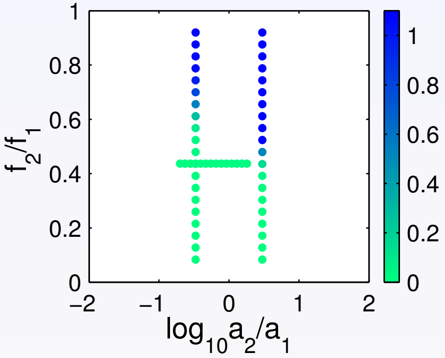
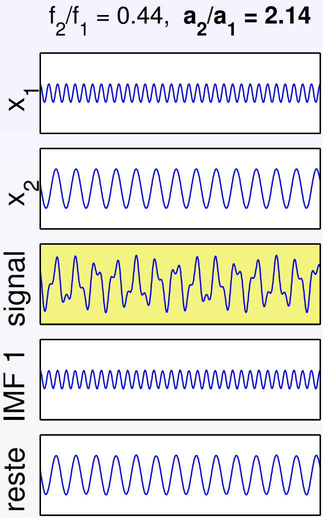


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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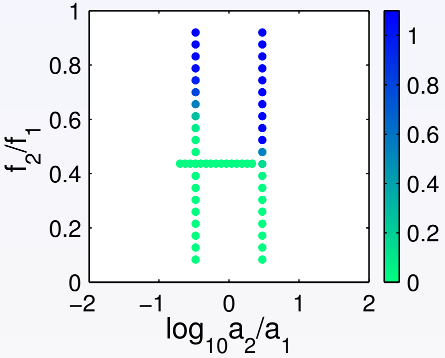
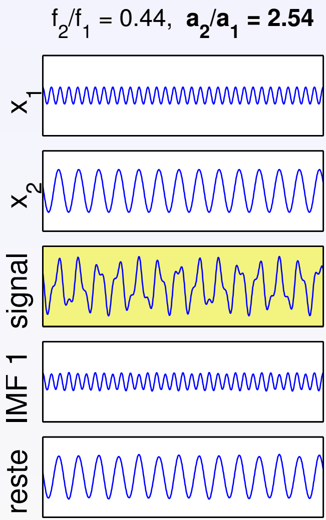


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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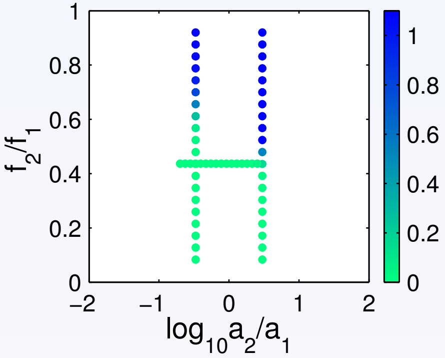
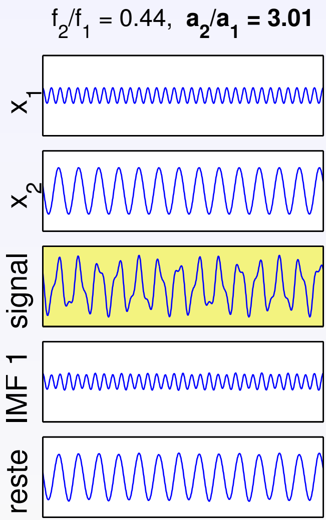


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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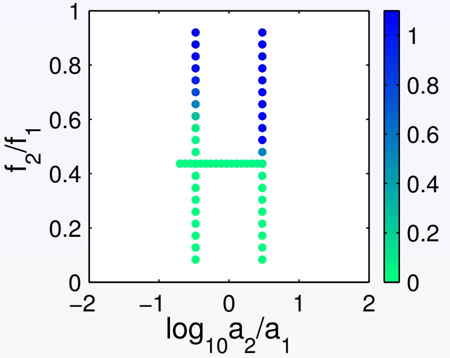
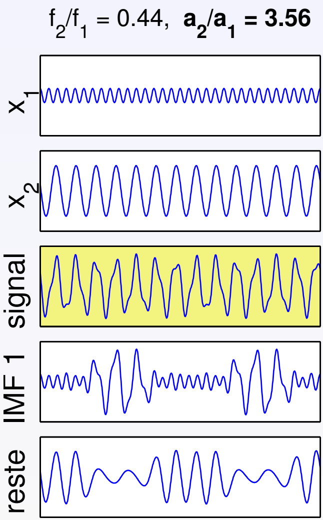
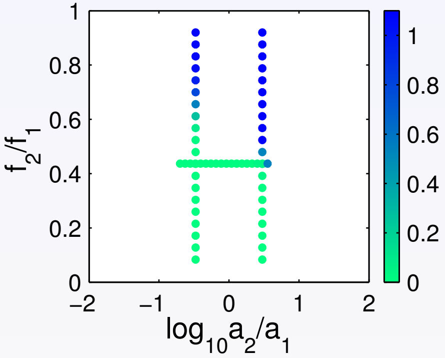


sum of two tones

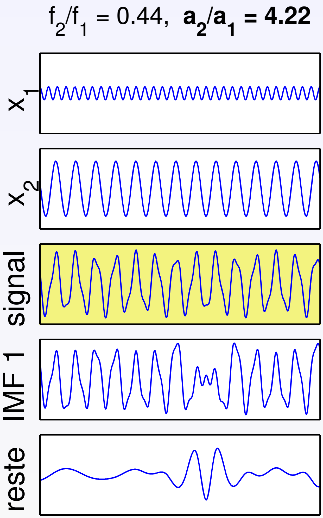


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation



sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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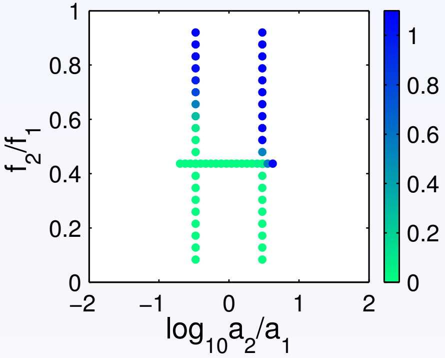
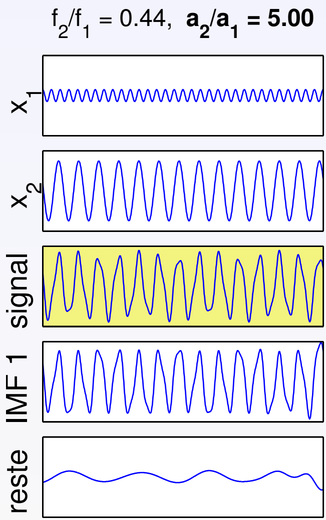
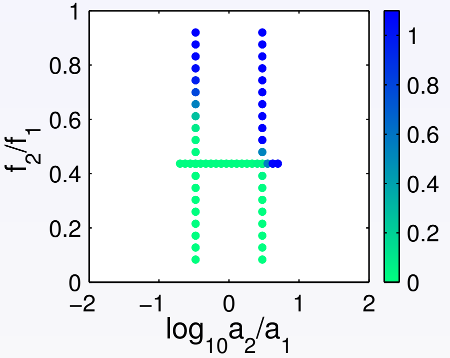


sum of two tones

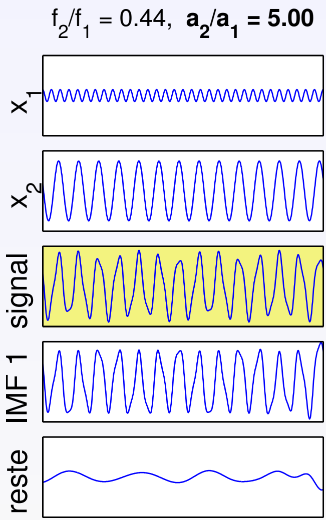


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation



sum of two tones



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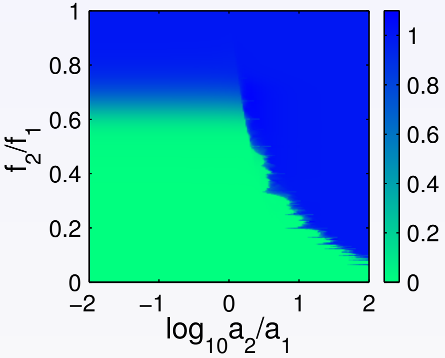
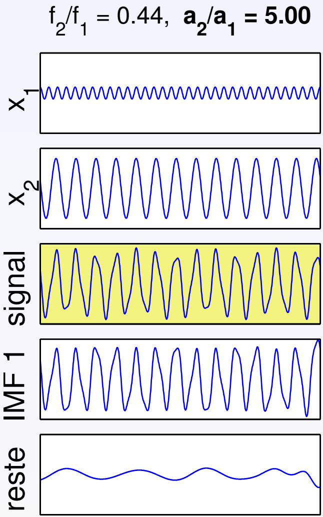


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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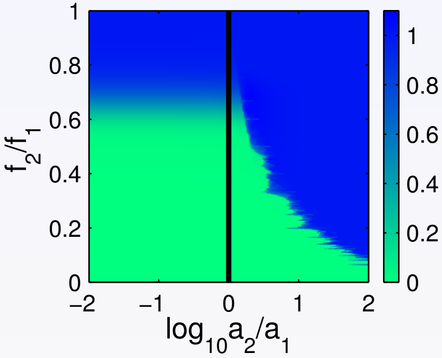
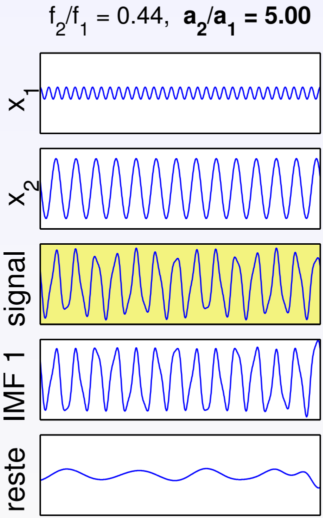


sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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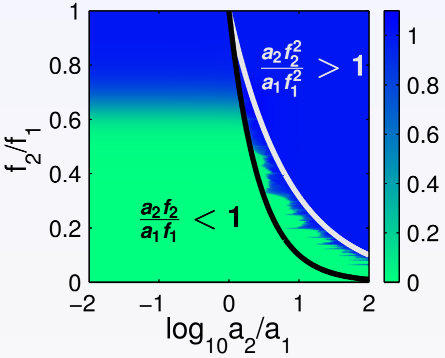
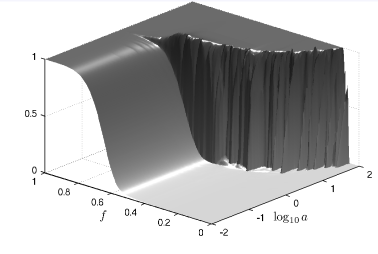


sum of two tones

- **Nonlinear** behaviour ⇒ dissymmetry of tones disentanglement w.r.t. amplitude ratio $a = a_2/a_1$, via the sign of $a - 1$:
  - *smooth variation* when $a < 1$ (HF dominant) & no a-dependence
  - *abrupt phase transition* when $a > 1$ (LF dominant) & strong a-dependence



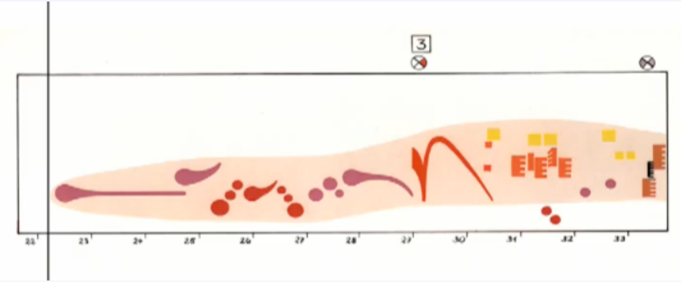
- **Data-driven** separation ⇒ good match to "beating effect" perception ⇒ connection with hearing?

concluding remarks

- Fourier: 200 years and still alive!
- basic ideas related to decompositions and frequency still central in “modern” approaches, whatever the variations (localized and/or evolutive tones, nonlinear techniques,...)
- time-frequency as a natural language

back to music notation

Rainer Wehinger' visual listening score created in the 70's to accompany Gyorgy Ligeti's *Artikulation*



http://www.youtube.com/watch?v=71hNl_skTZQ

(thanks to Laurent Chevillard & Gabriel Rilling)