Wavelets and Mathematical Scores

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to Ingrid Daubechies, 2011 Franklin Institute Laureate

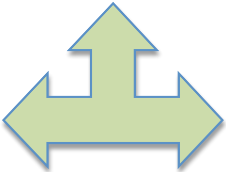
a “3-body system”

« physics »

(laws of Nature, real world applications)

« mathematics »

(models, proofs)



« computer science »

(algorithms)

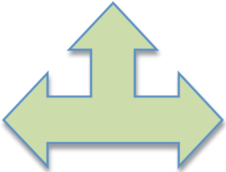
the Fourier example

« physics »

(heat equation)

« mathematics »

(harmonic analysis)



« computer science »

(Fast Fourier Transform)

Fourier analysis/synthesis

Fourier decomposition based on: $e_f(t) := \exp\{i2\pi ft\}$

$$x(t) \to X(f) = \langle x, e_f \rangle,\ s.t.\ x(t) = \int \langle x, e_f \rangle\, e_f(t)\, df$$

- mathematics: all waveforms are made of superimposed **everlasting, fixed frequency tones**
- physics (musical intuition): what about **notes** and **gliding frequencies**?

from tones to atoms

Way out

"localized tones" $\Rightarrow$ *switch to a 2-parameter group of transformations that include time*

$$x(t) \to T(t, \lambda) = \langle x, h_{t,\lambda} \rangle, \text{ s.t. } x(t) = \iint \langle x, h_{s,\lambda} \rangle \, h_{s,\lambda}(t) \, d\mu(s, \lambda)$$

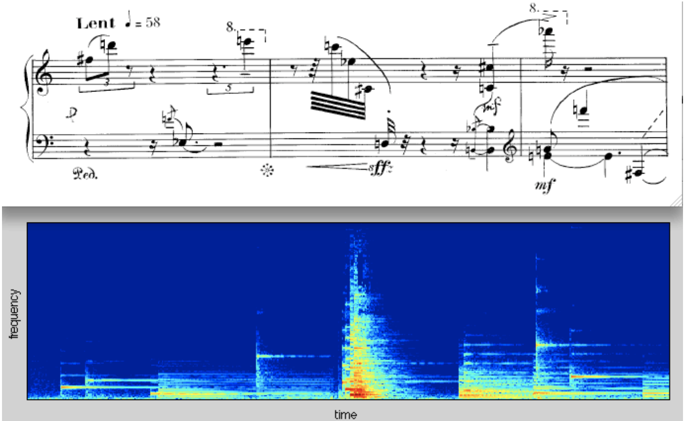
1. time-frequency: $\lambda = f$ and $h_{s,f}(t) = h(t - s)\, e_f(t)$

   $\to$ **short-time Fourier transform**

2. time-scale: $\lambda = a$ and $h_{s,a}(t) = |a|^{-1/2}\, h((s - t)/a)$

   $\to$ **wavelet transform**

a mathematical score



extending spectrum analysis

From stationarity. . .

"Wiener-Khintchine-Bochner" spectrum analysis:
$\Gamma_x(f) = \mathcal{F}\{\gamma_x\}(f)$*, with* $\gamma_x(\tau) := \langle x, \mathbf{T}_\tau x\rangle$ *a **time-independent** correlation*

. . . to nonstationarity

$\gamma_x \rightarrow$ ***time-frequency*** *correlation* $\langle x, \mathbf{T}_{\tau,\xi} x\rangle$ + *2D Fourier transform* $\Rightarrow$ ***Wigner-type transforms***

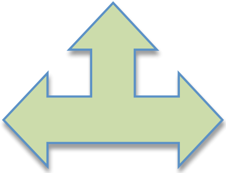
- *intrinsic definitions*
- *no dependence on a measurement device (window, wavelet)*

3 facets

« physics »

joint measurement of position and momentum

(Heisenberg, 1925)



« mathematics »

any Fourier pair of variables

(Weyl, 1927)

« computer science »

time and frequency

(Gabor, 1946 + ...)

classical formulation

Localization trade-off

based on a second-order (variance-type) measure:
$\Delta t_x = (\int t^2\,|x(t)|^2\,dt)^{1/2}$ *and* $\Delta f_x = (\int f^2\,|X(f)|^2\,df)^{1/2} \Rightarrow$

$$\Delta t_x\,\Delta f_x \geq \frac{\|x\|}{4\pi}\ (> 0)$$

- variations: same limitation with other measures of spread, e.g., entropy (Hirschman, 1957)
- common feature: **Gaussians** are minimizers

from 2 × 1 dimension to 1 × 2 dimensions

[time-frequency spreads or entropies (e.g., De Bruijn, 1967)]

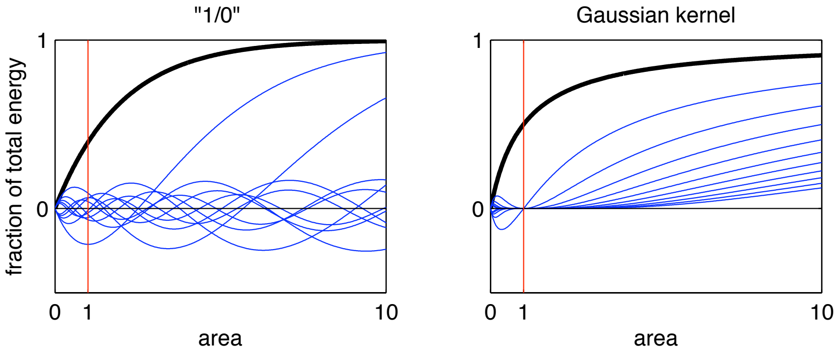
Joint energy concentration

$$\max_x \iint_D \rho_x(t,f)\,dt\,df\,?$$

D elliptic ⇒ Hermite functions eigenfunctions of the TF concentration operator for Wigner distributions, either

1. *on "1/0" domains (F., 1988; Lieb, 2010)*
2. *with Gaussian kernels, i.e., Gabor spectrograms (Daubechies, 1988)*

eigenvalues



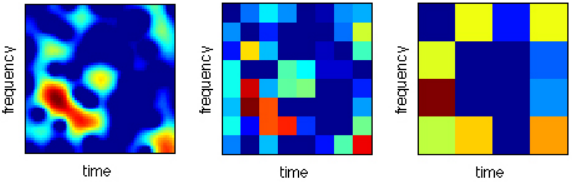
Gaussians as maximizers

no pointwise TF localization

Reproducing kernel identity

$$T(t', \lambda') = \iint \langle h_{t,\lambda}, h_{t',\lambda'} \rangle \, T(t, \lambda) \, d\mu(t, \lambda)$$

- $\langle h_{t,\lambda}, h_{t',\lambda'} \rangle \neq \delta(t - t') \, \delta(\lambda - \lambda') \Rightarrow$ **redundancy**
- time-*frequency* ($\lambda = f$) or time-*scale* ($\lambda = a$) **sampling**, in analogy with Shannon sampling for band-limited functions



Heisenberg revisited

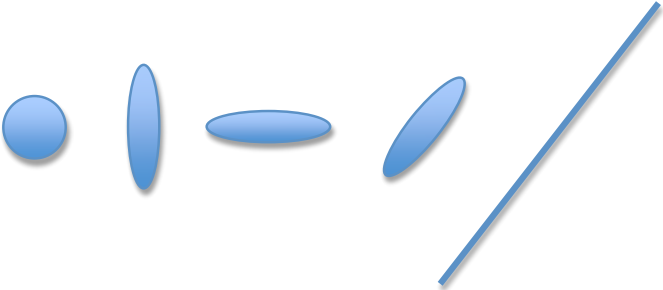
no pointwise localization does not mean no localization

Refined uncertainty relation (Schrödinger, 1935)

$$\Delta t_x \, \Delta f_x \geq \frac{\|x\|}{4\pi} \sqrt{1 + 16\pi^2 \left(\int t \, \dot{\varphi}(t) \, |x(t)|^2 \, dt \right)^2}$$

- ***"squeezed states"*** $\left\{\exp(\alpha t^2 + \beta t + \gamma); \mathrm{Re}\{\alpha\} \leq 0\right\}$ *as minimizers, with **linear "chirps"** as limiting form*
- ***perfect** localization for Wigner distribution, with possible extensions to nonlinear chirps*

energy ellipses



"chirp" signals

Model

multicomponent waveforms $x(t) = \sum_{k=1}^{K} a_k(t)\, e^{i\varphi_k(t)}$, *with*

- *amplitude modulations (AM)* $a_k(t)$
- *frequency modulations (FM)* $f_x(t) := \dot{\varphi}_k(t)/2\pi$

Aim

get a localized TF energy distribution of the form
$\rho(t, f) = \sum_{k=1}^{K} a_k^2(t)\, \delta\, (f - f_x(t))$

the duality “density/correlation”

Definition

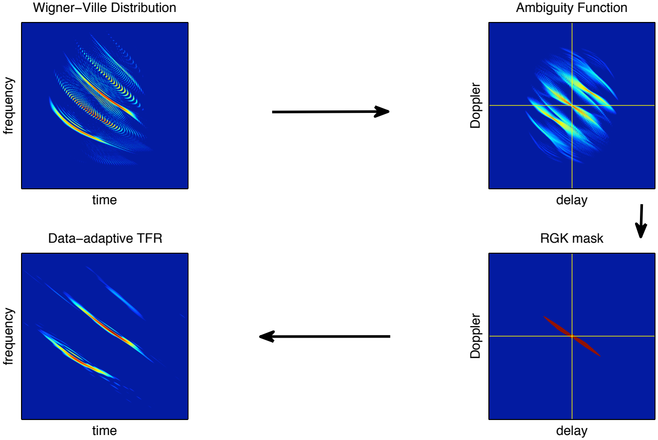
by definition, $W_x(t,f) \stackrel{2D-FT}{\longrightarrow} \mathcal{F}\{W_x\}(\xi,\tau) := A_x(\xi,\tau)$*:*
ambiguity function *(AF)*

Interpretation

given the TF shifts $(\mathbf{T}_{\xi,\tau}x)(t) := x(t-\tau)\, e^{-i2\pi\xi(t-\tau/2)}$*, we have* $A_x(\xi,\tau) = \langle x, \mathbf{T}_{\xi,\tau}x\rangle \Rightarrow$ ***AF = TF correlation****, with*

- *auto-terms neighbouring the origin of the plane*
- *cross-terms at a distance from the origin which equals the TF separation between components*

the other trade-off and “classical” solutions



approach 1 — reassignment(s)

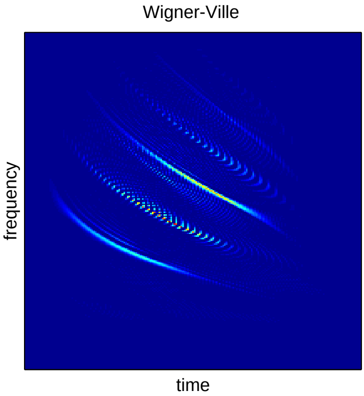
Observation

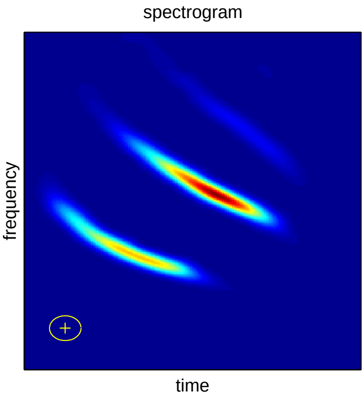
*(spectro/scalo)grams are **smoothed Wigner distributions***

Idea

- *move computed values to **local energy centroids***
- *3 versions*
  1. *“**hard**”: fixed point method (Kodera, Gendrin & De Villedary, 1976; Auger & F., 1995)*
  2. *“**differential**” : ODE (Auger, Chassande-Mottin, Daubechies & F., 1997)*
  3. *“**soft**”: iteration with damping à la Levenberg-Marquardt (Auger, Chassande-Mottin & F., 2011)*

reassignment in action





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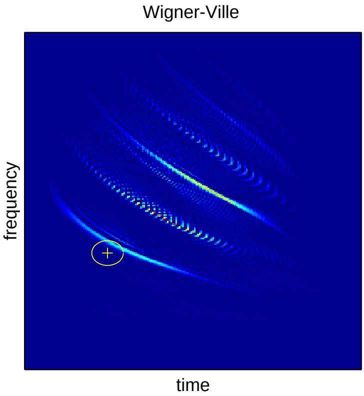
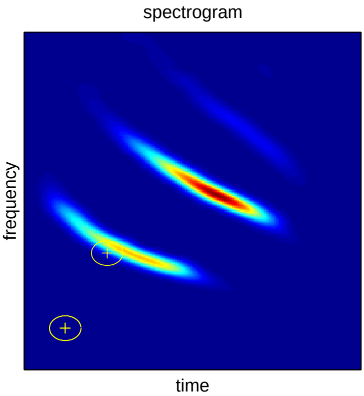
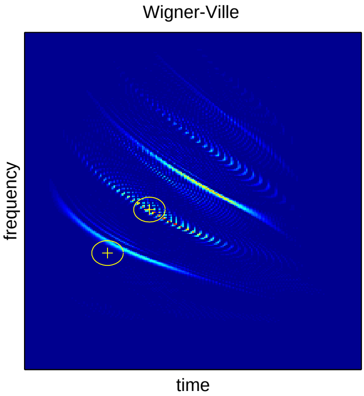
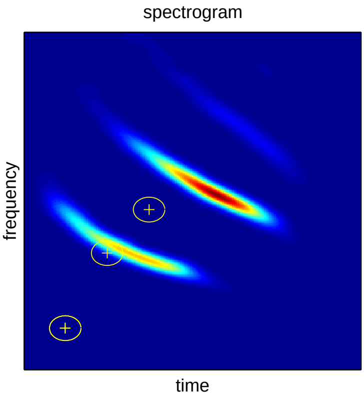


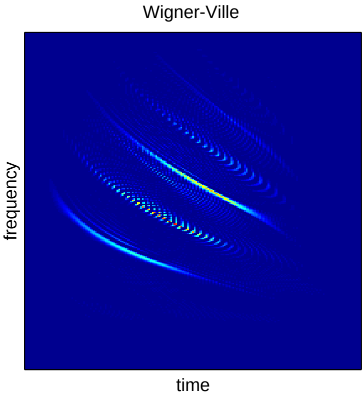


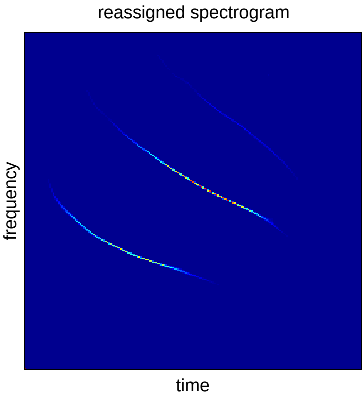
reassignment in action





reassignment in action





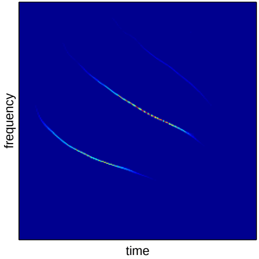
approach 2 — “compressed sensing”

Discrete-time

signal of dimension N $\Rightarrow$ TFD of dimension N^2 when computed over N frequency bins

Few components

$K \ll N \Rightarrow$ at most $KN \ll N^2$ non-zero values in the TF plane



Sparsity

minimizing ℓ_0-norm not feasible, but near-optimal solution by minimizing ℓ_1-norm (as in “compressed sensing”)

approach 2 — "compressed sensing"

Idea

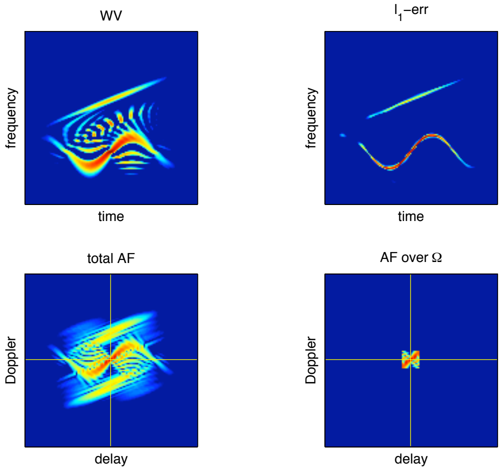
1. *select a domain Ω neighbouring the origin of the AF plane*
2. *solve the program*

$$\min_{\rho} \|\rho\|_1 \,;\, \mathcal{F}\{\rho\} - A_x = 0|_{(\xi,\tau)\in\Omega}$$

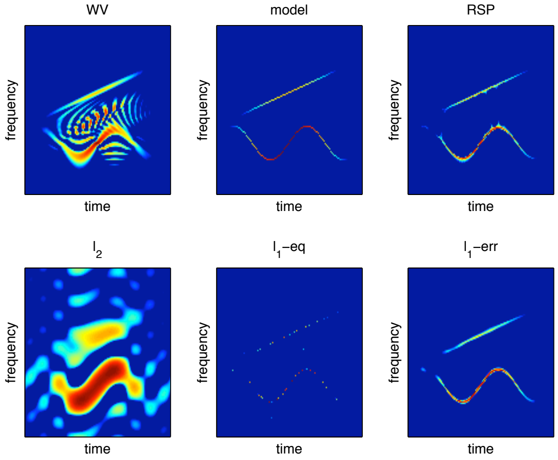
3. *the exact equality over Ω can be relaxed according to*

$$\min_{\rho} \|\rho\|_1 \,;\, \|\mathcal{F}\{\rho\} - A_x\|_2 \leq \epsilon|_{(\xi,\tau)\in\Omega}$$

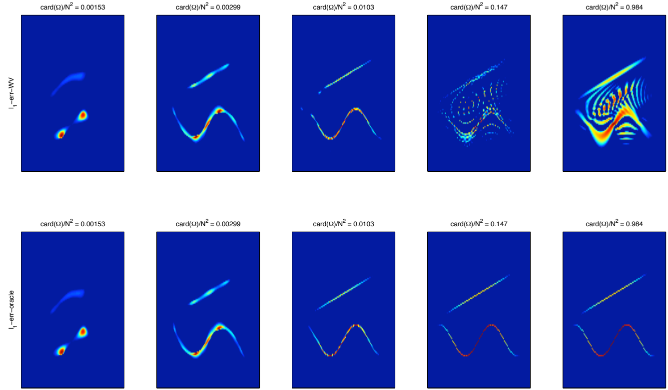
CS approach in action — principle



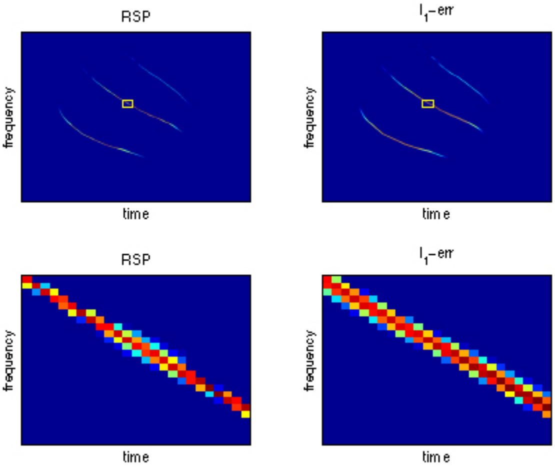
CS approach in action — comparison



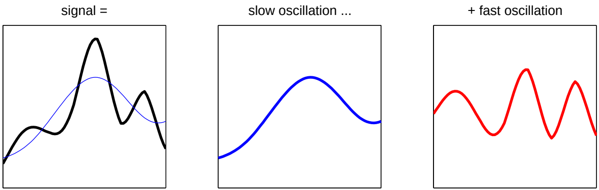
CS approach in action — convergence % “oracle”



bat chirp example



approach 3 — Empirical Mode Decomposition



Idea of "EMD" (Huang *et al.*, 1998)

signal = fast oscillation + slow oscillation

&

iteration

- separation "fast vs. slow" **data driven**
- **"local"** analysis based on neighbouring extrema
- **oscillation** rather than frequency

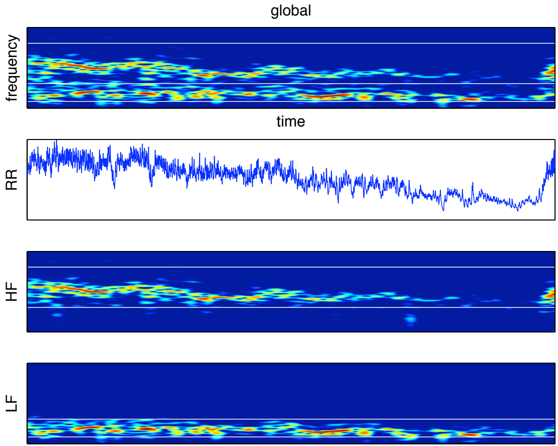
EMD algorithm

1. identify local maxima and local minima
2. deduce an upper envelope and a lower envelope by interpolation (cubic splines)
    1. subtract the mean envelope from the signal
    2. iterate until "mean envelope $= 0$" (*sifting*)
3. subtract the obtained mode from the signal
4. iterate on the residual

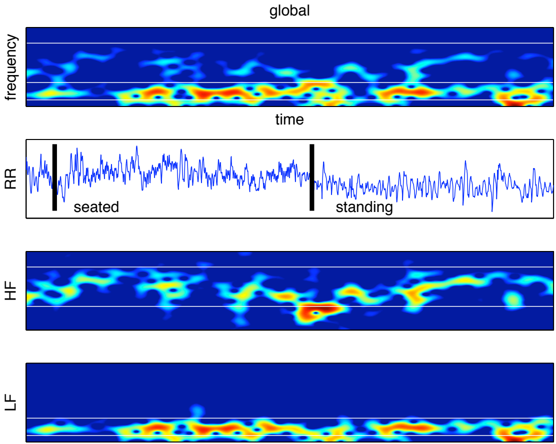
$$
\begin{aligned}
x(t) &= c_1(t) + r_1(t) \\
&= c_1(t) + c_2(t) + r_2(t) \\
&= \dots\dots\dots\dots\dots\dots \quad = \sum_{k=1}^{K} c_k(t) + r_K(t),
\end{aligned}
$$

with the $c_k(t)$'s referred to as **Intrinsic Mode Functions** (IMFs)

Heart Rate Variability example 1



Heart Rate Variability example 2



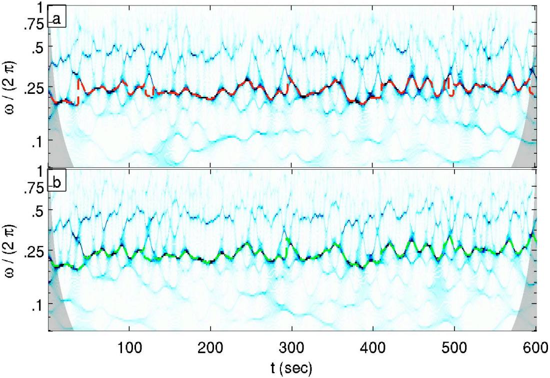
approach 4 — synchrosqueezing

Idea (Daubechies & Maes, 1996)

concentrate wavelet ***coefficients****, at* ***fixed times****, on the basis of* ***local frequency*** *information*

- guarantees a **sharply localized** representation (variant of reassignment)
- allows for a **reconstruction** of identified “modes”
- offers a **mathematically tractable** alternative to EMD (Daubechies, Lu & Wu, 2010 ; Wu, F. & Daubechies, 2011)

respiratory signal example



(a) via ECG and (b) reference

[courtesy of H.-T. Wu (Princeton)]

concluding remarks

1. **a unifying paradigm**
   time-frequency as a physically meaningful framework
2. **a computational perspective**
   wavelets instrumental in efficiently connecting theory with practice
3. **still many variations**
   Fourier limitations always apply $\Rightarrow$ no unique solution
   multiplicity of complementary approaches