ESTIMATING SINGULARITIES WITH REASSIGNED DISTRIBUTIONS

Eric Chassande-Mottin[1], *François Auger*[2] *and Patrick Flandrin*[3]

[1]Max Planck Institut für Gravitationsphysik, Albert Einstein Institut, Am Mühlenberg, 1,
D-14424 Golm, Germany eric@aei-potsdam.mpg.de
[2]GE44-LRTI, Bd de l'Université, BP 406,
44602 Saint-Nazaire Cedex, France francois.auger@univ-nantes.fr
[3]Ecole Normale Supérieure de Lyon, Labo. de Physique (UMR 5672 CNRS), 46 allée d'Italie,
69364 Lyon Cedex 07, France flandrin@ens-lyon.fr

ABSTRACT

Sharply localized time-frequency and time-scale distributions can be obtained by means of a nonlinear technique referred to as reassignment. Basics of reassignment are recalled in the case of both spectrograms and scalograms (i.e., short-time Fourier and wavelet-based energy densities, respectively), with emphasis on their usefulness for characterizing isolated Hölder singularities as well as oscillating singularities.

1. INTRODUCTION

Time-frequency analysis of nonstationary signals can be performed in many different ways, with techniques ranging from short-time Fourier or wavelet transforms to Wigner-type methods [6, 7, 14]. Whereas the former approaches exhibit poor localization properties, the latter ones are impaired by interference phenomena which limit the effectiveness of their sharper localization in multicomponent situations [10]. A nonlinear technique, referred to as *reassignment*, has been proposed to overcome both limitations [1, 12, 13]. In a nutshell, reassignment is a two-step process which consists in smoothing out oscillating interference terms while squeezing the localized terms which have been spread over the plane by the smoothing operation. Originally proposed only for spectrograms (short-time Fourier energy densities) [12, 13], the method has been generalized far beyond [1, 3], with a possible application to time-scale techniques such as scalograms (wavelet energy densities). The purpose of this paper is to address some specific issues related to scalogram reassignment, and especially to investigate how reassigned scalograms may be used for characterizing Hölder as well as oscillating singularities.

2. THE REASSIGNMENT PRINCIPLE

It is a well-known fact in time-frequency and time-scale analysis that quadratic energy distributions are faced with a conflict between their joint resolution and the importance of their interference terms [7, 10]. A way of getting some insight into this conflict is to start with the general frameworks offered by the Cohen's class and the affine class.

By definition [6, 7], Cohen's class $C_x(t,\omega)$ of finite energy signals $x(t) \in L^2(\mathbb{R})$ is the family of functions of time t and (angular) frequency ω that are given by[1] :

$$C_x(t,\omega) = \iint W_x(s,\xi)\,\Pi(s-t,\xi-\omega)\,\frac{ds\,d\xi}{2\pi}, \quad (1)$$

with $W_x(t,\omega)$ the *Wigner-Ville distribution* (WVD) :

$$W_x(t,\omega) := \int x(t+s/2)\,\overline{x(t-s/2)}\,e^{-i\omega s}\,ds, \quad (2)$$

and $\Pi(t,\omega)$ an arbitrary kernel function.

In a similar way [16], the affine class is the family of functions of time t and scale $a > 0$ given by :

$$\Omega_x(t,a) = \iint W_x(s,\xi)\,\Pi\left(\frac{s-t}{a}, a\xi\right)\frac{ds\,d\xi}{2\pi}. \quad (3)$$

In cases for which the arbitrary kernel $\Pi(t,\omega)$ is of low-pass type, the above distributions (1) and (3) make explicit the fact they both result from a *smoothing* of the WVD (2), with the consequence that the value that they take at a given point of the plane cannot be considered as pointwise. In fact, this value rather results from the summation of all Wigner-Ville contributions within some time-frequency domain defined as the essential time-frequency support of $\Pi(t,\omega)$, properly "centered" at the location of the considered point

[1]Throughout this paper, integrals run from $-\infty$ to $+\infty$, except for scale variables for which the integration range is $[0,\infty)$.

of interest. A whole distribution of values is therefore summarized by a single number, and this number is assigned to the geometrical center of the domain over which the distribution is considered. Reasoning with a mechanical analogy, the situation is as if the total mass of an object was assigned to its geometrical center, an arbitrary point which—except in the very specific case of an homogeneous distribution over the domain—has no reason to suit the actual distribution. A much more meaningful choice is to assign the total mass to the *center of gravity* of the distribution within the domain, and this is precisely what reassignment does : at each point where a distribution value is computed, we also compute the local centroïd of the WVD distribution, as seen through the time-frequency window defined by the local kernel, and the distribution value is moved from the point where it has been computed to this centroïd.

3. REASSIGNING SPECTROGRAMS AND SCALOGRAMS

General forms can be given for centroïds associated to any distribution [1] but, for a sake of simplicity, a special attention can be paid to spectrograms and scalograms, the only cases we will consider here.

Given a short-time window $h(t) \in L^2(\mathbb{R})$ and an admissible wavelet $\psi(t) \in L^2(\mathbb{R})$, the spectrogram and the scalogram of a signal $x(t) \in L^2(\mathbb{R})$ are respectively defined by $S_x^{(h)}(t,\omega) := |F_x^{(h)}(t,\omega)|^2$ and $\Sigma_x^{(\psi)}(t,a) := |T_x^{(\psi)}(t,a)|^2$, where $F_x^{(h)}(t,\omega)$ and $T_x^{(\psi)}(t,a)$ stand for its short-time Fourier transform (STFT) and continuous wavelet transform (CWT), defined respectively by :

$$F_x^{(h)}(t,\omega) := \int x(s)\, \overline{h(s-t)}\, e^{-i\omega(s-t/2)}\, ds\ ; \qquad (4)$$

$$T_x^{(\psi)}(t,a) := \frac{1}{\sqrt{a}} \int x(s)\, \overline{\psi\left(\frac{s-t}{a}\right)}\, ds. \qquad (5)$$

As written in (5), the CWT is a function of time and scale but one can remark that, under mild conditions on the analyzing wavelet $\psi(t)$ (namely that its spectrum is unimodal and characterized by some reference (angular) frequency $\omega_0 > 0$), it can also be expressed as a function of time and frequency $\omega > 0$, with the identification $\omega := \omega_0/a$.

Spectrograms and scalograms just happen to be special cases of (1) and (3), with $\Pi(t,\omega) \equiv W_h(t,\omega)$ or $W_\psi(t,\omega)$, respectively. As such, centroïds needed in the reassignment process could be directly computed from (1) and (3). This would be however a very unefficient way of action, which has been proved [1] to be replaced with much more computational profit by the equivalent procedure :

$$\hat{t}_x(t,\omega) = t + \mathrm{Re}\left\{F_x^{(\mathbf{T}h)}/F_x^{(h)}\right\}(t,\omega)\ ; \qquad (6)$$

$$\hat{\omega}_x(t,\omega) = \omega - \mathrm{Im}\left\{F_x^{(\mathbf{D}h)}/F_x^{(h)}\right\}(t,\omega), \qquad (7)$$

with $(\mathbf{T}h)(t) := t\,h(t)$ and $(\mathbf{D}h)(t) := (dh/dt)(t)$.

Scalogram reassignment is almost similar, although it is a two-step process. In fact, whereas the computation of the centroïd in time mimicks the spectrogram case, the associated centroïd in scale $\hat{a}_x(t,a)$ needs some intermediate step in frequency, from which a frequency-to-scale conversion is, then, achieved. From a practical point of view, an efficient evaluation of those centroïds requires the introduction of the two additional wavelets $(\mathbf{T}\psi)(t) := t\,\psi(t)$ and $(\mathbf{D}\psi)(t) := (d\psi/dt)(t)$, thanks to which we have [1] :

$$\hat{t}_x(t,a) = t + a\,\mathrm{Re}\left\{T_x^{(\mathbf{T}\psi)}/T_x^{(\psi)}\right\}(t,a)\ ; \qquad (8)$$

$$\hat{a}_x(t,a) = -\frac{a\,\omega_0}{\mathrm{Im}\left\{T_x^{(\mathbf{D}\psi)}/T_x^{(\psi)}\right\}(t,a)}. \qquad (9)$$

It is worth noting that the computation of the different CWT's involved in the reassignment process (three in the general case, and only two when using a Morlet wavelet $\psi_m(t)$ since we have $\mathbf{D}\psi_m \propto \mathbf{T}\psi_m$ in such a case) can be equipped with fast algorithms [8].

4. REASSIGNMENT AS SQUEEZING

Whereas there is no time-frequency curve onto which conventional spectrograms and scalograms can perfectly localize, their reassigned versions inherit automatically of the localization properties of the WVD on straight lines of the time-frequency plane. Within this model, letting the chirp rate go to $\pm\infty$ leads formally to idealized impulses for which we also have :

$$x(t) = \delta(t-t_0) \Rightarrow W_x(t,\omega) = \delta\left(t-t_0\right), \qquad (10)$$

contrasting with the situation of ordinary scalograms :

$$x(t) = \delta(t-t_0) \Rightarrow \Sigma_x^{(\psi)}(t,a) = \frac{1}{a}\left|\psi\left(\frac{t_0-t}{a}\right)\right|^2. \qquad (11)$$

In this latter case, the essential support of the scalogram corresponds to a time-scale domain (referred to as its *cone of influence*) that is limited by the two lines of equations $t = t_0 \pm a\,\Delta t_\psi/2$, where Δt_ψ stands for a measure of effective width of the wavelet $\psi(t)$.

It immediately follows from (8) that $\hat{t}_x(t,a) = t_0$ for any (t,a) and any $\psi(t)$, thus guaranteeing that the reassigned scalogram reduces to a Dirac distribution,

$\check{\Sigma}_x^{(\psi)}(t,a) \propto \delta(t-t_0)$, as does the WVD in (10). Reassignement acts therefore as a squeezing operator that permits to end up with a perfectly localized distribution in the case of a perfectly localized impulse.

5. SINGULARITY CHARACTERIZATION FROM REASSIGNED SCALOGRAMS

Scalograms are known to provide a tool simultaneously adapted for the detection and the characterization of singularities [14, 15]. Inspired by the impulse example sketched above, it is therefore natural to consider reassigned scalograms of singularities, the underlying motivation being that reassignment methods may improve the contrast in the representation of singularities and therefore their detection.

In a first step, we will limit ourselves to isolated Hölder singularities whose frequency structure can be written as (details about the construction of this family and the explicit form of A_ν can be found in [3]) :

$$X(\omega) = A_\nu |\omega|^{-\nu-1}. \tag{12}$$

Labelling $x^{(\alpha)}(t)$ the fractional derivative of order α of a signal $x(t)$, it turns out [9] that isolated Hölder singularities of the type (12) have the property that their CWT is equal to a rescaled version of the wavelet fractional derivative of order $\alpha = -\nu - 1$:

$$T_x^{(\psi)}(t,a) = A_{-\alpha-1}\, a^{-(\alpha+1/2)}\, i^\alpha\, \overline{\psi^{(\alpha)}(-t/a)}. \tag{13}$$

One can see in (13) two important characteristics of the scalogram structure of Hölder singularities. First, the energy is almost entirely concentrated in the support of $|\psi^{(\alpha)}(-t/a)|$, which defines in the time-scale plane a cone-shaped domain centered around $t=0$. Second, from the restriction of (13) at time $t=0$,

$$\log|T^{(\psi)}(0,a)|^2 = \log|A_\nu \psi^{(\alpha)}(0)|^2 + (2\nu+1)\log a, \tag{14}$$

one can get a simple estimate of ν by measuring the slope along the scale axis in a log-log diagram [15].

In order to go further, the wavelet $\psi(t)$ has to be specified, and it proves convenient to make use of a *Klauder wavelet* [11], defined in the time- domain as

$$\kappa_{\beta,\gamma}(t) = C_{\beta,\gamma}\,(\gamma - it)^{-(\beta+1)}, \tag{15}$$

with $C_{\beta,\gamma}$ a suitable normalization constant. One can check [5] that the Klauder wavelet family is covariant to fractional differentiation, as well as stable by multiplication by t and by differentiation. Combining these properties, we can get the algebraic form of the three wavelet transforms involved in (8) and (9), leading finally to (see [3, 5] for details) :

$$\hat{t}(t,a) = \frac{\alpha}{\alpha+\beta}\, t\ ; \tag{16}$$

$$\hat{a}(t,a) = \frac{\omega_0}{\alpha+\beta+1}\left(\gamma a + \frac{t^2}{\gamma a}\right), \tag{17}$$

with the reference frequency of the Klauder wavelet equal to $\omega_0 = (\beta+1/2)/\gamma$.

Therefore, although the Klauder wavelet is of infinite support in time, it leads to a reassigned scalogram whose time-scale support is *strictly* limited to a cone centered at the time of occurrence $t=0$ of the singularity. The sharpness of this cone is controlled by both the singularity strength (through α) and the chosen wavelet (through β and γ). More precisely, it can be shown that, for any fixed β and γ (i.e., for any fixed Klauder wavelet), the angle θ of the cone goes to zero as $\theta \sim [(\gamma/\beta)(\beta+1)/(\beta+1/2)] \times \alpha$ when $\alpha \to 0$, as expected from the result we obtained in the impulse case. Conversely, we also get that θ goes to zero as $\theta \sim \gamma\alpha/\beta$ when α is fixed and $\beta \to \infty$. In such a limit, the Klauder wavelet converges to a Morlet wavelet, thus evidencing that a Morlet scalogram perfectly localizes Hölder singularities, even in cases where $\alpha \neq 0$.

The explicit evaluation of the reassigned scalogram can finally be achieved by inverting the reassignment operators (16) and (17). The central result is that, at the time of occurrence $t=0$ of the singularity, we get

$$\check{\Sigma}_x^{(\kappa)}(0,\hat{a}) \propto \hat{a}^{-(2\alpha+1)}, \tag{18}$$

equation from which we conclude that (i), as for the scalogram, the reassigned scalogram undergoes a power-law behaviour with respect to scales, and (ii) the exponent $-(2\alpha+1) = 2\nu+1$ of this power-law is the same as in the scalogram case (see (14)). This means that the measurement of the Hölder exponent ν can be possibly done with a reassigned scalogram.

6. OSCILLATING SINGULARITY SIGNATURE FROM REASSIGNED SPECTROGRAMS

Whereas Hölder singularities naturally occur in signals or images as signatures of discontinuities or breakpoints, more complex (non-Hölder) singularities may also be encountered. Those singularities include an oscillating part, a typical example being provided by the gravitational waves radiated from two coalescing astrophysical objects [4]. In such cases, the signal model (which generalizes (12)) is that of a power-law chirp of the form

$$X_k(\omega) = C|\omega|^{-\nu-1}\,\exp\{i\Psi_k(\omega)\}, \tag{19}$$

with $\Psi_k(\omega) = -(t_0\omega + c\omega^k)$ if $k < 0$ and the extension $\Psi_0(\omega) = -(t_0\omega + c\log\omega)$.

A well-established theory has been developed for the analysis of such signals [2], with specific distributions $B_X^{(k)}(t,\omega)$ explicitly tailored to the group delays $t_{X_k}(\omega) := -(\partial\Psi_k/\partial\omega)(\omega)$, i.e., such that

$$B_{X_k}^{(k)}(t,\omega) \propto \delta\left(t - t_{X_k}(\omega)\right). \tag{20}$$

The computational burden associated to such distributions is however quite heavy, whereas a reassigned spectrogram may serve as a much simpler, yet very accurate, substitute (see, e.g., [4]). The reason is that reassignment operates basically in a local fashion, with a nearly optimal localization as long as the time-frequency trajectory of the chirp (i.e., its group delay or instantaneous frequency curve) can be considered as *locally linear* within the analysis time-frequency window. A way of guaranteeing at best this approximation is to adapt the short-time window length to the chirp rate, at the time-frequency point where the group delay has a maximum curvature. An explicit evaluation shows that the window length has to be chosen as

$$\Delta t_h = \left(\frac{3-k}{3-2k}\right)^{1/4} \Delta t_0, \tag{21}$$

where Δt_0 stands for the duration of the Gaussian window whose WVD has circular isocontours.

7. CONCLUSION

Initially introduced for improving spectrogram readability, reassignment techniques have been generalized to large classes of time-frequency and time-scale distributions. The case of singularities, discussed here, is one of the instances in which reassignment has moreover proved to be relevant not only for a qualitative "image" enhancement, but also for a quantitative parameter estimation. Other aspects along this line are discussed in [3].

Software — Matlab codes for computing reassigned time-frequency and time-scale distributions are available as part of a Toolbox, freely distributed on the Internet :
`http://iut-saint-nazaire.univ-nantes.fr/~auger/tftb.html`.

8. REFERENCES

[1] F. Auger, P. Flandrin. Improving the readability of time-frequency and time-scale representations by the reassignment method. *IEEE Trans. Signal Proc.*, SP-43(5):1068–1089, 1995.

[2] J. Bertrand, P. Bertrand. A class of affine Wigner distributions with extended covariance properties. *J. Math. Phys.*, 33(7):2515–2527, 1992.

[3] E. Chassande-Mottin. *Méthodes de réallocation dans le plan temps-fréquence pour l'analyse et le traitement de signaux non stationnaires.* Thèse de Doctorat, Université de Cergy-Pontoise (France), 1998.

[4] E. Chassande-Mottin, P. Flandrin. On the time-frequency detection of chirps. *Appl. Comp. Harm. Anal.*, 6(9):252–281, 1999.

[5] E. Chassande-Mottin, P. Flandrin. Reassigned scalograms and singularities. Proc. 3rd European Congress of Mathematics, Barcelona, 2000.

[6] L. Cohen. *Time-Frequency Analysis.* Prentice Hall, Englewoods Cliffs (NJ), 1995.

[7] P. Flandrin. *Time-Frequency/Time-Scale Analysis.* Academic Press, San Diego (CA), 1999.

[8] P. Flandrin, E. Chassande-Mottin, P. Abry. Reassigned scalograms and their fast algorithms. Proc. SPIE–95, 2569:152–158, San Diego, 1995.

[9] P. Guillemain, R. Kronland-Martinet. Characterization of acoustic signals through continuous linear time-frequency representations. *Proc. IEEE*, 84(4):561–587, 1996.

[10] F. Hlawatsch, P. Flandrin. The interference structure of Wigner distributions and related time-frequency signal representations In *The Wigner Distribution : Theory and Applications in Signal Processing* (W. Mecklenbräuker, F. Hlawatsch, *eds.*), 59–133, Elsevier, 1998.

[11] J. R. Klauder. Path integrals for affine variables. In J.-P. Antoine and E. Tirapegui, editors, *Functional Integration: Theory and Applications*, 101–119, Plenum Press, New York (NY), 1980.

[12] K. Kodera, C. De Villedary, R. Gendrin. A new method for the numerical analysis of nonstationary signals. *Phys. Earth and Plan. Int.*, 12:142–150, 1976.

[13] K. Kodera, R. Gendrin, C. De Villedary. Analysis of time-varying signals with small BT values. *IEEE Trans. on Acoust., Speech, and Signal Proc.*, ASSP-26(1):64–76, 1978.

[14] S. Mallat. *A Wavelet Tour of Signal Processing.* Academic Press, New York (NY), 1998.

[15] S. Mallat, W. L. Hwang. Singularity detection and processing with wavelets. *IEEE Trans. on Info. Theory*, IT-38(2):617–643, 1992.

[16] O. Rioul, P. Flandrin. Time-scale energy distributions : a general class extending wavelet transforms. *IEEE Trans. on Signal Proc.*, SP-40(7):1746–1757, 1992.