“Chirps” everywhere

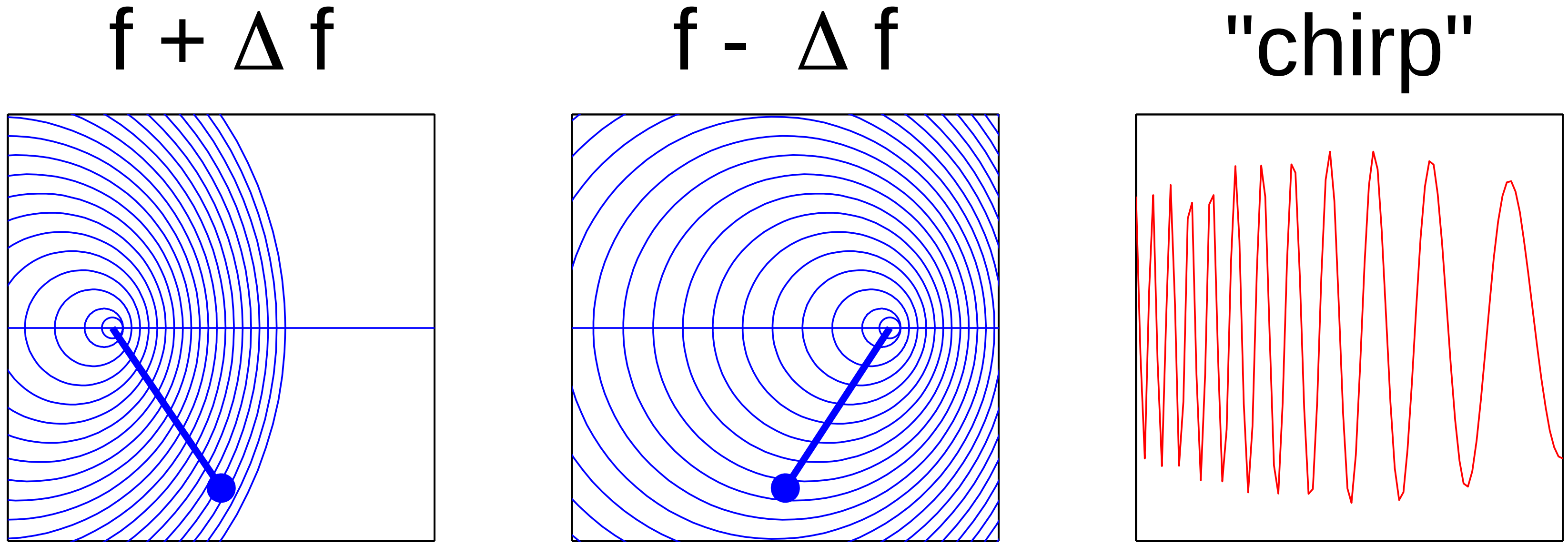
Patrick Flandrin*
CNRS — École Normale Supérieure de Lyon

*thanks to
Pierre-Olivier Amblard (LIS Grenoble), François Auger (Univ. Nantes), Pierre Borgnat (ENS Lyon), Eric Chassande-Mottin (Obs. Nice), Franz Hlawatsch (TU Wien), Paulo Gonçalvès (INRIAlpes), Olivier Michel (Univ. Nice) and Jeffrey C. O’Neill (iConverse)

observation

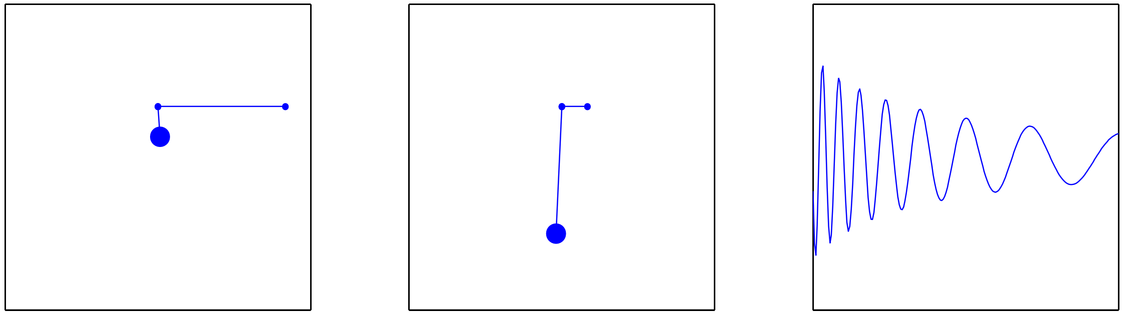
Doppler effect

Motion of a monochromatic source $\Rightarrow$ *differential* perception of the emitted frequency $\Rightarrow$ “chirp”.



Pendulum

$$\ddot{\theta}(t) + (g/L)\,\theta(t) = 0$$



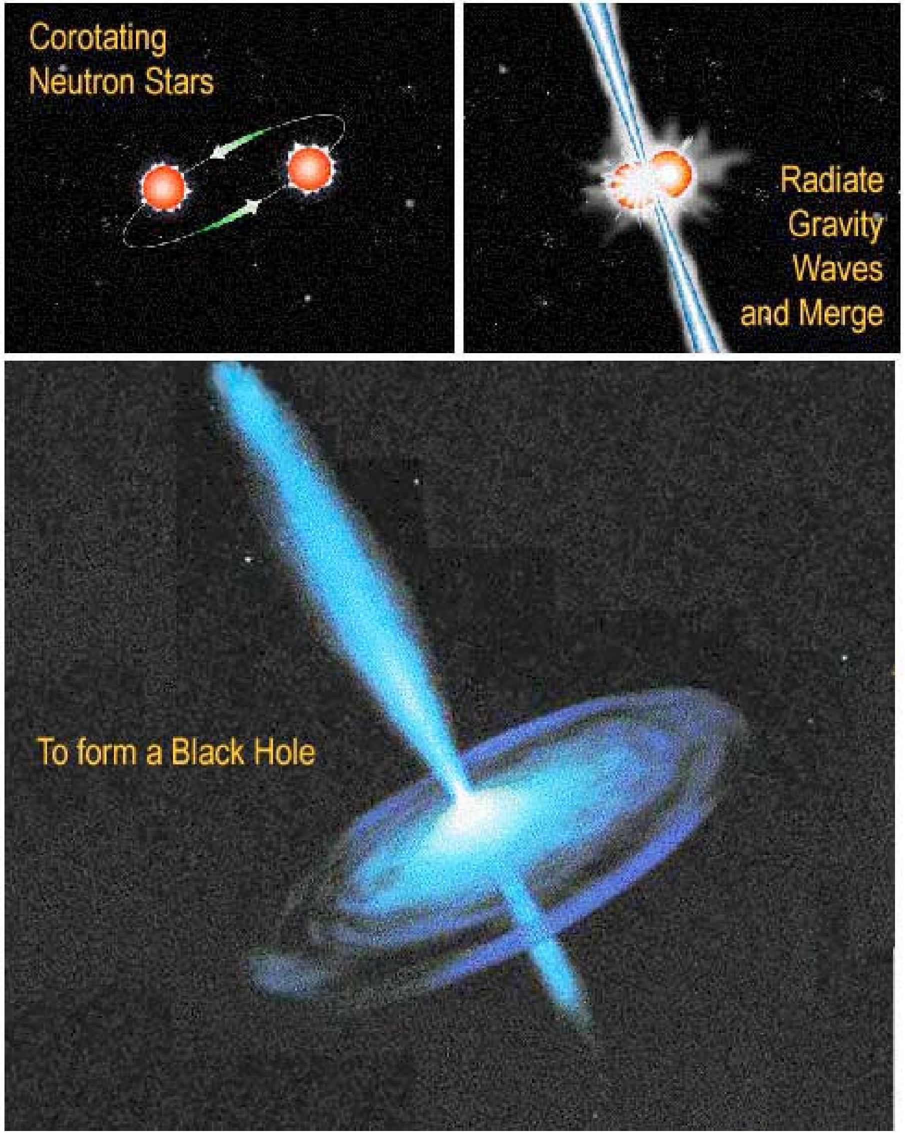
Fixed length $L = L_0$ — Small oscillations are sinusoïdal, with *fixed* period $T_0 = 2\pi\sqrt{L_0/g}$.

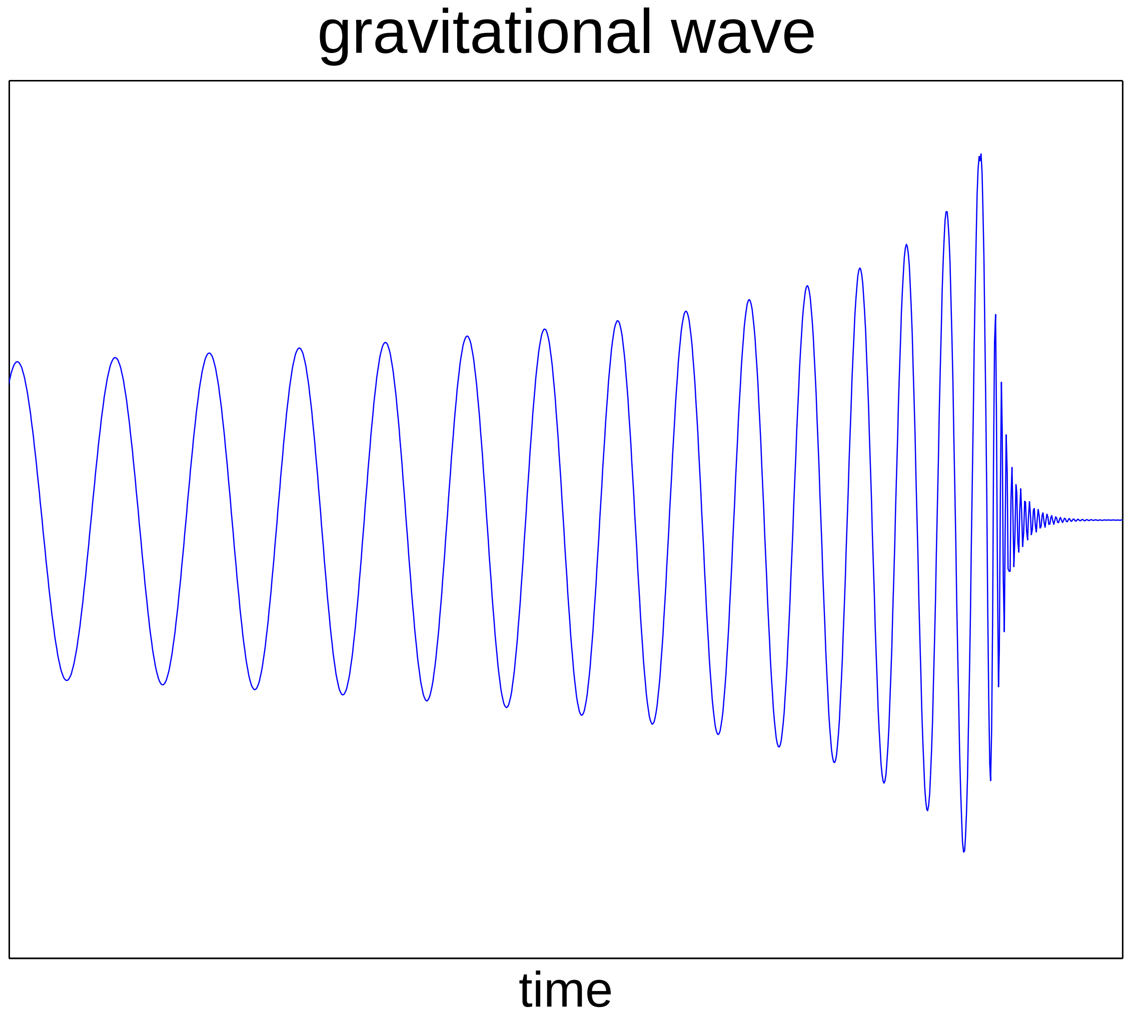
"Slowly" varying length $L = L(t)$ — Small oscillations are quasi-sinusoïdal, with *time-varying* pseudo-period $T(t) \sim 2\pi\sqrt{L(t)/g}$.

Gravitational waves

Theory — Though predicted by general relativity, *gravitational waves* have never been observed directly. They are "space-time vibrations," resulting from the acceleration of moving masses $\Rightarrow$ most promising sources in *astrophysics* (e.g., coalescence of binary neutrons stars).

Experiments — Several large instruments (VIRGO project for France and Italy, LIGO project for the USA) are currently under construction for a direct *terrestrial* evidence via *laser interferometry*.


Corotating
Neutron Stars
Radiate
Gravity
Waves
and Merge
To form a Black Hole


gravitational wave
time

VIRGO

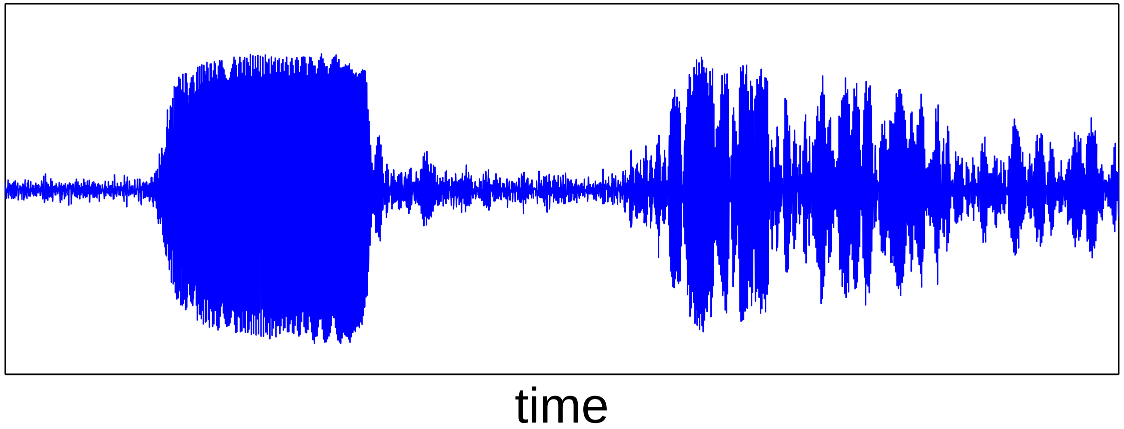
Bat echolocation

System — *Active* system for navigation, “natural sonar” .

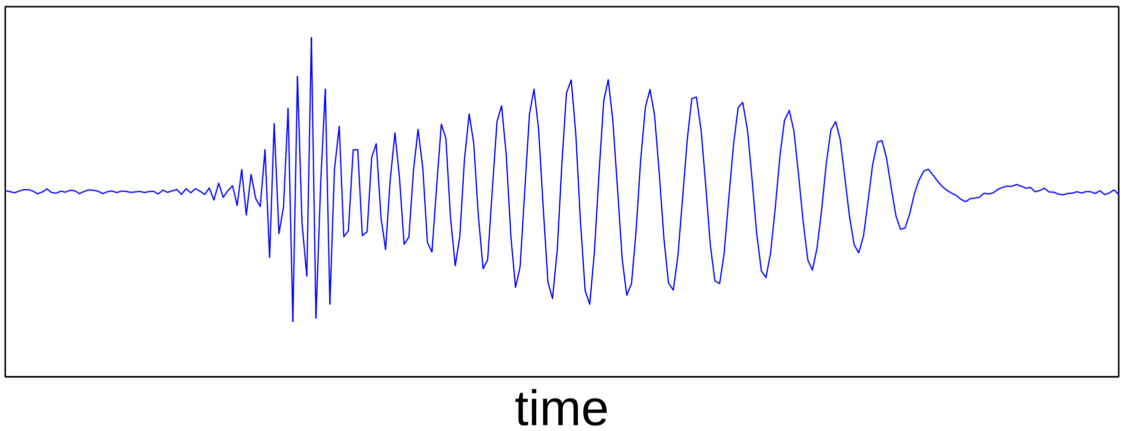
Signals — Ultrasonic acoustic waves, transient (some ms) and “wide band” (some tens of kHz between 40 and 100kHz).

Performance — Nearly optimal, with *adaptation* of emitted waveforms to multiple tasks (detection, estimation, classification, interference rejection,. . .).

bat echolocation call + echo



bat echolocation call (heterodyned)



More examples

Waves and vibrations — Bird songs, music (“glissando”), speech, geophysics (“whistling atmospherics” , vibroseis), wide band pulses propagating in a dispersive medium, radar, sonar,. . .

Biology and medicine — EEG (seizure), uterine EMG (contractions),. . .

Desorder and critical phenomena — Coherent structures in turbulence, accumulation of precursors in earthquakes, “speculative bubbles” prior a financial krach,. . .

Mathematics — Riemann and Weierstrass functions, . . .

description

Chirps

Definition — *We will call "chirp" any complex signal of the form* $x(t) = a(t)\,\exp\{i\varphi(t)\}$, *where* $a(t) \geq 0$ *is a low-pass amplitude whose evolution is slow as compared to the oscillations of the phase* $\varphi(t)$.

Slow evolution? — Usual heuristic conditions assume that:

1. $|\dot{a}(t)/a(t)| \ll |\dot{\varphi}(t)|$: the amplitude is *quasi-constant* at the scale of one pseudo-period $T(t) = 2\pi/|\dot{\varphi}(t)|$.

2. $|\ddot{\varphi}(t)|/\dot{\varphi}^2(t) \ll 1$: the pseudo-period $T(t)$ is itself *slowly varying* from one oscillation to the next.

Chirp spectrum

Stationary phase — In the case where the phase derivative $\dot{\varphi}(t)$ is monotonic, one can approximate the chirp spectrum

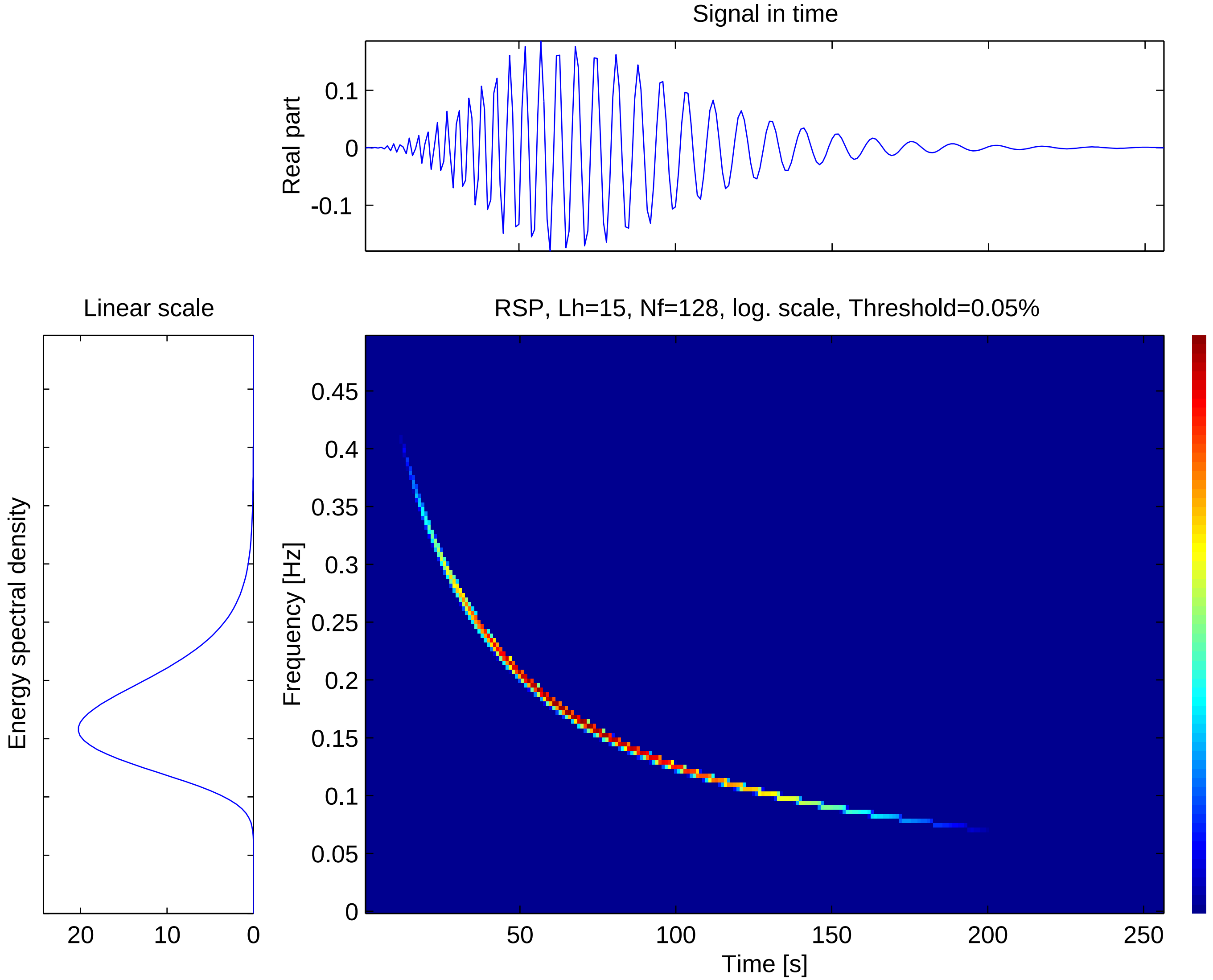
$$X(f) = \int_{-\infty}^{+\infty} a(t)\, e^{i(\varphi(t) - 2\pi f t)}\, dt$$

by its *stationary phase approximation* $\tilde{X}(f)$. We get this way:

$$|\tilde{X}(f)|^2 \propto \frac{a^2(t_s)}{|\ddot{\varphi}(t_s)|},$$

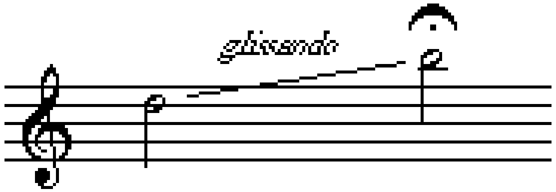
with t_s such that $\dot{\varphi}(t_s) = 2\pi f$.

Interpretation — The "instantaneous frequency" curve $\dot{\varphi}(t)$ defines a one-to-one correspondence between one time and one frequency. The chirp spectrum follows by weighting the *visited frequencies* by the corresponding *times of occupancy*.


Signal in time
Real part
0.1
0
-0.1
Linear scale
Energy spectral density
20
10
0
RSP, Lh=15, Nf=128, log. scale, Threshold=0.05%
Frequency [Hz]
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
50
100
150
200
250
Time [s]

representation

Time-frequency



Idea — Give a mathematical formulation to musical notation

Objective — Write the “musical score” of a signal

Constraint — Get a *localized* representation in chirp cases:

$$\rho(t,f) \sim a^2(t)\,\delta\left(f - \dot{\varphi}(t)/2\pi\right).$$

Local methods and localization

The example of the short-time Fourier transform — One defines the *local* quantity:

$$F_x^{(h)}(t, f) = \int_{-\infty}^{\infty} x(s)\, \overline{h(s-t)}\, e^{-i2\pi fs}\, ds.$$

Measure — Such a representation results from an interaction between the analyzed signal and some *apparatus* (the window $h(t)$).

Adaptation — Analysis adapted to *impulses* if $h(t) \to \delta(t)$ and to *spectral lines* if $h(t) \to 1 \Rightarrow$ adapting analysis to *chirps* requires $h(t)$ to be *(locally) dependent on the signal*.

Self-adaptation of local methods

Matched filtering — If the window $h(t)$ is the *time-reversed* signal $x_-(t) := x(-t)$, one gets $F_x^{(x_-)}(t,f) = W_x(t/2, f/2)/2$, where

$$W_x(t,f) := \int_{-\infty}^{+\infty} x(t+\tau/2)\,\overline{x(t-\tau/2)}\,e^{-i2\pi f\tau}\,d\tau,$$

is the so-called *Wigner-Ville Distribution* (Wigner, '32; Ville, '48).

Linear chirps — The WVD localizes *perfectly* on *straight lines* in the TF plane:

$$x(t) = \exp\{i2\pi(f_0 t + \alpha t^2/2)\} \Rightarrow W_x(t,f) = \delta\,(f - (f_0 + \alpha t))\,.$$

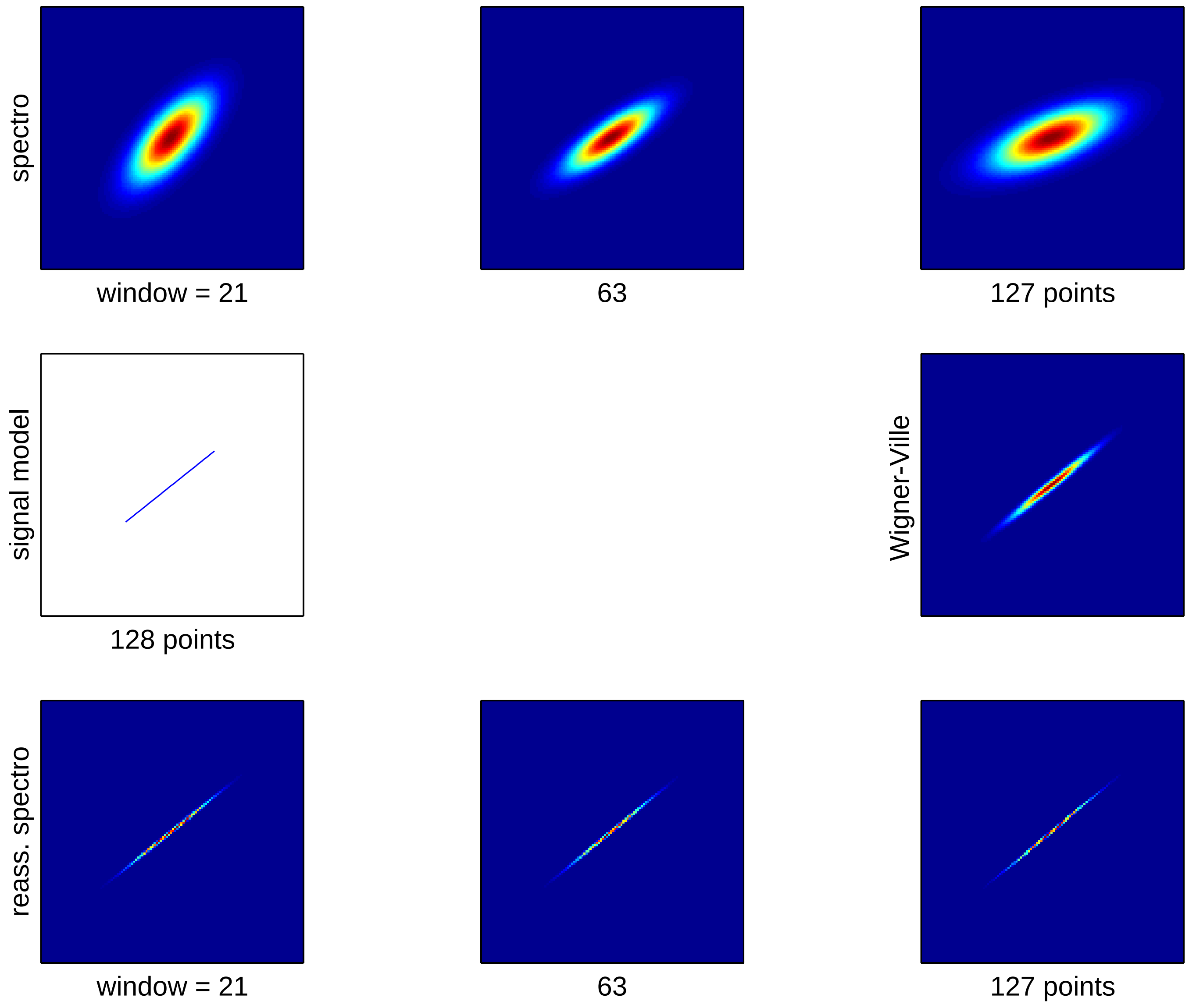
Remark — Localization via self-adaptation ends up in a *quadratic* transformation (energy distribution).

Beyond linear chirps

Global approach — The principle of self-adaptation via *phase compensation* can be extended to *non linear* chirps (Bertrand & Bertrand, '84 ; Gonçalvès et F., '94).

Limitations — Specific models and heavy computational burden.

Local approach — Spectrogram/scalogram = *smoothed* WVD $\Rightarrow$ localized distributions via *reassignment* towards *local* centroïds (Kodera et al., '76 ; Auger & F., '94).

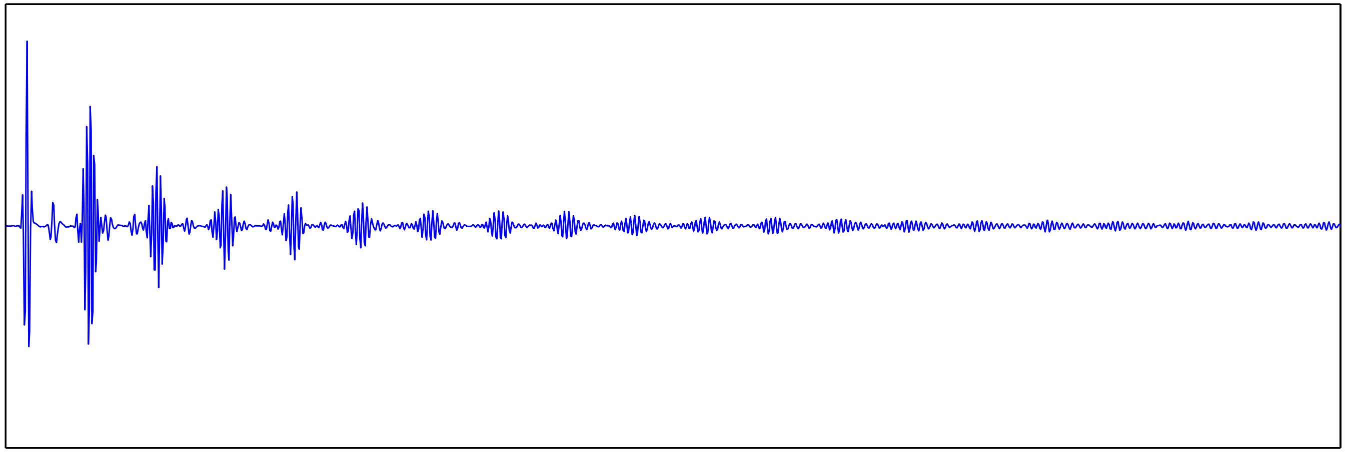

spectro
window = 21
63
127 points
signal model
128 points
Wigner-Ville
reass. spectro
window = 21
63
127 points

manipulation

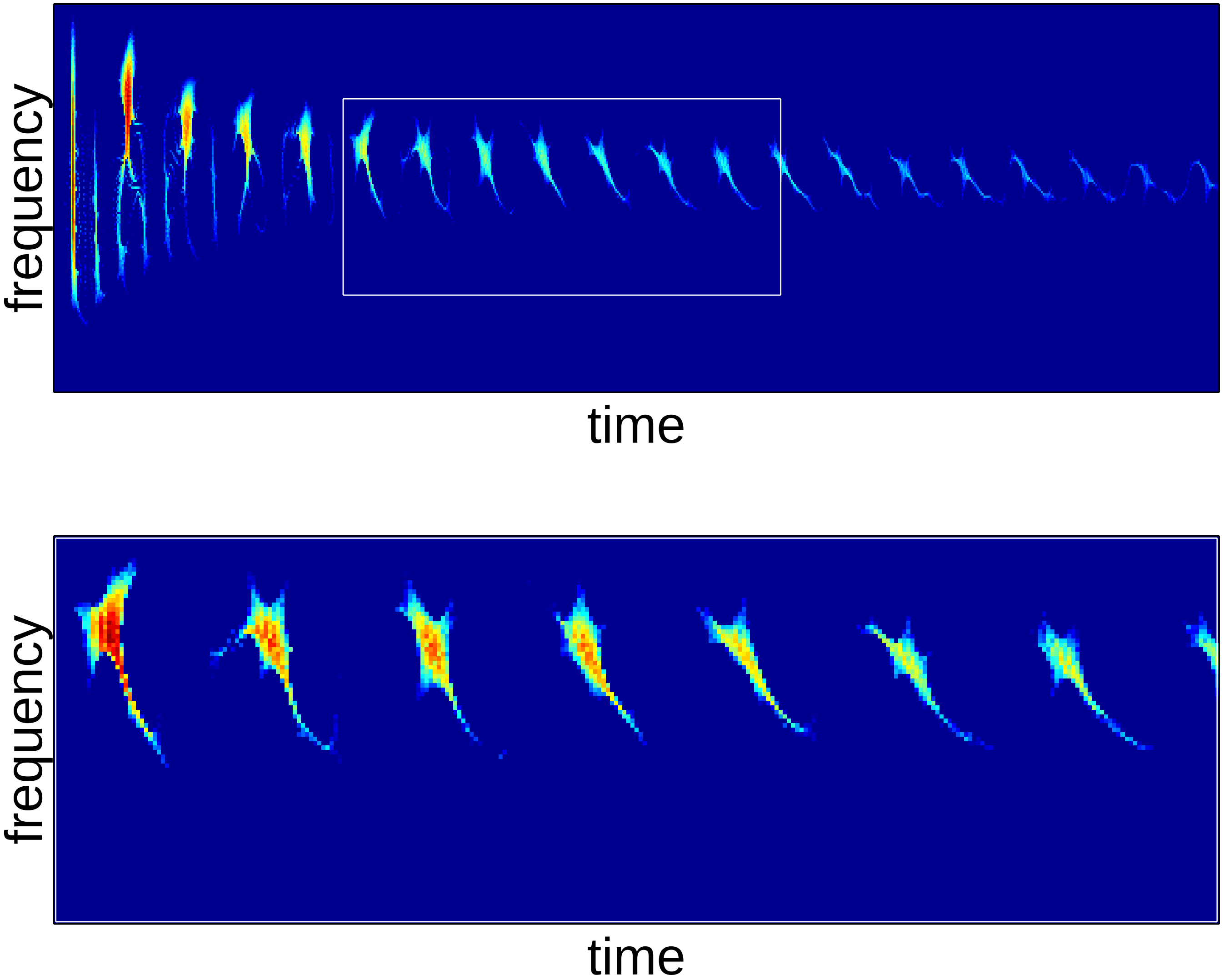
Chirps and dispersion

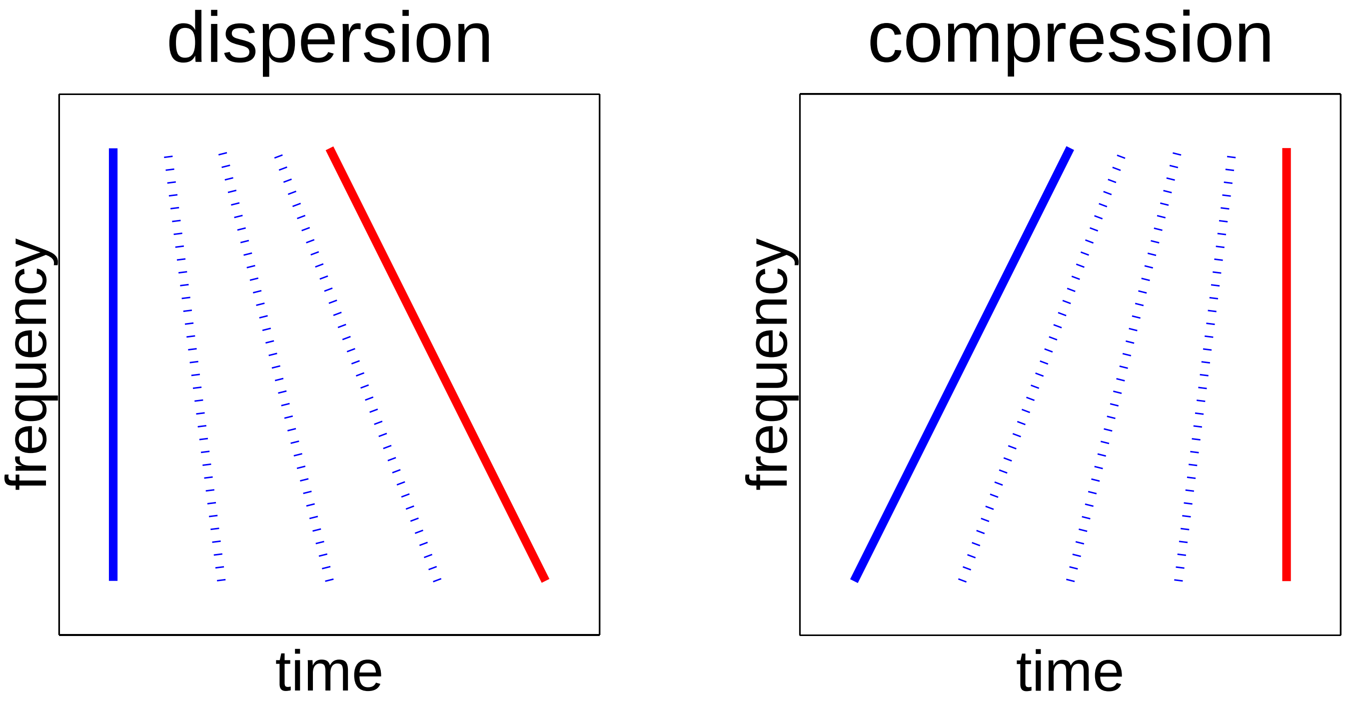
Example — Acoustic backscattering of an ultrasonic wave on a thin spherical shell ⇒ frequency dispersion of elastic surface waves.

acoustic backscaterring



acoustic backscaterring




dispersion
frequency
time
compression
frequency
time

Pulse compression

Limitation — Correlation radius $\sim 1/\text{spectral bandwidth}$, $\forall$ signal duration.

"Reception" — Post-processing by *matched filtering* (radar, sonar, vibroseismics, non destructive evaluation).

"Emission" — Pre-processing by *dispersive grating* (production of ultra-short laser pulses).

Chirps and detection/estimation

Optimality — Matched filtering, maximum likelihood, contrast,. . . : basic ingredient = *correlation* "observation — template".

Time-frequency interpretation — *Unitarity* of a time-frequency distribution $\rho_x(t, f)$ guarantees the equivalence:

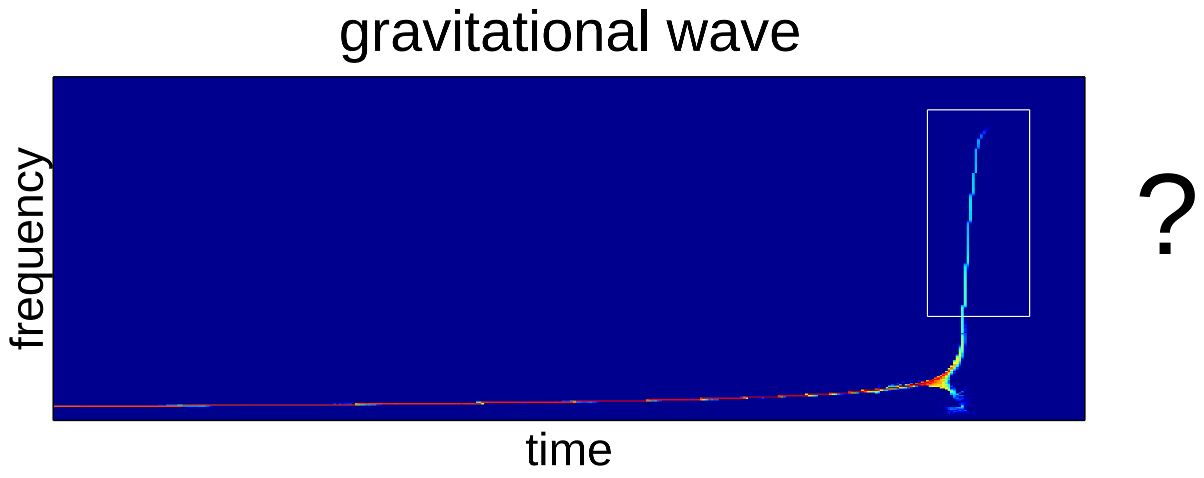
$$|\langle x, y\rangle|^2 = \langle\langle \rho_x, \rho_y \rangle\rangle.$$

Chirps — Unitarity + localization $\Rightarrow$ detection/estimation via *path integration* in the plane.

Time-frequency detection?

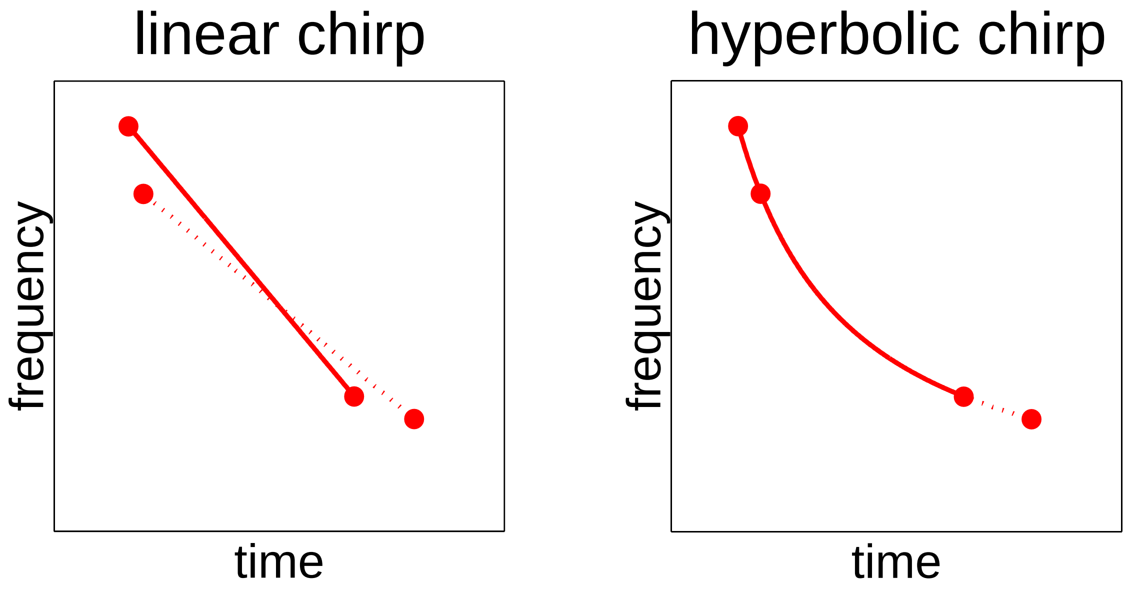
Language — Time-frequency offers a natural *language* for dealing with detection/estimation problems *beyond* nominal situations.

Robustness — *Uncertainties* in a chirp model can be incorporated by replacing the integration *curve* by a *domain* (example of post-newtonian approximations in the case of gravitational waves).

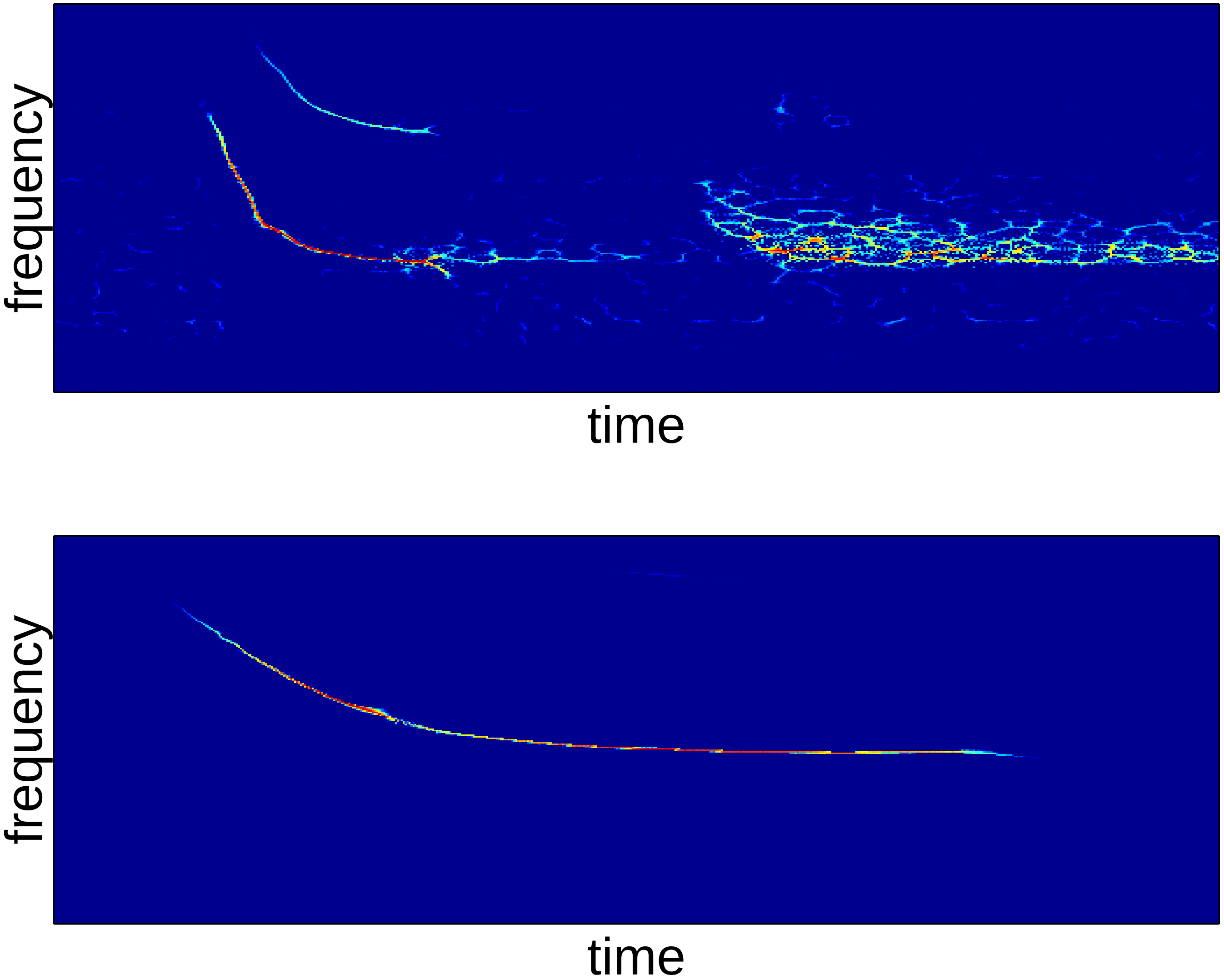


Doppler tolerance

Signal design — Specification of performance by a *geometrical* interpretation of the time-frequency structure of a chirp.



bat echolocation calls (+ echo)



modeling

Chirps and “atomic” decompositions

Fourier — The usual Fourier Transform (FT) can be formally written as $(\mathcal{F}x)(f) := \langle x, e_f \rangle$, with $e_f(t) := \exp\{i2\pi ft\}$, so that:

$$x(t) = \int_{-\infty}^{+\infty} \langle x, e_f \rangle \, e_f(t) \, df.$$

Extensions — Replace complex exponentials by chirps, considered as *warped* versions of monochromatic waves, or by “chirplets” (chirps of short duration) $\Rightarrow$ *modified short-time FTs or wavelet transforms modifiées*.

Modified TFs — Example

Mellin Transform — A *Mellin Transform* (MT) of a signal $x(t) \in L^2(\mathbb{R}^+, t^{-2\alpha+1}dt)$ can be defined as the projection:

$$(\mathcal{M}x)(s) := \int_0^{+\infty} x(t)\, t^{-i2\pi s-\alpha}\, dt = \langle x, c \rangle.$$

- Analysis on *hyperbolic* chirps $c(t) := t^{-\alpha} \exp\{i2\pi s \log t\}$.

- $\dot{\varphi}_c(t)/2\pi = s/t \Rightarrow$, the Mellin parameter s can be interpreted as a *hyperbolic chirp rate*.

- The MT can also be seen as a form of *warped* FT, since $\tilde{x}(t) := e^{(1-\alpha)t}\, x(e^t) \Rightarrow (\mathcal{M}x)(s) = (\mathcal{F}\tilde{x})(s)$.

"Chirplets"

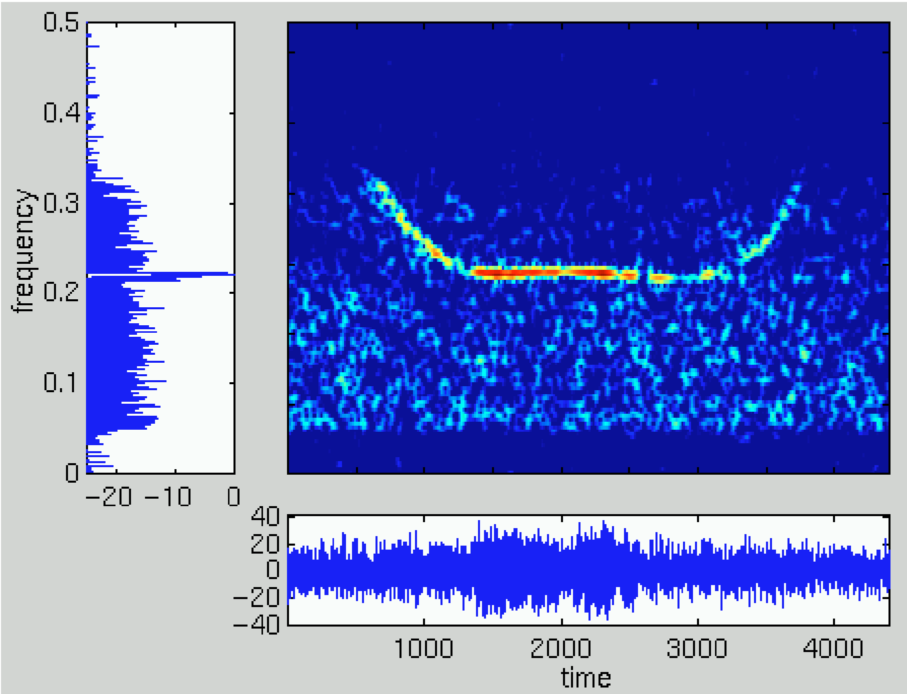
From "gaborets" and "wavelets" to "chirplets" — *Localization* + *modulation* lead to 4-parameter representations such as, e.g., $\langle x, x_{t,f,\alpha,\gamma}\rangle$ with

$$x_{t,f,\alpha,\gamma}(s) \propto \exp\{-\pi(\gamma + i\alpha)(s-t)^2 + i2\pi f(s-t)\}.$$

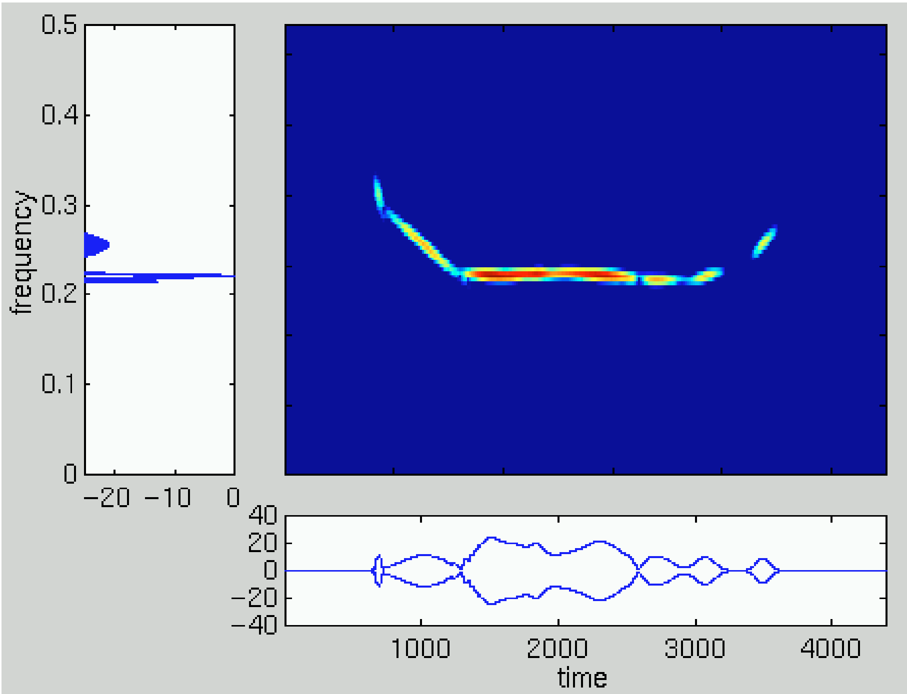
Decomposition as estimation — Constitutive chirplets can be sequentially *identified* by "matching (or basis) pursuit" techniques (Mallat & Zhang, '93; Chen & Donoho, '99; Bultan, '99; Gribonval, '99). They can also be *estimated* in the maximum likelihood sense (O'Neill & F., '98–'00).

"Parametric" limitation — Necessary trade-off between *dictionary size* and *algorithmic complexity*.

"Chirplet" decomposition — An example



signal + noise



8 atoms

Chirps and self-similarity

Dilation — Given $H, \lambda > 0$, let $\mathcal{D}_{H,\lambda}$ be the operator acting on processes $\{X(t), t > 0\}$ as $(\mathcal{D}_{H,\lambda} X)(t) := \lambda^{-H} X(\lambda t)$.

Self-similarity — A process $\{X(t), t > 0\}$ is said to be *self-similar* of parameter H (or "H-ss") if, for any $\lambda > 0$,

$$\{(\mathcal{D}_{H,\lambda} X)(t), t > 0\} \stackrel{d}{=} \{X(t), t > 0\}.$$

Self-similarity and stationarity — Self-similar processes and stationary processes can be put in a one-to-one correspondence (Lamperti, '62).

Lamperti

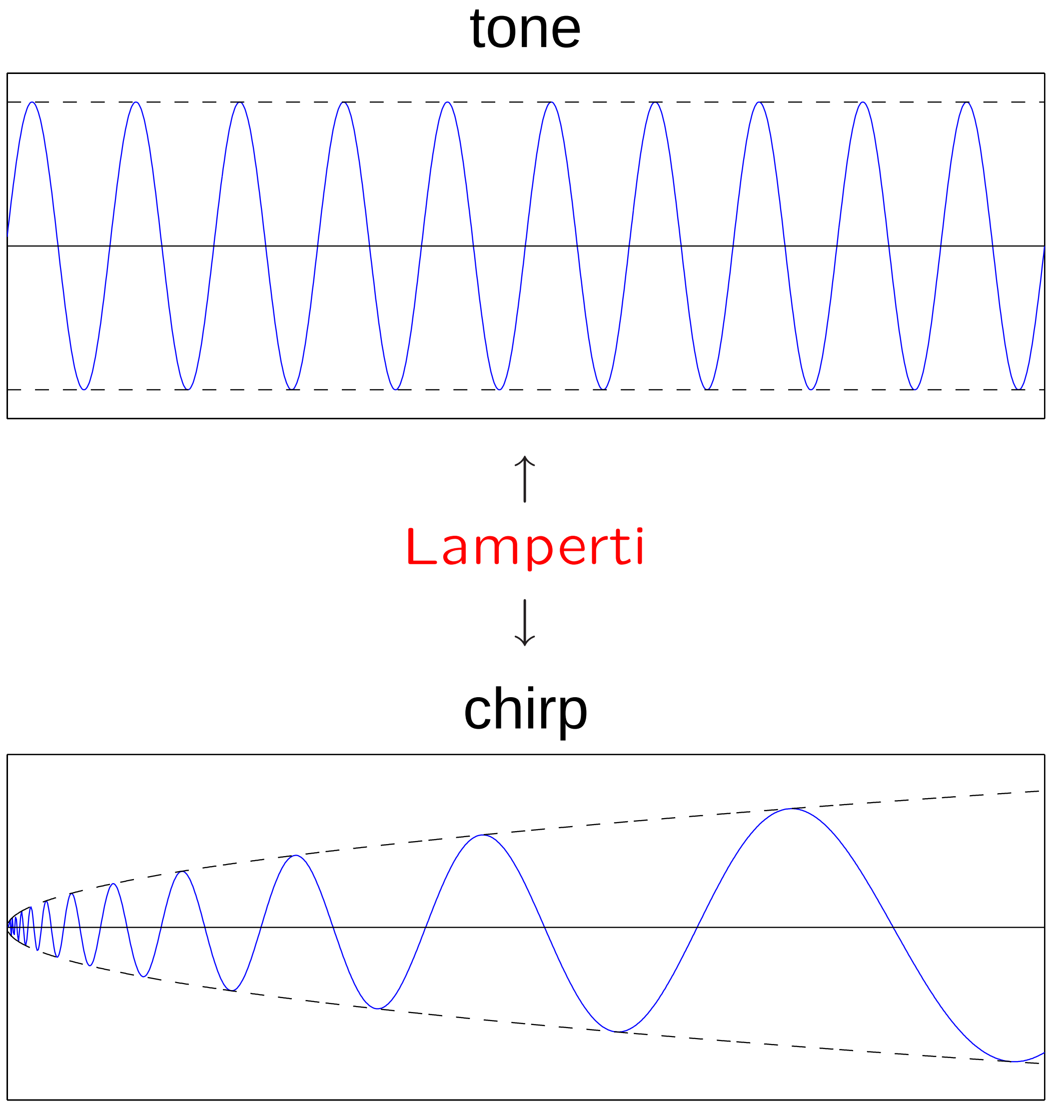
Definition — Given $H > 0$, the *Lamperti transformation* $\mathcal{L}_H$ acts on $\{Y(t), t \in \mathbb{R}\}$ as:

$$(\mathcal{L}_H Y)(t) := t^H Y(\log t), t > 0,$$

and its inverse $\mathcal{L}_H^{-1}$ acts on $\{X(t), t > 0\}$ as :

$$(\mathcal{L}_H^{-1} X)(t) := e^{-Ht} X(e^t), t \in \mathbb{R}.$$

Theorem — If $\{Y(t), t \in \mathbb{R}\}$ is stationary, its Lamperti transform $\{(\mathcal{L}_H Y)(t), t > 0\}$ is H-ss. Conversely, if $\{X(t), t > 0\}$ is H-ss, its (inverse) Lamperti transform $\{(\mathcal{L}_H^{-1} X)(t), t \in \mathbb{R}\}$ is stationary.


tone
Lamperti
chirp

“Spectral” representations

Fourier — (Harmonisable) *stationary* processes admit a spectral representation based on Fourier modes (monochromatic waves):

$$Y(t) = \int_{-\infty}^{+\infty} e^{i2\pi ft}\, d\xi(f).$$

Mellin — (Multiplicatively harmonisable) *self-similar* processes admit a corresponding representation based on Mellin modes (hyperbolic chirps):

$$X(t) = \int_{-\infty}^{+\infty} t^{H+i2\pi f}\, d\xi(f).$$

Weierstrass functions as an example

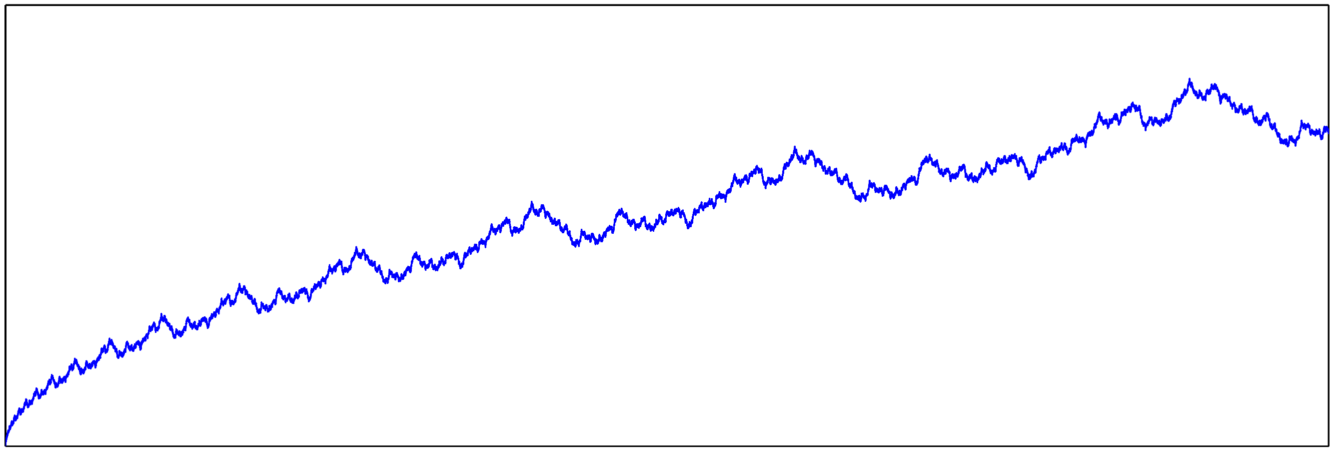
Discrete invariance — For any i.i.d. phases $\varphi_n \in \mathcal{U}(0, 2\pi)$ and any 2π-periodic g, functions of the form

$$W_{H,\lambda}(t) = \sum_{n=-\infty}^{\infty} \lambda^{-Hn} \, g(\lambda^n t) \, e^{i\varphi_n},$$

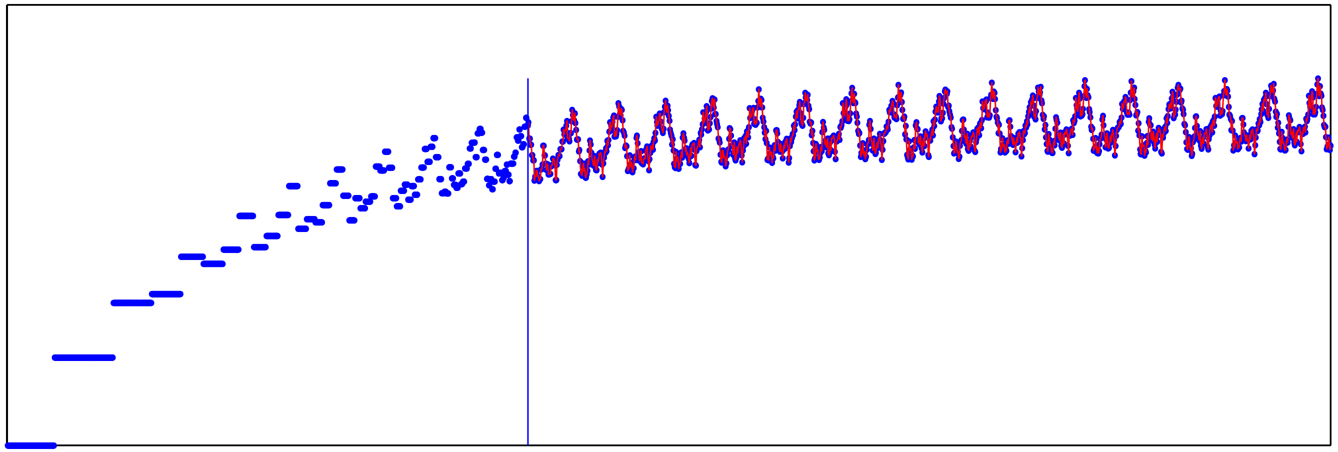
have a *discrete scale invariance* for any fixed λ, i.e., they are H-ss only for $\mu = \lambda^k, k \in \mathbb{Z}$.

Consequence —Such "(H, λ)-DSI" processes have *cyclostationary* Lamperti images and they can be represented on a *discrete* basis of Mellin chirps (Borgnat et al., '01).

Weierstrass function (H = 0.5)



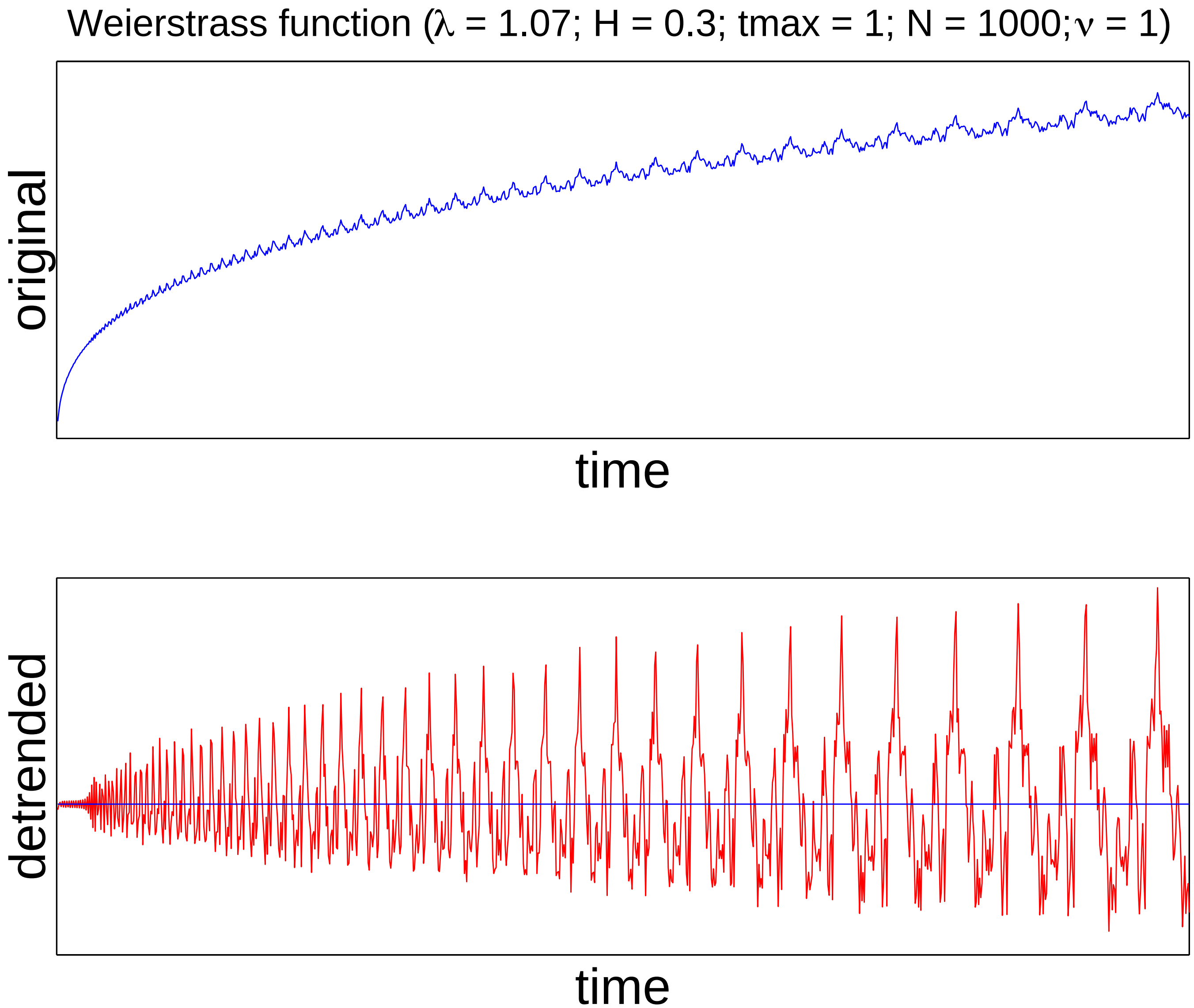
"delampertized" Weierstrass function



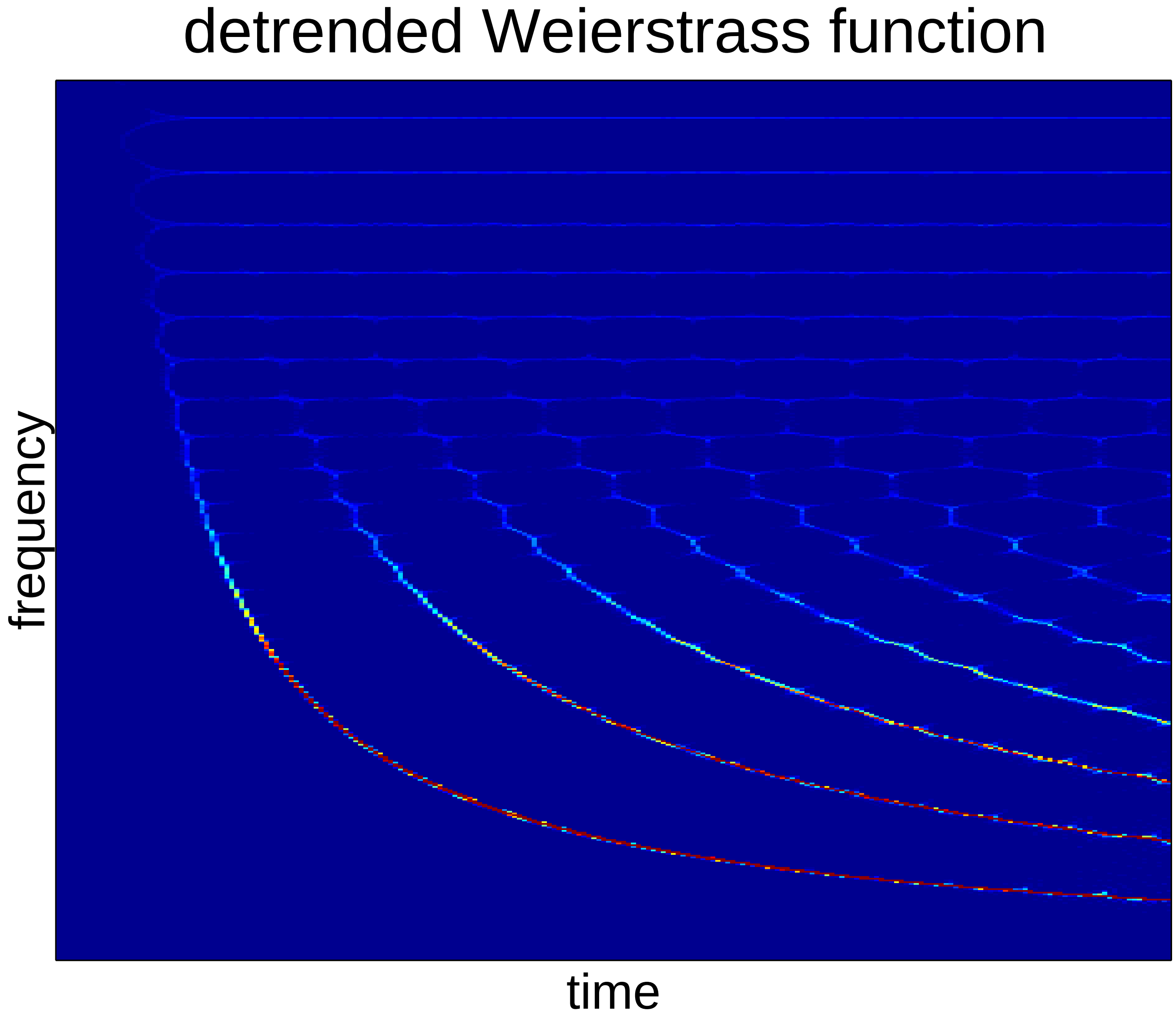
Weierstrass-Mandelbrot

Fourier — In the case where $g(t) = 1 - \exp it$, one gets the so-called Weierstrass-Mandelbrot function, whose usual representation is given by a superposition of Fourier modes (in geometrical progression).

Mellin — An equivalent representation exists (Berry and Lewis, '80), as superposition of Mellin modes, i.e., of hyperbolic chirps.



Weierstrass function (λ = 1.07; H = 0.3; tmax = 1; N = 1000;ν = 1)
original
time
detrended
time


detrended Weierstrass function
frequency
time

Chirps and power laws

A general model — $C_{\alpha,\beta}(t) = a\, t^{\alpha} \exp\{i(b\, t^{\beta} + c)\}$.

Example — Newtonian approximation of the *inspiraling* part of gravitational waves $\rightarrow (\alpha, \beta) = (-1/4, 5/8)$.

Typology — At $t = 0$: divergence of *amplitude* if $\alpha < 0$, of "*instantaneous frequency*" if $\beta < 1$ and of *phase* if $\beta < 0$.

Oscillating singularities. The case $(\alpha > 0, \beta < 0)$ is beyond a simple Hölder characterization $\Rightarrow$ development of specific tools (2-*microlocal analysis*, *wavelets*).

The Riemann function as an example

Definition — $\sigma(t) := \sum_{n=1}^{\infty} n^{-2} \sin \pi n^2 t$

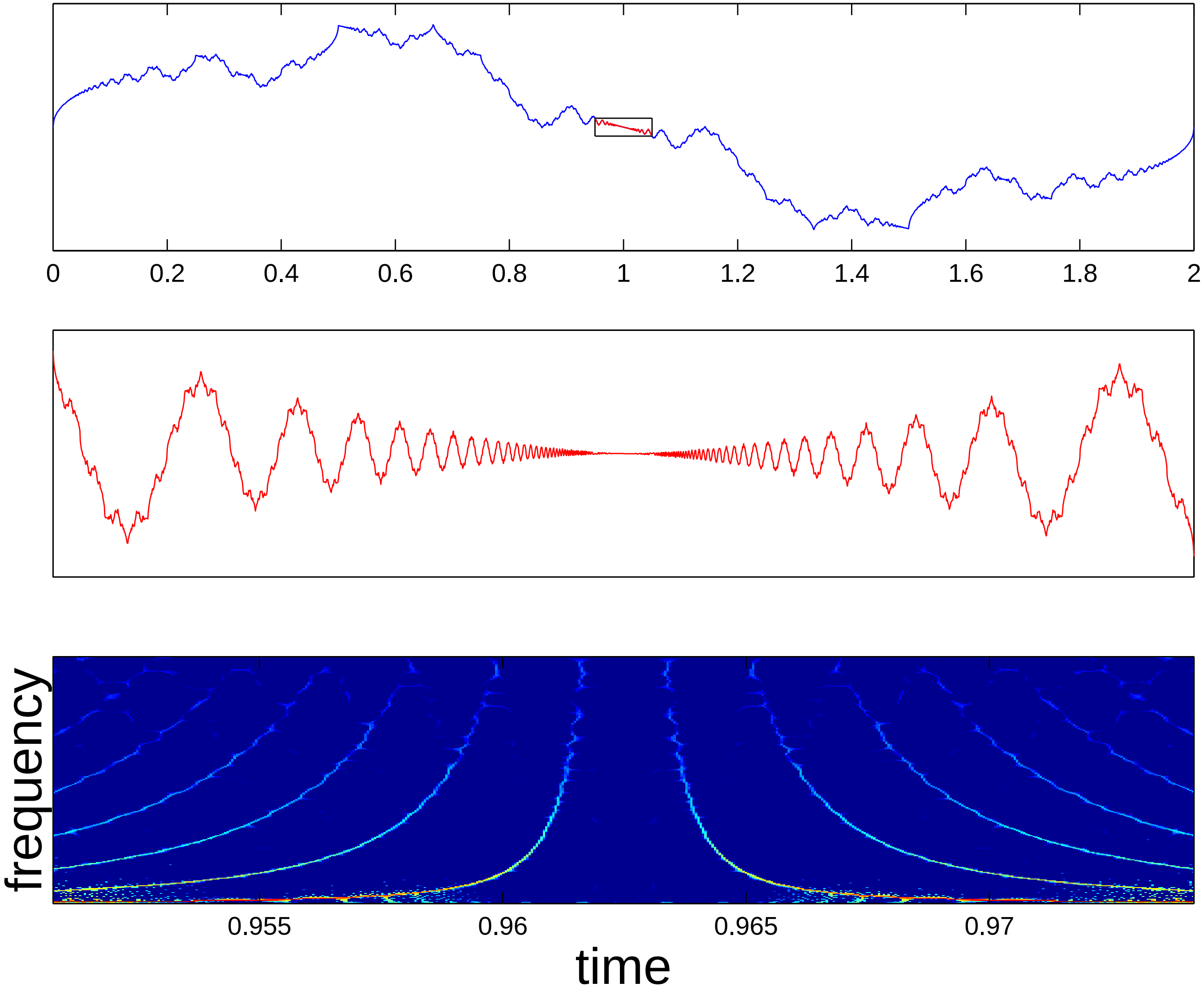
Differentiability — $\sigma(t)$ happens to be non-differentiable if $t \neq t_0 = (2p+1)/(2q+1)$, $p, q \in \mathbb{N}$ (Hardy, '16) *but* differentiable in $t = t_0$ (Gerver, '70).

Local chirps — One can show (Meyer, '96) that, in the vicinity of $z = 1$, the holomorphic version of Riemann function reads

$$\sigma(1+z) = \sigma(1) - \pi z/2 + \sum_{n=1}^{\infty} K_n(z)\, C_{3/2,-1}(z),$$

leading to $\sigma(1+t) = \sigma(1) - \pi t/2 + O(|t|^{3/2})$ when $t \to 0$.

Riemann function



conclusion

Chirps and time-frequency

Signals — Chirps "everywhere"

Representations — Natural description framework = the time-frequency plane

Models — "Chirps = time-frequency trajectories" $\Rightarrow$ the notion of instantaneous frequency can be approached as a by-product of representations in the plane (e.g., "ridges", or fixed points of reassignment operators)