RAMANUJAN SUMS FOR SIGNAL PROCESSING OF LOW FREQUENCY NOISE

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Abstract

An aperiodic (low frequency) spectrum may originate from the error term in the mean value of an arithmetical function such as Möbius function or Mangoldt function, which are coding sequences for prime numbers. In the discrete Fourier transform (and the FFT) the analyzing wave is periodic and not well suited to represent the low frequency regime. In place we introduce a new signal processing tool based on the Ramanujan sums $c_q(n)$, well adapted to the analysis of arithmetical sequences with many resonances p/q. The sums are quasiperiodic versus the time n of the resonance and aperiodic versus the order q of the resonance. New results arise from the use of this Ramanujan-Fourier transform (RFT) in the context of arithmetical and experimental signals.

Keywords: signal processing, $1/f$ noise, number theory

1 INTRODUCTION

"In this age of computers, it is very natural to replace the continuous with the finite. One thinks nothing about replacing the real line R with a finite circle (i.e. a finite ring $\mathcal{Z}/q\mathcal{Z}$) and similarly one replaces the real Fourier transform with the fast Fourier transform" [1].

In this paper our claim is that the discrete Fourier transform (and the FFT) is well suited to the analysis of periodic or quasi periodic sequences, but fails to discover the constructive features of aperiodic sequences, such as low frequency noise. This claim is not new and led to alternative time series analysis methods such as Poincaré maps [2] (i.e. one dimensional return maps of the form $x_{n+1} = f(x_n)$ or more general multidimensional maps), fractal or wavelet analysis methods [3] and autoregressive moving average (ARMA) models [4] to mention a few. These methods appeared in diverse contexts: turbulence, financial, ecological, physiological and astrophysical data. For stochastic sequences such as $1/f$ electronic noise only small progress was obtained thanks to these techniques [2].

Here we introduce still another approach by considering the time series as an arithmetical sequence, that is a discrete sequence $x(n)$, $n = 1 \cdots t$, in which generic arithmetical functions (such as $\sigma(n)$: the sum of divisors of n, or $\varphi(q)$: the number of irreducible fractions of denominator q, or the Möbius function $\mu(n)$, or the Mangoldt function $\Lambda(n)$(see below)...) may be hidden.

Recently we published a number of papers which emphasize the connection between frequency measurements and arithmetic [2],[5],[6]. The standard heterodyne method, which compares one oscillator of frequency $f(n)$ at time n to a reference oscillator of frequency f_0, leads to irreducible fraction p_i/q_i of index i given from continued fraction expansions of $\nu = f(n)/f_0$ and beat signals of frequencies $F(n) = f_0 q_i|\nu - p_i/q_i|$. Jumps between fractions of index $i,\ i \pm 1,\ i \pm 2 \cdots$ were clearly identified as a source of white or $1/f$ frequency noise in such frequency counting measurements [5]. A phase locked loop was characterized as well, leading to a possible relationship between $1/f$ noise close to baseband and arithmetical sequences of prime number theory [6].

2 THE DISCRETE FOURIER TRANSFORM

The discrete Fourier transform (DFT) or its fast analogue (the FFT) are well known signal processing tools. It extends the conventional Fourier analysis to sequences with finite period q ($q = 2^l$ with l integer for the FFT).

In the DFT one starts with the roots of unity of the form $\exp(2i\pi\frac{p}{q})$, $p = 1 \dots q$ and the signal analysis is performed thanks to the n^{th} power

$$e_p(n) = \exp(2i\pi\frac{p}{q}n). \tag{1}$$

(In the mathematical language one says that $e_p(n)$ is a character of $G = Z/qZ$, it is a group homomorphism from the additive group G into the multiplicative group of complex numbers of norm 1). The DFT of the time series $x(n)$ is defined as

$$\hat{x}(p) = \sum_{n=1}^{q} x(n) e_p(-n), \tag{2}$$

and they are a number of relations such as the inversion formula

$$x(n) = \frac{1}{q} \sum_{p=1}^{q} \hat{x}(p) e_p(n), \tag{3}$$

the Parseval formula (conservation of energy)

$$\sum_{n=1}^{q} |x(n)|^2 = \frac{1}{q} \sum_{p=1}^{q} |\hat{x}(p)|^2, \tag{4}$$

the orthogonality relations between the characters

$$\sum_{n=1}^{q} e_p(n) e_r(n) = q\delta_p(r) = \begin{cases} q \text{ if } p \equiv r (mod\ q) \\ 0 \text{ otherwise,} \end{cases} \tag{5}$$

and the convolution formula

$$\widehat{x * y} = \hat{x}\hat{y}, \tag{6}$$

where $*$ means the convolution.

Here is a short table of useful DFT.

$x(n)$	$\hat{x}(p)$
1	$q\ \delta_0(p)$
$e_l(n)$	$q\ \delta_l(p)$
$\delta_l(n)$	$e_l(-p)$
$\frac{1}{2}(\delta_1 + \delta_{-1})(n)$	$\cos(2\pi p/q)$
$L_q(-n)$	$L_q(n)\ \hat{L}_q(-1)$

In particular, as it is well known, an oscillating signal $e_l(n) = \exp(2i\pi \frac{l}{q} n)$ of frequency l/q transforms to a line at $p = l$ in the DFT spectrum. Inversely a line at $n = l$ in the time series transforms to an oscillating signal $\exp(-2i\pi \frac{l}{q} p)$ of frequency l/q.

A Gaussian transforms to a Gaussian through the Fourier integral. Not so well known is that the role of the Gaussian is played by the Legendre symbol in the context of the DFT [1],[7]. Let us define the Legendre symbol $L_q(n) = \left(\frac{n}{q}\right)$ for an odd prime q as follows

$$\left(\frac{n}{q}\right) = \begin{cases} 0 \text{ if } q \text{ divides } n, \\ +1 \text{ if } n \text{ is a square modulo q} \\ \quad (x^2 \equiv n\ (mod\ q) \text{ has a solution}), \\ -1 \text{ otherwise.} \end{cases} \tag{7}$$

There are a number of relations attached to the Legendre symbol

$$\left(\frac{n}{q}\right) = n^{\frac{q-1}{2} (mod\ q)},$$
$$\left(\frac{-1}{q}\right) = (-1)^{\frac{q-1}{2}},$$
$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}.\frac{q-1}{2}} \text{ for distinct odd primes } p, q,$$
$$\left(\frac{2}{q}\right) = (-1)^{\frac{q^2-1}{8}}.$$

The invariant relation for the DFT on $\mathcal{Z}/q\mathcal{Z}$ is

$$\hat{L}_q(-n) = g\ L_q(n) \text{ with } g = \hat{L}_q(-1). \tag{8}$$

The Fourier coefficient at position n equals the coefficient of the original sequence up to a constant factor $g = \hat{L}_q(-1) = \sum_{p=1}^{q} \left(\frac{p}{q}\right) \exp(2i\pi \frac{p}{q})$ and $g^2 = (-1)^{\frac{q-1}{2}} q$.

3 THE RAMANUJAN FOURIER TRANSFORM

Ramanujan sums $c_q(n)$ are defined as the sums of the n^{th} powers of the q^{th} primitive roots of the unity [10]

$$c_q(n) = \sum_{\substack{p=1 \\ (p,q)=1}}^{q} \exp(2i\pi \frac{p}{q} n), \tag{9}$$

where $(p, q) = 1$ means that p and q are coprimes. It may be observed that the $c_q(n)$ are the sums over the primitive characters $e_p(n)$. The sums were introduced by Ramanujan to play the role of base functions over which typical arithmetical functions $x(n)$ may be projected

$$x(n) = \sum_{q=1}^{\infty} x_q c_q(n). \tag{10}$$

It should be observed that the infinite expansion with $q \to \infty$ reminds the Fourier series analysis, rather than the discrete Fourier transform, which is taken with a finite q. As a typical example the function $\sigma(n)$ (the sum of divisors of n) expands with a RFT coefficient $\sigma_q = \frac{\pi^2 n}{6} \frac{1}{q^2}$

$$\sigma(n) = \frac{\pi^2 n}{6} \{1 + \frac{(-1)^n}{2^2} + \frac{2\cos(2n\pi/3)}{3^2} + \frac{2\cos(n\pi/2)}{4^2} + \cdots\}. \tag{11}$$

For functions $x(n)$ having a mean value

$$A_v(x) = \lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} x(n), \tag{12}$$

one obtains the inversion formula

$$x_q = \frac{1}{\varphi(q)} M(x(n) c_q(n)). \tag{13}$$

It will be called Ramanujan-Fourier transform (RFT) in the rest of the paper. This follows from a number of important relations. There is the multiplicative property of Ramanujan sums

$$c_{qq'}(n) = c_q(n)c_{q'}(n) \text{ if } (q,q') = 1, \tag{14}$$

and the orthogonality property

$$\begin{array}{l} \sum_{n=1}^{qq'} c_q(n)c_{q'}(n) = 1 \text{ if } q \neq q' \\ \sum_{n=1}^{q} c_q^2(n) = q\varphi(q) \text{ otherwise,} \end{array} \tag{15}$$

which remind us (5). It is relatively easy to evaluate Ramanujan sums from basic functions of number theory. Let us denote (q,n) the greatest common divisor of q and n. Using the unique prime number decomposition of q and n

$$q = \prod_i q_i^{\alpha_i} \quad (q_i \text{ prime}),$$

$$n = \prod_k n_k^{\beta_k} \quad (n_k \text{ prime}), \tag{16}$$

$$\tag{17}$$

one gets the number $\varphi(q)$ of irreducible fractions of denominator q, also called Euler totient function

$$\varphi(q) = q\prod_i (1 - \frac{1}{q_i}), \tag{18}$$

and a coding of prime numbers from the Möbius function $\mu(n)$ which is defined as

$$\mu(n) = \begin{cases} 0 \text{ if } n \text{ contains a square } \beta_k > 1, \\ 1 \text{ if } n = 1, \\ (-1)^k \text{ if } n \text{ is the product} \\ \qquad \text{of } k \text{ distinct primes.} \end{cases} \tag{19}$$

Ramanujan sums are evaluated from

$$c_q(n) = \mu\left(\frac{q}{(q,n)}\right)\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,n)}\right)}. \tag{20}$$

Note that for $(q,n) = 1$, $c_q(n) = \mu(q)$. The first values are given from

$$\begin{array}{c} c_1 = \overline{1};\ c_2 = \overline{-1,1};\ c_3 = \overline{-1,-1,2} \\ c_4 = \overline{0,-2,0,2};\ c_5 = \overline{-1,-1,-1,4}\cdots \end{array} \tag{21}$$

where the bar indicates the period. For instance $c_3(1) = -1,\ c_3(2) = -1,\ c_3(3) = 2,\ c_3(4) = -1\ldots$ A short table of known Ramanujan-Fourier transforms is given below In the table the function $\varphi_2(q)$ generalizes Euler function

$$\varphi_2(q) = q^2\prod_i (1 - \frac{1}{q_i^2}) \tag{22}$$

$x(n)$	x_q
$\frac{\sigma(n)}{n}$	$\frac{\pi^2}{6}\frac{1}{q^2}$
$\frac{\varphi(n)}{n}$	$\frac{6}{\pi^2}\frac{\mu(q)}{\varphi_2(q)}$
$b(n) = \frac{\varphi(n)\Lambda(n)}{n}$	$\frac{\mu(q)}{\varphi(q)}$
$C(n)$	$\left(\frac{\mu(q)}{\varphi(q)}\right)^2$

In $b(n)$ the Mangoldt function $\Lambda(n)$ is defined as

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^\alpha, p \text{ a prime} \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

According to Hardy and Littlewood (1922) the number of pair of primes of the form $p, p+h$ is

$$\pi_h(x) = C(h)\frac{x}{\ln^2(x)}, \tag{24}$$

with

$$C(h) = \begin{cases} 2C_2\prod_{p|h}\frac{p-1}{p-2}, & \text{if } h \text{ odd} \\ 0\ , & \text{if } h \text{ even.} \end{cases} \tag{25}$$

where $p > 2$ is a prime, and the notation $p|h$ means p divides h. The parameter $C_2 \simeq 0.660...$ is the twin prime constant. It was recently conjectured [10] that this problem of prime pairs is also related to a Wiener-Khintchine formula

$$A_v(b(n)b(n+h)) = C(h). \tag{26}$$

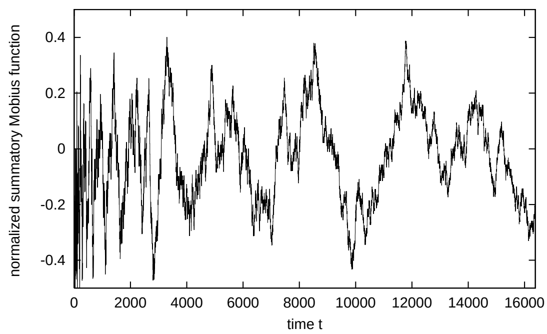

Figure 1: The normalized summatory Möbius function $M(t)/t^{1/2}$

4 LOW FREQUENCY NOISE FROM ARITHMETICAL FUNCTIONS

The idea which subtends our new signal processing based on the Ramanujan-Fourier transform (RFT) is that experimental signals may hide arithmetical features. It is thus very important to master the low frequency effects due to generic arithmetical functions such as Möbius function, Mangoldt function and so on [5],[6].

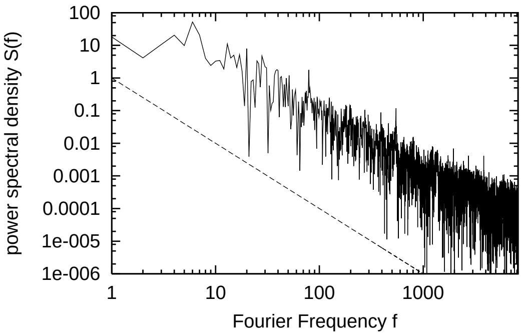

Figure 2: The power spectral density (FFT) of the normalized summatory Möbius function $M(t)/t^{1/2}$ in comparison to the power law $1/f^2$ (dotted line).

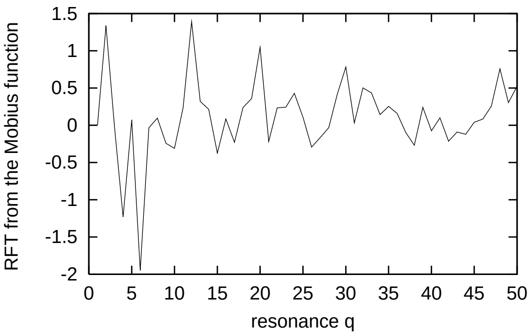

Figure 3: The Ramanujan-Fourier transform (RFT) of the normalized summatory Möbius function shown in Fig.1.

4.1 On the summatory Möbius function

Let us consider the summatory function

$$M(t)=\sum_{n=1}^{t}\mu(n)=O(t^{\frac{1}{2}+\varepsilon}),\ \text{whatever}\ \varepsilon. \qquad (27)$$

The asymptotic dependance assumes the Riemann hypothesis [5]. The normalized summatory function $M(t)/t^{1/2}$ is shown on Fig.1. The corresponding power spectral density is on Fig.2; it looks like the FFT of a random walk.

The RFT of $M(t)/t^{1/2}$ is shown on Fig.3. It shows a signature with well defined peaks which is reminiscent of the function $\mu(q)/\varphi(q)$ shown below on Fig 8.

4.2 Results related to the Mangoldt function

Riemann hypothesis can also be studied thanks to the summatory Mangoldt function

$$\psi(t)=\sum_{n=1}^{t}\Lambda(t)=t(1+\varepsilon_\psi(t)). \qquad (28)$$

The error term represented on Fig.4 can be expressed analytically from the singularities (the pole and the zeros) of the Riemann zeta function [5]. Fig.5 shows the FFT of the error term $\varepsilon_\psi(t)$: it roughly behaves as $1/f$ noise.

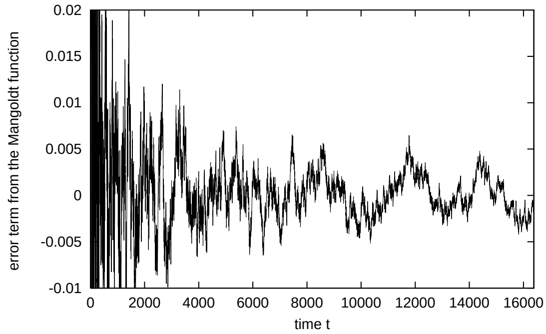

Figure 4: Error term in the Mangoldt function $\Lambda(n)$.

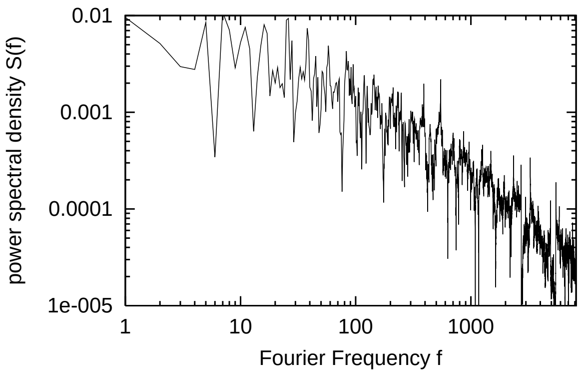

Figure 5: Power spectral density (FFT) of the error term of Mangoldt function $\Lambda(n)$.

Hardy found that the RFT of the modified Mangoldt function $b(n)=\Lambda(n)\varphi(n)/n$ equals $\mu(q)/\varphi(q)$. It is thus interesting to look at the summatory function

$$B(t)=\sum_{n=1}^{t}\Lambda(n)\varphi(n)/n=t(1+\varepsilon_B(t)). \qquad (29)$$

The error term (Fig.6) is found to follow approximately the power law

$$S_B(t)\sim f^{-2\alpha} \qquad (30)$$

with $\alpha=(\sqrt(5)-1)/2=1/(1+1/(1+1/[1+...)))$, as shown in Fig.7. This spectrum shows a possible connexion between α and $\mu(q)$ and thus a possible relationship between the theory of diophantine approximations for quadratic irrational numbers such as α and prime number theory. The RFT of $\varepsilon_B(t)$ looks similar to the one $\mu(q)/\varphi(q)$ of the new Mangoldt function $b(n)$.

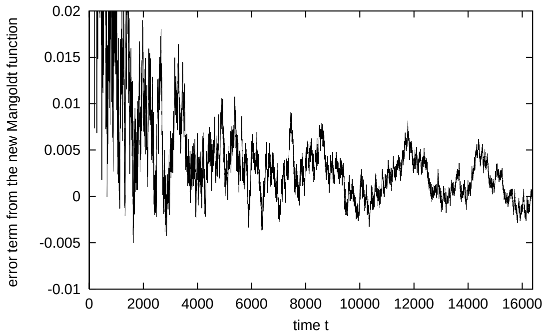

Figure 6: Error term in the new Mangoldt function $b(n)$.

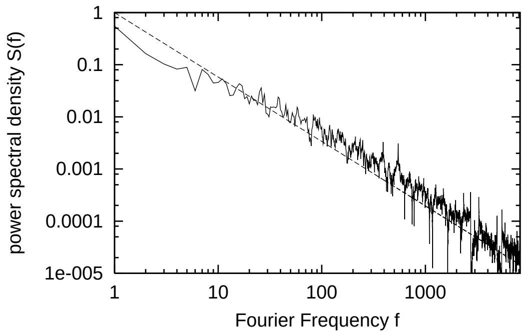

Figure 7: Power spectral density (FFT) of the error term in new Mangoldt function $b(n)$ in comparison to the power law $1/f^{\alpha}$, with $\alpha = (\sqrt(5)-1)/2$ the golden mean.

5 LOW FREQUENCY NOISE FROM EXPERIMENTAL DATA

Our final goal in using the Ramanujan-Fourier transform is to discover known arithmetical rules behind experimental sequences.

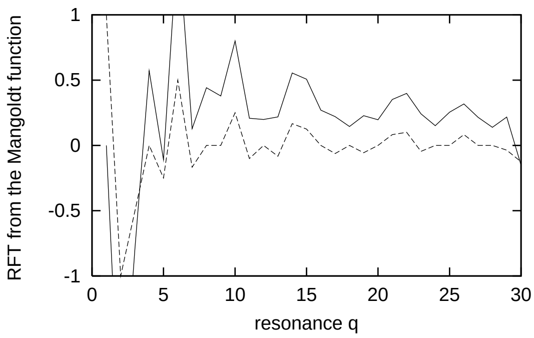

Figure 8: Ramanujan-Fourier transform (RFT) of the error term (upper curve) of new Mangoldt function $b(n)$ in comparison to the function $\mu(q)/\varphi(q)$(lower curve).

5.1 Low frequency noise from galactic nuclei

We give an example taken from astronomy. The observation of variability in astronomical systems may lead to valuable information on the physical nature of the observed system. In particular Seyfert galaxies are a subset of galaxies which exhibit evidence for highly energetic phenomena in their nuclei: they are called active galactic nuclei or AGN. They are thought to be powered by accretion onto massive black holes at their centers. X-rays are created mainly in high temperature, high density regimes, and since matter is fairly transparent to high energy X-rays, monitoring X-ray emission from AGNs provides a view into the core and may be used to understand the accretion process there.

Here we used a sample of data taken from the EXOSAT archive by M. Koenig and available at http://astro.uni-tuebingen.de/groups/time/ (sample mkn766-85.dat.outZRM) (see Fig.9). The power spectral density exhibits a 1/f low frequency noise as well as white noise as shown in Fig.10. The corresponding RFT analysis shown in Fig.11 shows a well defined signature reminiscent of the RFT signature of Mangoldt function, that is $\mu(q)/\varphi(q)$. That may be an indication that many resonance processes occur between the black hole and the matter to be accreted, a process which may be described from prime number theory.

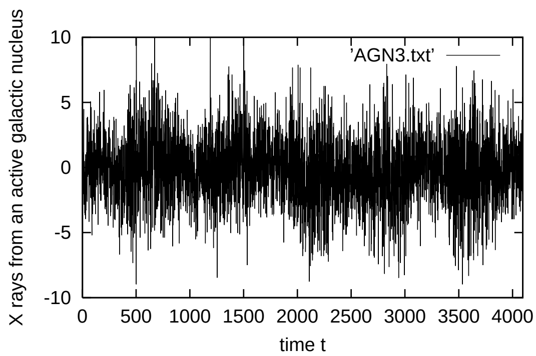

Figure 9: X ray variability from an active galactic nucleus (AGN).

5.2 Low frequency noise close to phase locking

Our second example is taken from the study of radiofrequency oscillators close to phase locking. We recently demonstrated a relation between phase locking, $1/f$ frequency noise and prime numbers [6]. According to that approach the coupling coefficient between the oscillators could be described from a Mangoldt function, leading to desynchronization effects and $1/f$ frequency noise. The RFT should be able to support that conjecture. Fig.12, 13, 14 show the beat note close to phase locking of 5 MHz oscillators, the $1/f$ noise calculated

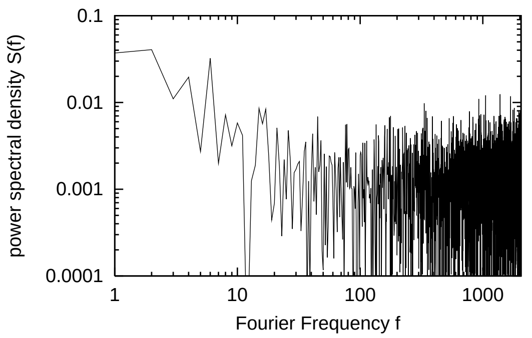

Figure 10: Power spectral density (FFT) of X ray variability from an AGN.

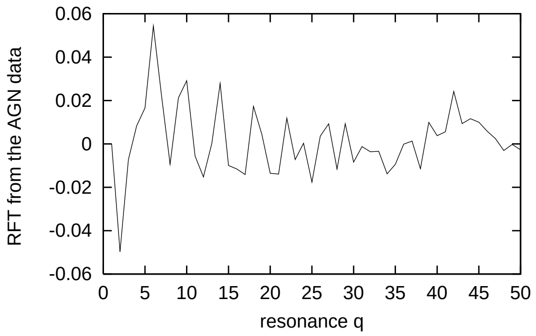

Figure 11: Ramanujan-Fourier transform (RFT) of X ray variability of an AGN.

from the FFT and the corresponding RFT. The Ramanujan sums, in addition to others tools taken from number theory, should become essential for non linear signal processing of low frequency noise.

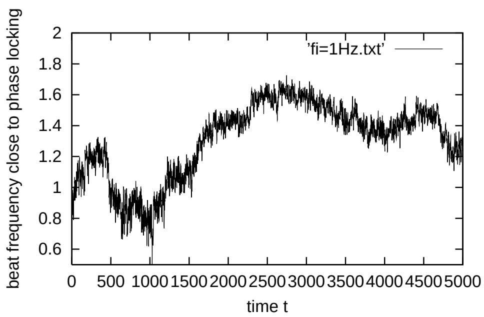

Figure 12: Beat frequency between two radiofrequency oscillators close to phase locking.

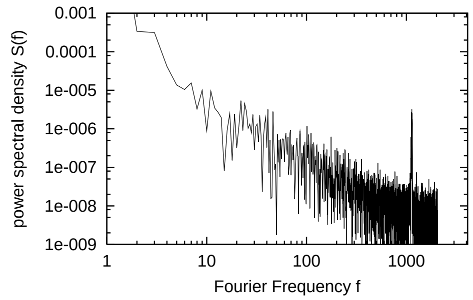

Figure 13: FFT of the beat frequency for two oscillators close to phase locking.

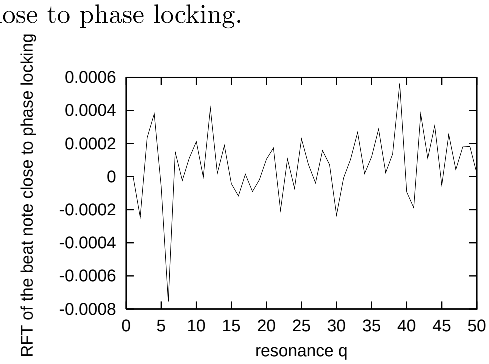

Figure 14: RFT of the beat frequency for two oscillators close to phase locking.

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