Elements of time-frequency analysis

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observing

"chirps"

- **Waves and vibrations** — Bird songs, bats, music ("glissando"), speech, "whistling atmospherics", tidal waves, gravitational waves, wide-band impulses propagating in a dispersive medium, pendulum, diapason (string, pipe) with time-varying length, vibroseismics, radar, sonar, Doppler effect...
- **Biology and medicine** — EEG (epilepsy), uterine EMG (contractions),...
- **Disorder and critical phenomena** — Coherent structures in turbulence, accumulation of earthquake precursors, "speculative bubbless" prior a financial crash,...
- **Mathematical special functions** — Weierstrass, Riemann...

describing

definition

Definition

A "chirp" is any complex-valued signal reading $x(t) = a(t)\exp\{i\varphi(t)\}$, where $a(t) \geq 0$ is a low-pass amplitude whose evolution is slow as compared to the phase oscillations $\varphi(t)$.

Slow evolution? — Usual heuristic conditions assume that:

1. $|\dot{a}(t)/a(t)| \ll |\dot{\varphi}(t)|$: the amplitude is **alomost constant** at the scale of a pseudo-period $T(t) = 2\pi/|\dot{\varphi}(t)|$.
2. $|\ddot{\varphi}(t)|/\dot{\varphi}^2(t) \ll 1$: the pseudo-period $T(t)$ is itself **slowly varying** from one oscillation to the enxt.

modulations

- **Monochromatic wave** — In the case of a harmonic model $x(t) = a\cos(2\pi f_0 t + \varphi_0)$, observing $x(t)$ leads in an **unambiguous** way to the amplitude a and to the frequency f_0.
- **Amplitude and frequency modulations** — Moving to an **evolutive** model amounts (intuitively) to achieve the transformation $a\cos(2\pi f_0 t + \varphi_0) \to a(t)\cos\varphi(t)$ with $a(t)$ variable and $\varphi(t)$ nonlinear. In an **observation** context, the unicity of the representation is however lost since

$$a(t)\cos\varphi(t) = \left[\frac{a(t)}{b(t)}\right]\,[b(t)\cos\varphi(t)] =: \tilde{a}(t)\cos\tilde{\varphi}(t)$$

for any function $0 < b(t) < 1$.

Fresnel

Monochromatic wave — The **real-valued** harmonic model can be written as

$$x(t) = a\,\cos(2\pi f_0 t + \varphi_0) = Re\left\{a\,\exp i(2\pi f_0 t + \varphi_0)\right\},$$

with

$$a\,\exp i(2\pi f_0 t + \varphi_0) = x(t) + i\,(\mathbf{H}x)(t)$$

and where **H** is the **Hilbert transform** (quadrature).

Interpretation

A monochromatic wave (prototype of a "stationary" deterministic signal) is described, in the complex plane, by a **rotating vector** *whose modulus and rotation speed are* **constant** *along time.*

instantaneous amplitude and frequency

Generalisation — A wave **modulated** in amplitude and in frequency (prototype of a "nonstationary" deterministic signal) is described, in the complex plane, by a **rotating vector** whose modulus and rotation speed are **varying** along time, complexification mimicking the "stationary" case:

$$x(t) \rightarrow z_x(t) := x(t) + i\,(\mathbf{H}x)(t).$$

Definition (Ville, '48)

The **instantaneous amplitude** *and* **frequency** *follow from this complex-valued representation, called* **analytic signal**, *as :*

$$a_x(t) := |z_x(t)| \quad ; \quad f_x(t) := (d/dt)\arg z_x(t)/2\pi.$$

[freqinst1.m]

limitations

- **Multiple components** — By construction, the instantaneous frequency can only attach **one** frequency value at a given time $\Rightarrow$ weighted average in the case of multicomponent signals.

  [freqinst2.m]

- **Trends** — Same problem with a monocomponent signal **with a DC component** or a very low frequency **trend**.

  [freqinsttrend.m]

  Possible improvement with an "osculating" Fresnel representation (Aboutajdine *et al.*, '80).

  [freqinstosc.m]

- **Noise** — **Differential** definition very sensitive to additive noise, even faint.

  [freqinst1b.m]

stationarity

Definition

A process $\{x(t), t \in \mathbb{R}\}$ is said to be (second order) **stationary** *if its statistical properties (of orders 1 and 2) are independent of some absolute time.*

- **Mean value** — The expectation $\mathbb{E}\{x(t)\}$ is constant ($\rightarrow 0$)
- **Covariance** — The covariance function $r_x(t, t') := \mathbb{E}\{x(t)\,\overline{x(t')}\}$ is such that

$$r_x(t, t') =: \gamma_x(t - t').$$

spectral representation

Result (Cramér)

$$x(t) = \int e^{i2\pi ft}\, dX(f)$$

with $\mathbb{E}\{dX(f)\,\overline{dX(f')}\} = \delta(f-f')\, d\Gamma_x(f)\, df'$

- **Simplification** — $d\Gamma_x(f)$ abs. cont. wrt Lebesgue
  $\Rightarrow d\Gamma_x(f) =: S_x(f)\, df$ with $S_x(f)$ **power spectral density**.
- **Duality (Bochner, Wiener, Khintchine)** — One thus gets

$$r_x(\tau) = \int e^{i2\pi f\tau}\, d\Gamma_x(f) \left(= \int e^{i2\pi f\tau}\, S_x(f)\, df \right).$$

nonstationarit(y/ies)

- **Spectral representation** — Always valid, but **without** the orthogonality of spectral increments $\Rightarrow$ the spectral distribution is no more diagonal but a function of **two** frequencies.
- **Covariance** – Depends explicitly of **two** times (e.g., one **absolute** time and one **relative** time).

Interpretation

The "power spectrum density" becomes **time-dependent** $\Rightarrow$ *time-frequency.*

chirp spectrum

Stationary phase — In the case where the phase derivative $\dot{\varphi}(t)$ is monotonous, one can approach a chirp spectrum

$$X(f) = \int a(t)\, e^{i(\varphi(t) - 2\pi f t)}\, dt$$

by its **stationary phase approximation** $\tilde{X}(f)$, leading to

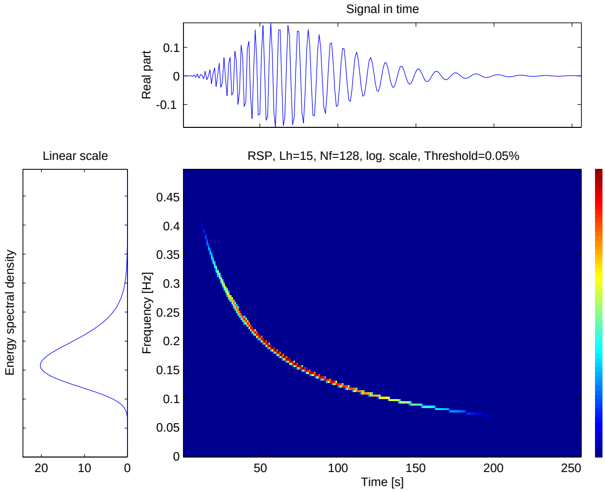
$$|\tilde{X}(f)| \propto a(t_s)\, |\ddot{\varphi}(t_s)|^{-1/2},$$

with t_s such that $\dot{\varphi}(t_s) = 2\pi f$.

Interpretation

The "instantaneous frequency" curve $\dot{\varphi}(t)$ puts in a one-to-one correspondence one time and one frequency. The spectrum follows by weighting the **visited frequencies** *by the corresponding* **residence durations**.

time-frequency interpretation



representing

intuition

Idea

Give a mathematical sense to musical notation

Aim

Write the "musical score" of a signal with multiple, evolutive components with that additional constraint of getting, in the case of an isolated chirp $x(t) = a(t) \exp\{i\varphi(t)\}$*, a* **localized** *representation*

$$\rho(t, f) \sim a^2(t)\, \delta\left(f - \dot{\varphi}(t)/2\pi\right).$$

local methods and localization

- **The example of the short-time FT** — One defines the **local** quantity

$$F_x^{(h)}(t, f) = \int x(s)\, \overline{h(s-t)}\, e^{-i2\pi fs}\, ds,$$

where $h(t)$ is some short-time observation window.

- **Measurement** — The representation results from an interaction between the signal and a **measurement device** (the window $h(t)$).
- **Trade-off** — A short window favors the "resolution" in time at the expense of the "resolution" in frequency, and vice-versa.

[spectrodemo.m]

adaptation

- **Chirps** — Adaptation to **pulses** if $h(t) \to \delta(t)$ and to **tones** if $h(t) \to 1 \Rightarrow$ adapting the analysis to arbitrary **chirps** suggests to make $h(t)$ **(locally) depending on the signal**.
- **Linear chirp** — In the linear case $f_x(t) = f_0 + \alpha t$, the equivalent frequency width δf_S of the **spectrogram** $S_x^{(h)}(t,f) := |F_x^{(h)}(t,f)|^2$ behaves as:

$$\delta f_S \approx \sqrt{\frac{1}{\delta t_h^2} + \alpha^2\, \delta t_h^2}$$

for a window $h(t)$ with an equivalent time width $\delta t_h \Rightarrow$ minimum for $\delta t_h \approx 1/\sqrt{\alpha}$ (but α **unknown**...).

self-adaptation and Wigner-Ville distribution

- **Matched filtering** — If one takes for the window $h(t)$ the **time-reversed** signal $x_-(t) := x(-t)$, one readily gets that $F_x^{(x_-)}(t,f) = W_x(t/2, f/2)/2$, where

$$W_x(t,f) := \int x(t+\tau/2)\,\overline{x(t-\tau/2)}\,e^{-i2\pi f\tau}\,d\tau$$

  is the **Wigner-Ville Distribution** (Wigner, '32; Ville, '48).
- **Linear chirps** — The WVD **perfectly** localizes on **straight lines** of the plane:

$$x(t) = \exp\{i2\pi(f_0 t + \alpha t^2/2)\} \Rightarrow W_x(t,f) = \delta\left(f - (f_0 + \alpha t)\right).$$

- **Remark** — Localization via self-adaptation leads to a **quadratic** transformation (energy distribution).

interpretation

- **Mirror symmetry** — Indexing the analyzed signalwrt a local frame as $x_t(s) := x(s+t)$, one gets :

$$W_x(t,f) := \int \left[x_t(+\tau/2)\, \overline{x_t(-\tau/2)} \right]\, e^{-i2\pi f\tau}\, d\tau,$$

[WVdemo.m]

- **Phase signal** — If $x_t(s) = \exp\{i\varphi_t(s)\}$, $W_x(t,f)$ is, as a function of t, the FT od a phase signal $\Phi_t(\tau) := \varphi_t(+\tau/2) - \varphi_t(-\tau/2)$, with "instantaneous frequency"

$$\tilde{f}_{x_t}(\tau) = \frac{1}{2\pi}\frac{\partial}{\partial\tau}\Phi_t(\tau) = \frac{1}{2}\left[f_{x_t}(+\tau/2) + f_{x_t}(-\tau/2)\right]$$

- **Localization** — It follows that $\tilde{f}_{x_t}(\tau) = f_0$ if $f_{x_t}(\tau) = f_0 + \alpha\,\tau$, for **any** modulation rate α.

[spectrovsWV.m]

further properties

- **Energy**

$$\iint W_x(t,f)\,dt\,df = \|x\|^2$$

- **Marginals**

$$\int W_x(t,f)\,dt = |X(f)|^2; \int W_x(t,f)\,df = |x(t)|^2$$

- **Unitarity ("Moyal's formula)**

$$\iint W_x(t,f)\,W_y(t,f)\,dt\,df = |\langle x, y\rangle|^2$$

- **Conservation of supports, covariance wrt scaling, linear filtering and modulation, etc.**

further properties

- **Local moments**

$$\int f\, W_x(t,f)\, df/|x(t)|^2 = f_x(t); \int t\, W_x(t,f)\, dt/|X(f)|^2 = t_x(f)$$

Interpretation

$W_x(t,f)$ **quasi-probability (joint) density** *of energy in time and frequency :*

$$W_x(t,f) = W_x(t|f) \int W_x(t,f)\, dt = W_x(f|t) \int W_x(t,f)\, df$$

$$f_x(t) = \mathbb{E}\{f|t\}; t_x(f) = \mathbb{E}\{t|f\}$$

- **Limitation** — $W_x(t,f) \in \mathbb{R}$ but $\notin \mathbb{R}_+$.

interferences

- **Quadratic superposition** — For any pair of signals $\{x(t), y(t)\}$ and coefficients (a, b), one gets

$$W_{ax+by}(t,f) = |a|^2\, W_x(t,f) + |b|^2\, W_y(t,f) + 2\, Re\left\{a\, \overline{b}\, W_{x,y}(t,f)\right\},$$

with

$$W_{x,y}(t,f) := \int x(t+\tau/2)\, \overline{y(t-\tau/2)}\, e^{-i2\pi f\tau}\, d\tau$$

- **Drawback** — Interferences between **disjoint** component reduce readability.
- **Advantage** — Inner interferences between **coherent** components guarantee localization.

interferences

- **Janssen's formula (Janssen, '81)** — It follows from the**unitarity** of $W_x(t, f)$ that:

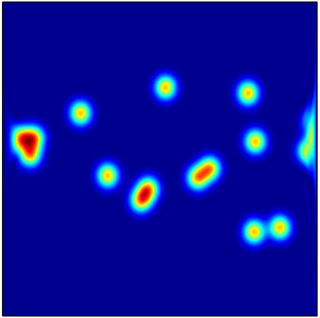
$$|W_x(t,f)|^2 = \iint W_x\left(t+\frac{\tau}{2}, f+\frac{\xi}{2}\right) W_x\left(t-\frac{\tau}{2}, f-\frac{\xi}{2}\right) d\tau\, d\xi$$

- **Geometry (Hlawatsch & F., '85)** — Contributions located in any two points of the plane plan interfere to create a third contribution
    1. midway of the segment joining the two components
    2. oscillating (positive and negativ values) in a direction perpendicular to this segment
    3. with a "frequency" proportional to their "time-frequency distance".

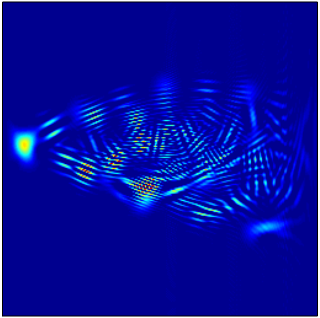
`[WV2trans.m, WVinterf.m]`

interferences and readability

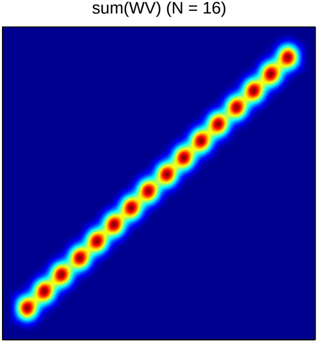
somme des WV (N = 16)

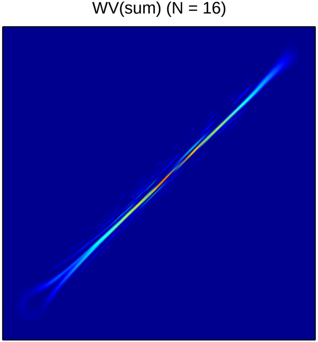


WV de la somme (N = 16)



interferences and localization





classes of quadratic distributions

Observation

Many quadratic distributions have been proposed in the literature since more than half a century (e.g., spectrogram and DWV): **none fully extends the notion of spectrum density to the nonstationary case**.

Principle of conditional unicity — Classes of quadratic distributions of the form $\rho_x(t,f) = \langle x, \mathbf{K}_{t,f}x\rangle$ can be constructed based on **covariance requirements** :

$$\begin{array}{ccc} x(t) & \rightarrow & \rho_x(t,f) \\ \downarrow & & \downarrow \\ (\mathbf{T}x)(t) & \rightarrow & \rho_{\mathbf{T}x}(t,f) = (\tilde{\mathbf{T}}\rho_x)(t,f) \end{array}$$

classes of quadratic distributions

- **Cohen's class** — Covariance wrt **shifts** $(\mathbf{T}_{t_0,f_0}x)(t) = x(t-t_0)\exp\{i2\pi f_0 t\}$ leads to **Cohen's class** (Cohen, '66) :

$$C_x(t,f) := \iint W_x(s,\xi)\,\Pi(s-t,\xi-f)\,ds\,d\xi,$$

  with $\Pi(t,f)$ "arbitrary" (and to be specified via additional constraints).
- **Variations** — Other choices possibles, e.g., $(\mathbf{T}_{t_0,f_0}x)(t) = (f/f_0)^{1/2}x(f(t-t_0)/f_0) \rightarrow$ **affine class** (Rioul & F, '92), etc.

an alternative interpretation of Cohen's class

- **Duality between distribution and correlation** — In the "stationary" case, the **frequency** energy distribution can be estimated as the Fourier image of the **time** correlation $\langle x, \mathbf{T}_\tau x \rangle$, possibly weighted.
- **Extension** — In the "nonstationary" case, one must consider a **time-frequency correlation** $A_x(\xi, \tau) \propto \langle x, \mathbf{T}_{\tau,\xi} x \rangle$ (**ambiguity function**) which, after weighting and Fourier transformation, leads again to Cohen's class:

$$C_x(t, f) = \iint \varphi(\xi, \tau)\, A_x(\xi, \tau)\, e^{-i2\pi(\xi t + \tau f)}\, d\xi\, d\tau.$$

[WVvsAF.m]

why “Cohen-type” classes?

- **Unification** — Specifying a kernel (i.e., $\Pi(t, f)$) **defines** a distribution: unifying framework or most propositions of the literature (Wigner-Ville, spectrogram, Page, Levin, Rihaczek, etc.).
- **Parameterization** — Properties of a distribution are directly connected with **admissibility** conditions of the associated kernel $\Rightarrow$ simplified possibility of **evaluation** and **design**.

an example of definition

Spectrogram — If we consider the case of the **spectrogram** with window $h(t)$, one can write:

$$\begin{array}{rcl} S_x^{(h)}(t,f) & = & \left| \int x(s)\, \overline{h(s-t)}\, e^{-i2\pi fs}\, ds \right|^2 \\ & = & |\langle x, \mathbf{T}_{t,f}\, h\rangle|^2 \\ & = & \iint W_x(s,\xi)\, W_{\mathbf{T}_{t,f}h}(s,\xi)\, ds\, d\xi \\ & = & \iint W_x(s,\xi)\, W_h(s-t,\xi-f)\, ds\, d\xi \end{array}$$

$\Rightarrow$ a spectrogram is a member of Cohen's class, with kernel

$$\Pi(t,f) = W_h(t,f)$$

an example of admissibility constraint

Marginal in time — If one wants to have $\int C_x(t,f)\,df = |x(t)|^2$, one can write:

$$\begin{aligned}
\int C_x(t,f)\,df &= \int \left(\iint \varphi(\xi,\tau)\,A_x(\xi,\tau)\,e^{-i2\pi(\xi t+\tau f)}\,d\xi\,d\tau\right)\,df \\
&= \int \varphi(\xi,0)\,A_x(\xi,0)\,e^{-i2\pi\xi t}\,d\xi \\
&= \int \varphi(\xi,0)\,\left(\int |x(\theta)|^2\,e^{i2\pi\xi\theta}\,d\theta\right)\,e^{-i2\pi\xi t}\,d\xi \\
&= \int |x(\theta)|^2\,\left(\int \varphi(\xi,0)\,e^{i2\pi\xi(\theta-t)}\,d\xi\right)\,d\theta
\end{aligned}$$

$\Rightarrow$ the associated kernel must necessarily satisfy

$$\varphi(\xi,0)=1, \forall\xi$$

(true for Wigner-Ville but not for spectrograms)

Cohen's class and smoothing

- **Spectrogram** — Given a low-pass window $h(t)$, one gets the **smoothing** relation:

$$S_x^{(h)}(t,f) := |F_x^{(h)}(t,f)|^2 = \iint W_x(s,\xi)\, W_h(s-t,\xi-f)\, ds\, d\xi$$

- **From Wigner-Ville to spectrograms** — A generalization amounts to choose a smoothing function $\Pi(t,f)$ allowing for a **continuous** and **separable** transition between Wigner-Ville and a spectrogram (**smoothed pseudo-Wigner-Ville** distributions) :

$$\begin{array}{ccccccc} Wigner-Ville & \ldots & \rightarrow & PWVL & \ldots & \rightarrow & spectrogram \\ & & & & & & \\ \delta(t)\,\delta(f) & & & g(t)\,H(f) & & & W_h(t,f) \end{array}$$

[WV2Smovie.m]

time-frequency spectrum

Definition (Martin, '82)

One of the most "natural" extensions of the power spectrum density is given by the **Wigner-Ville Spectrum** *:*

$$\mathbf{W}_x(t,f) := \int r_x\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right)\, e^{-i2\pi f\tau}\, d\tau$$

- **Interpretation** — FT of a **local** correlation.
- **Properties** — PSD if $x(t)$ stationary, marginals, etc.
- **Relation with the WVD** — Under simple conditions, one has $\mathbf{W}_x(t,f) = \mathbb{E}\{W_x(t,f)\}$.

estimation of the Wigner-Ville spectrum

Aim

Approach $\mathbb{E}\{W_x(t,f)\}$ on the basis of only one realization.

- **Assumption** — **Local** stationnarity (in time and in frequency).
- **Estimators** — Smoothing of the DWV :

$$\hat{\mathbf{W}}_x(t,f) = (\Pi ** W_x)(t,f)$$

i.e., Cohen's classe.

- **Properties** — **Statistical** (bias-variance) and **geometrical** (localization) trade-offs, both controlled by $\Pi(t,f)$.

global vs. local

- **Global approach** — The Wigner-Ville Distribution localizes perfectly on **straight lines** of the plane (linear chirps). One can construct other distributions localizing on more general **curves** (ex.: **Bertrand**'s distributions adapted to hyperbolic chirps).
- **Local approach** — A different possibility consists in revisiting the smoothing relation defining the spectrogram and in considering localization wrt the instantaneous frequency as it can be measured **locally**, at the scale of the short-time window $\Rightarrow$ **reassignment**.

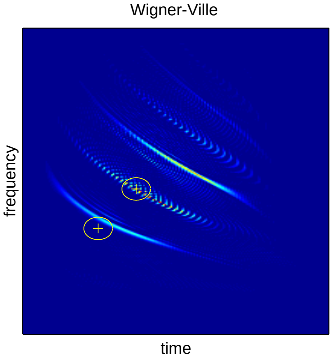
reassignment

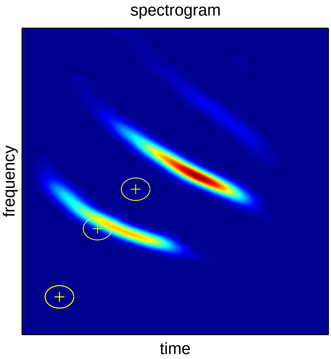
- **Principle** — The key idea is (1) to replace the **geometrical** center of the smoothing time-frequency domain by the **center of mass** of the WVD over this domain, and (2) to **reassign** the value of the smoothed distribution to this local centroïd:

$$S_x^{(h)}(t,f) \mapsto \iint S_x^{(h)}(s,\xi)\,\delta\left(t-\hat{t}_x(s,\xi), f-\hat{f}_x(s,\xi)\right)\,ds\,d\xi.$$

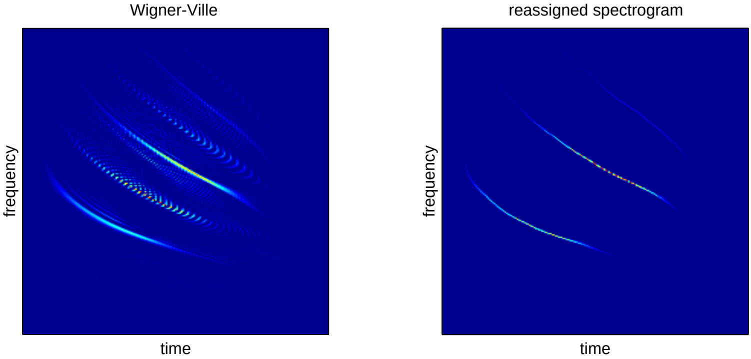
- **Remark** — Reassignment has been first introduced for the only spectrogram (Kodera *et al.*, '76), but its principle has been further generalized to **any** distribution resulting from the smoothing of a localizable mother-distribution (Auger & F., '95).

reassignment





reassignment



reassignment in action

- **Spectrogram — Implicit** computation of the local centroïds (Auger & F., '95) :

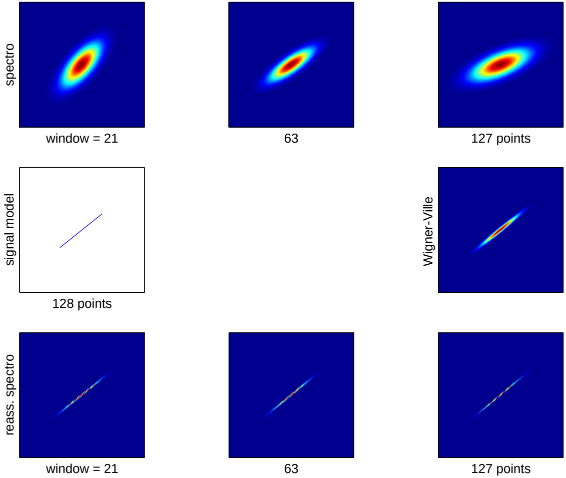
$$\hat{t}_x(t,f) = t + Re\left\{\frac{F_x^{(\mathcal{T}h)}}{F_x^{(h)}}\right\}(t,f)$$

$$\hat{f}_x(t,f) = f - Im\left\{\frac{F_x^{(\mathcal{D}h)}}{F_x^{(h)}}\right\}(t,f),$$

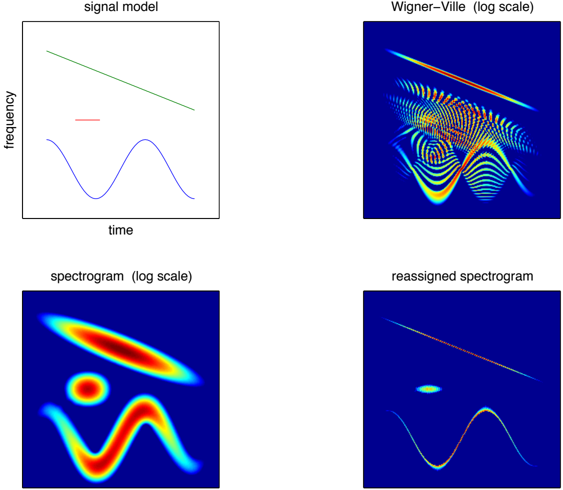
with $(\mathcal{T}h)(t) = t\,h(t)$ and $(\mathcal{D}h)(t) = (dh/dt)(t)/2\pi$.

- **Beyond spectrograms —** Possible generalizations to other smoothings (smoothed pseudo-Wigner-Ville, scalogram, etc.).

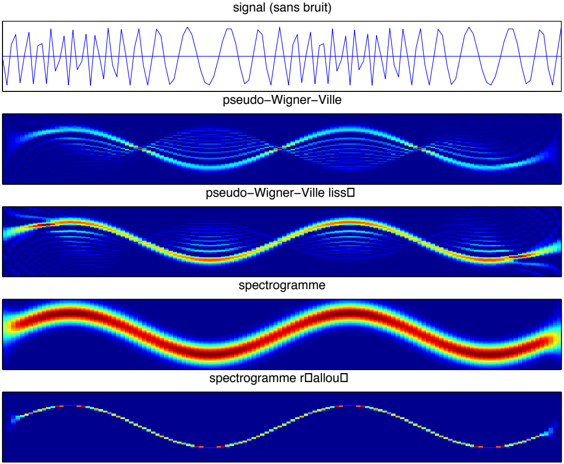
independence wrt window size



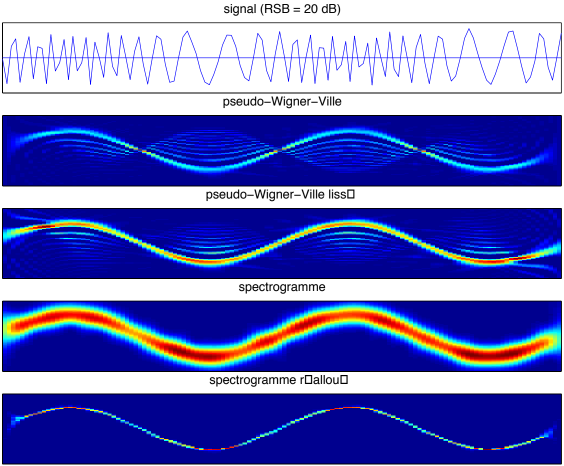
an example of comparison



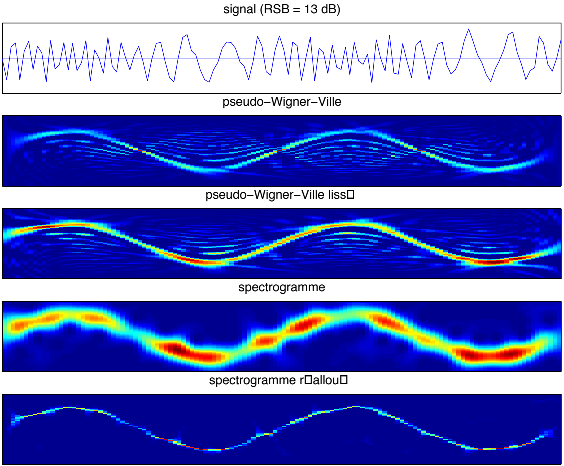
comparison with noise



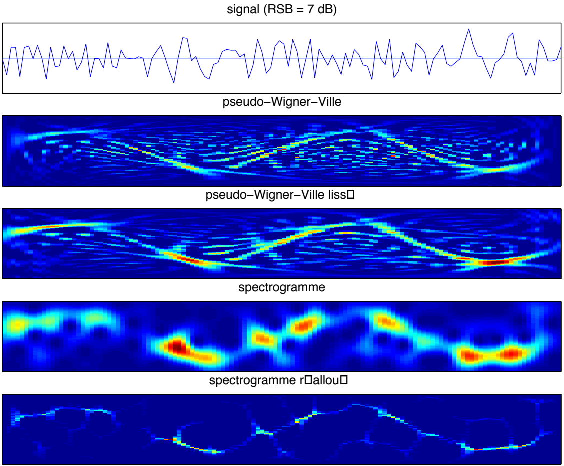
comparison with noise



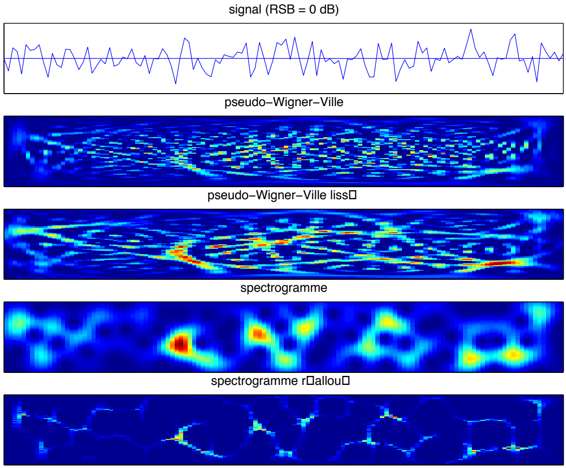
comparison with noise



comparison with noise



comparison with noise



reassignment and estimation

- **Advantage** — Very good properties of **localization** for chirps ($>$ spectrogram).
- **Limitation** — High **sensitivity to noise** ($<$ spectrogram).

Aim

Reduce fluctuations while preserving localization.

Idea (Xiao & F., '06)

Adopt a **multiple windows** *approach.*

back to spectrum estimation

- **Stationary processes** — The **power spectrum density** can be viewed as:

$$\mathbf{S}_x(f) = \lim_{T \to \infty} \mathbb{E} \left\{ \frac{1}{T} \left| \int_{-T/2}^{+T/2} x(t)\, e^{-i2\pi ft}\, dt \right|^2 \right\}$$

- **In practice** — Only one, finite duration, realization $\Rightarrow$ crude periodogram (squared FT) $=$ **non consistent estimator with large variance**

classical way out (Welch, '67)

- **Principle** — Method of **averaged periodograms**

$$\hat{\mathbf{S}}_{x,K}^{(W)}(f) = \frac{1}{K} \sum_{k=1}^{K} S_x^{(h)}(t_k, f)$$

with $t_{k+1} - t_k$ of the order of the width of the window $h(t)$.

- **Bias-variance trade-off** — Given T (finite), increasing $K \Rightarrow$ **reduces variance**, but **increases bias**

multitaper solution (Thomson, '82)

- **Principle** — Computing

$$\hat{\mathbf{S}}_{x,K}^{(T)}(f) = \frac{1}{K}\sum_{k=1}^{K} S_x^{(h_k)}(0,f)$$

with $\{h_k(t), k \in \mathbb{N}\}$ a family of orthonormal windows extending over the whole support of the observation $\Rightarrow$ **reduced variance, without sacrifying bias**

- **Nonstationary extension — Multitaper spectrogram**

$$\hat{\mathbf{S}}_{x,K}^{(T)}(f) \rightarrow S_{x,K}(t,f) := \frac{1}{K}\sum_{k=1}^{K} S_x^{(h_k)}(t,f)$$

- **Limitation** — Localization controlled by most spread spectrogram.

Multitaper reassignment

Idea

Combining the advantages of reassignment (wrt localization) with those of multitapering (wrt fluctuations) :

$$S_{x,K}(t,f) \rightarrow RS_{x,K}(t,f) := \frac{1}{K}\sum_{k=1}^{K} RS_x^{(h_k)}(t,f)$$

1. **coherent averaging of chirps** (localization independent of the window)
2. **incoherent averaging of noise** (different TF distributions for different windows)

in practice

- **Choice of windows — Hermite functions**

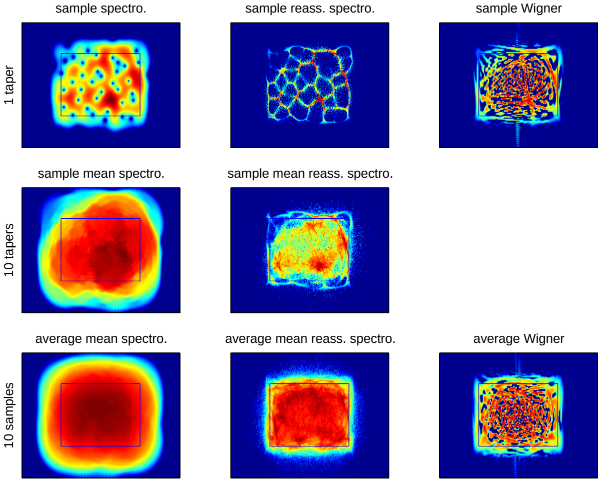
$$h_k(t) = (-1)^k \frac{e^{-t^2/2}}{\sqrt{\pi^{1/2} 2^k k!}} (\mathcal{D}^k \gamma)(t); \gamma(t) = e^{t^2}$$

rather than **Prolate Spheroidal Wave functions**

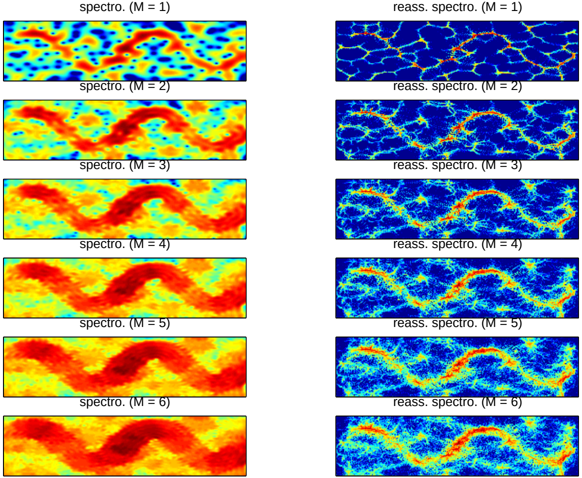
- **Two main reasons**
  1. WVD with **elliptic symmetry** and **maximum concentration** in the plane.
  2. **recursive** computation of $h_k(t)$, $(\mathcal{T} h_k)(t)$ and $(\mathcal{D} h_k)(t) \Rightarrow$ better implementation in **discrete-time**. In particular:

$$(\mathcal{D} h_k)(t) = (\mathcal{T} h_k)(t) - \sqrt{2(k+1)}\, h_{k+1}(t)$$

example 1


sample spectro.
sample reass. spectro.
sample Wigner
1 taper
sample mean spectro.
sample mean reass. spectro.
10 tapers
average mean spectro.
average mean reass. spectro.
average Wigner
10 samples

example 2

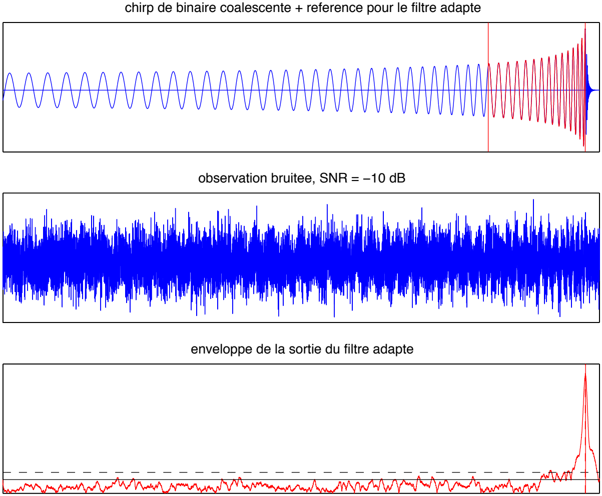

spectro. (M = 1)
reass. spectro. (M = 1)
spectro. (M = 2)
reass. spectro. (M = 2)
spectro. (M = 3)
reass. spectro. (M = 3)
spectro. (M = 4)
reass. spectro. (M = 4)
spectro. (M = 5)
reass. spectro. (M = 5)
spectro. (M = 6)
reass. spectro. (M = 6)

detection/estimation of chirps

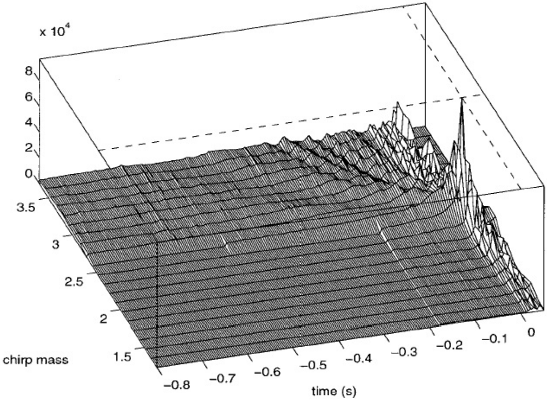
- **Optimality** — Matched filtering, maximum likelihood, contrast,...: basic ingredient = **correlation** "received signal — copy of emitted signal".
- **Time-frequency interpretation** — **Unitarity** of a time-frequency distribution $\rho_x(t, f)$ guarantees the equivalence:
$$|\langle x, y\rangle|^2 = \langle\langle \rho_x, \rho_y\rangle\rangle.$$
- **Chirps** — Unitarity + localization $\Rightarrow$ detection/estimation via **path integration** in the plane (e.g., Wigner-Ville and linear chirps).

[detectTF.m]

VIRGO example

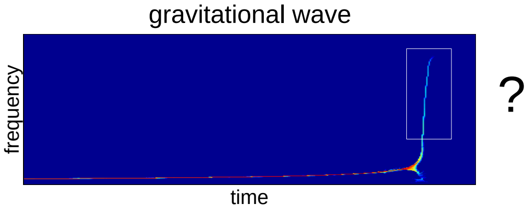


VIRGO example (Chassande-Mottin & F., '98)



time-frequency detection ?

- **Language** — The time-frequency viewpoint offers a natural **language** for addressing detection/estimation problems **beyond** nominal situations.
- **Robustness** — Incorporation of **uncertainties** in the chirp model by replacing the integration **curve** by a **domain** (example of post-newtonian approximations in the case of gravitational waves).



interpretation example: Doppler-tolerance

- **Localization of a moving target** — When estimating a delay by matched filtering with some unknown Doppler effect, estimations of delay and Doppler are coupled $\Rightarrow$ **bias** and **contrast loss** at the detector output.
- **Addressed problem** — Suppress bias on delay and minimize contrast loss.
- **Signal design** — Specification of performance via a **geométric** interpretation of the time-frequency structure of a chirp.

[dopptol.m, faTFdopp.m]

monographs

- L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, 1995.
- S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 1997.
- R. Carmona, H.L. Hwang & B. Torrésani, *Practical Time-Frequency Analysis*, Academic Press, 1998.
- F. Hlawatsch, *Time-Frequency Analysis and Synthesis of Linear Signal Spaces*, Kluwer, 1998.
- P. Flandrin, *Time-Frequency/Time-Scale Analysis*, Academic Press, 1999.

collective books

- A. Papandreou-Suppappola (*ed.*), *Applications in Time-Frequency Signal Processing*, CRC Press, 2003.
- B. Boashash (*ed.*), *Time-Frequency Signal Analysis and Processing*, Elsevier, 2003.
- Ch. Doncarli & N. Martin (*eds.*), *Décision dans le Plan Temps-Fréquence*, Traité IC2, Hermes, 2004.
- F. Auger & F. Hlawatsch (*eds.*), *Temps-Fréquence — Concepts et Outils*, Traité IC2, Hermes, 2005.

preprints & Matlab codes

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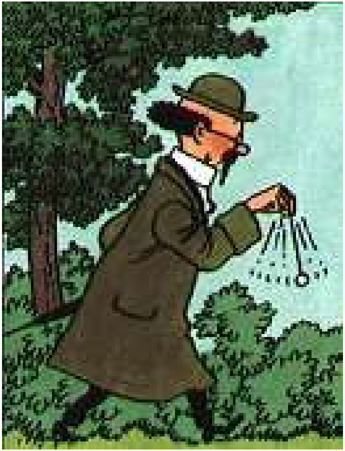
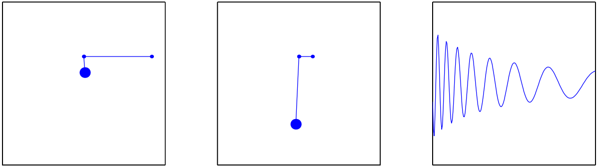
pendulum



[expendule.m]

pendulum

$$\ddot{\theta}(t) + (g/L)\,\theta(t) = 0$$



- **Constant length** — $L = L_0 \Rightarrow$ small oscillations are sinusoidal, with **constant** period $T_0 = 2\pi\sqrt{L_0/g}$.
- **"Slowly" varying length** — $L = L(t) \Rightarrow$ small oscillations are quasi-sinusodal, with **varying** pseudo-period $T(t) \sim 2\pi\sqrt{L(t)/g}$.

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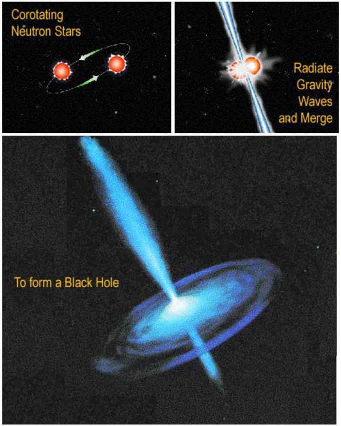
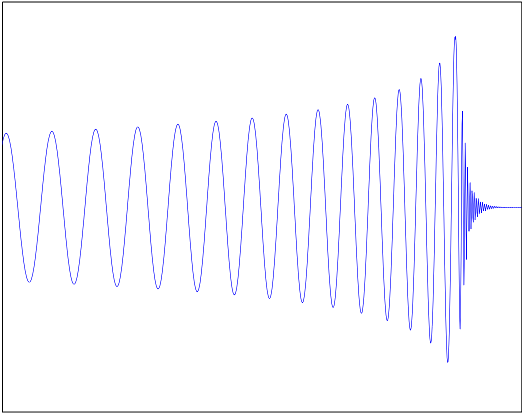
gravitational waves



[binaire.m]

gravitational waves

gravitational wave



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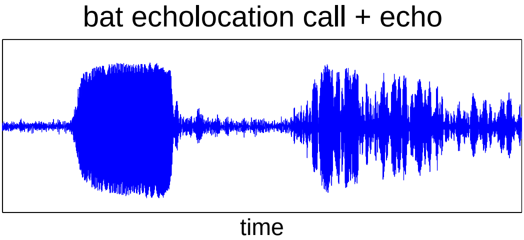
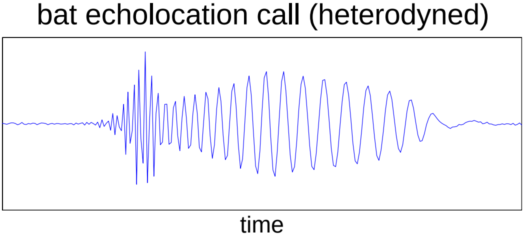
bat echolocation

[chauvesouris.m]

bat echolocation





bat echolocation

- **System** —(**Active**) navigation system, natural sonar
- **Signals** — Ultrasound acoustic waves, **transient** (a few ms) and **"wideband"** (some tens of kHz between 40 and 100kHz)
- **Performance** — Close to optimality, with **adaption** of the waveforms to multiple tasks (detection, estimation, recognition, interference rejection,. . .)

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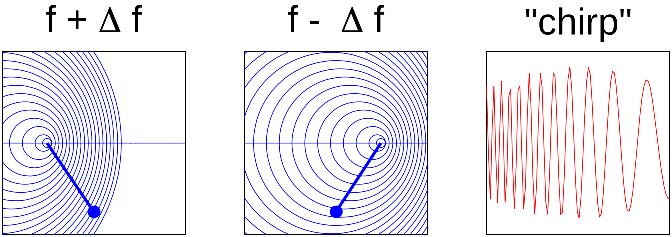
Doppler effect

[exdoppler.m]

Doppler effect

- **Moving monochromatic source — Differential** perception of the emitted frequence.



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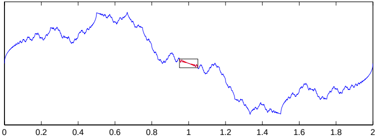
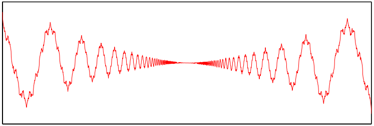
Riemann function

Riemann function

$$\sigma(t) := \sum_{n=1}^{\infty} n^{-2} \sin \pi n^2 t$$





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