Empirical Mode Decomposition

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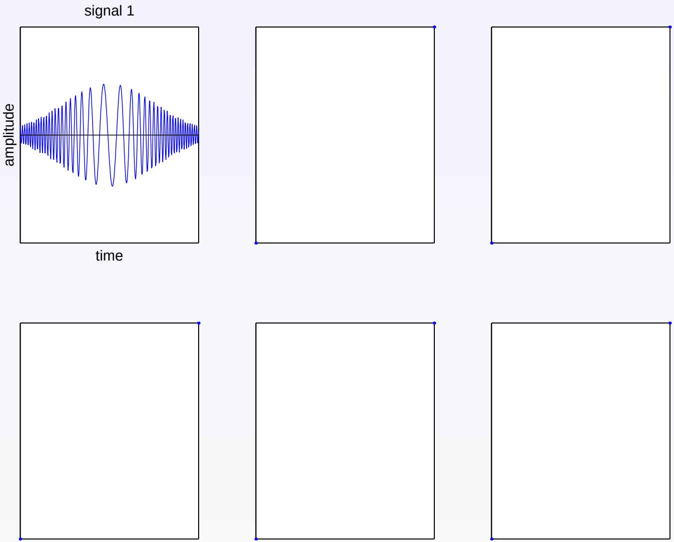
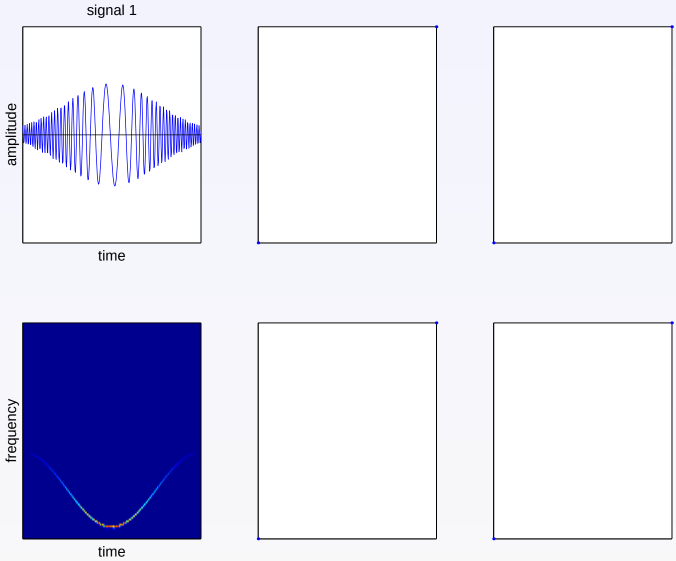
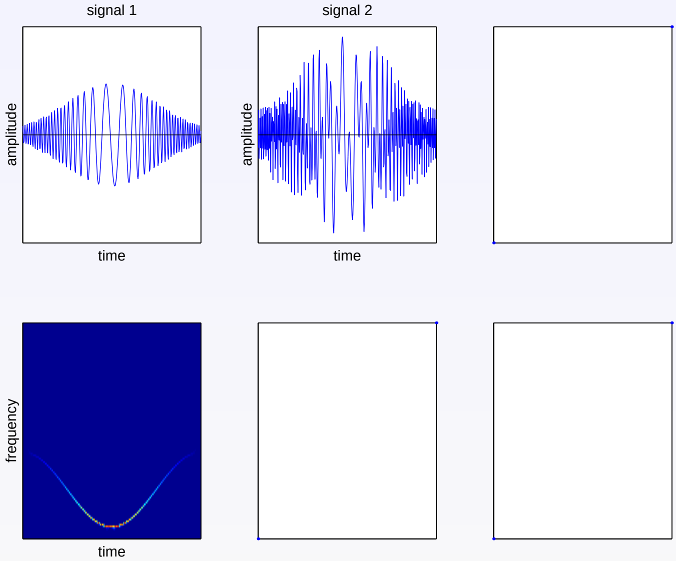
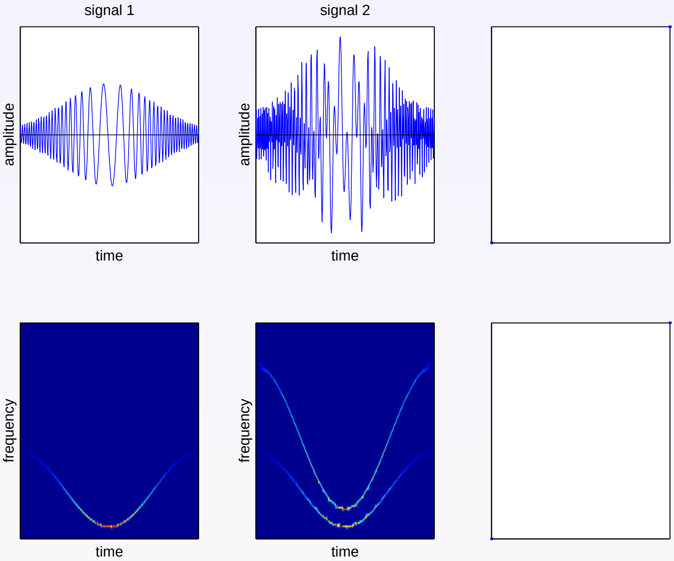
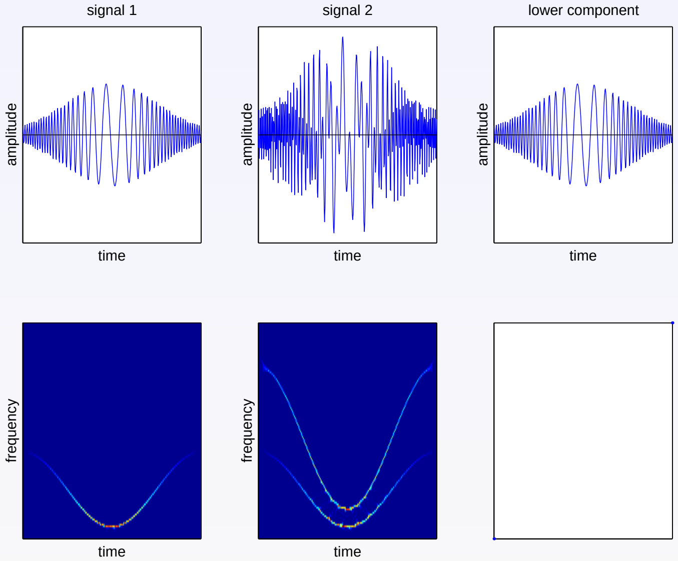
thanks to Gabriel Rilling, Paulo Gonçalves, Azadeh Moghtaderi & Pierre Borgnat

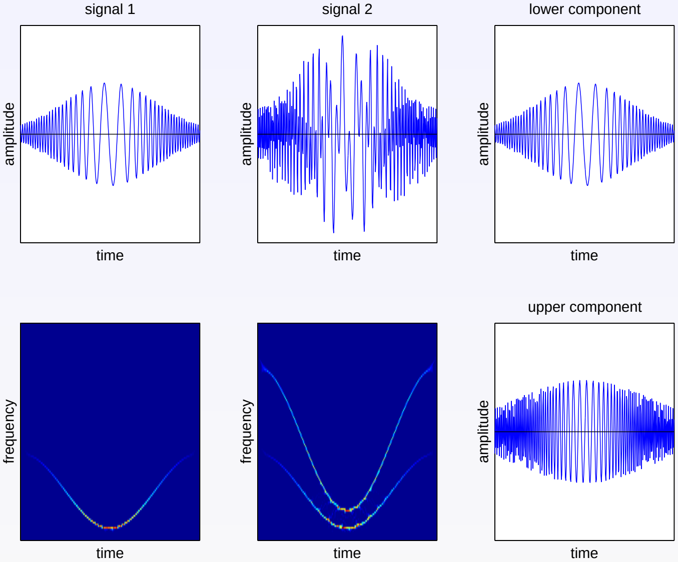

signal 1
amplitude
time

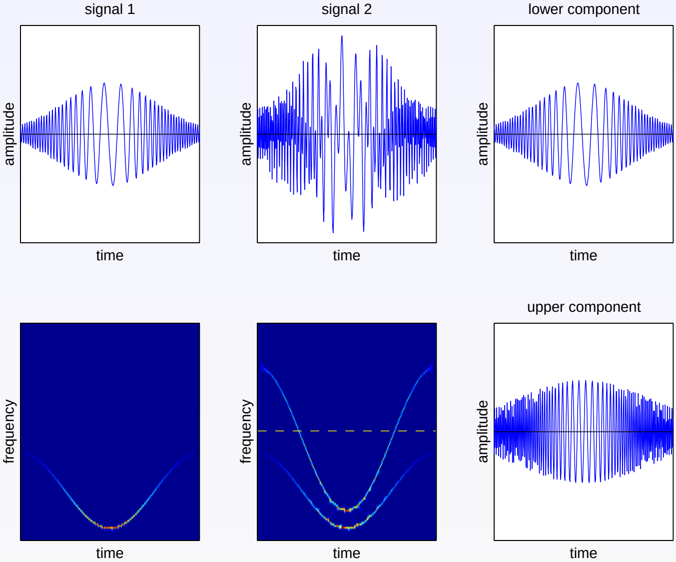

signal 1
amplitude
time
frequency
time

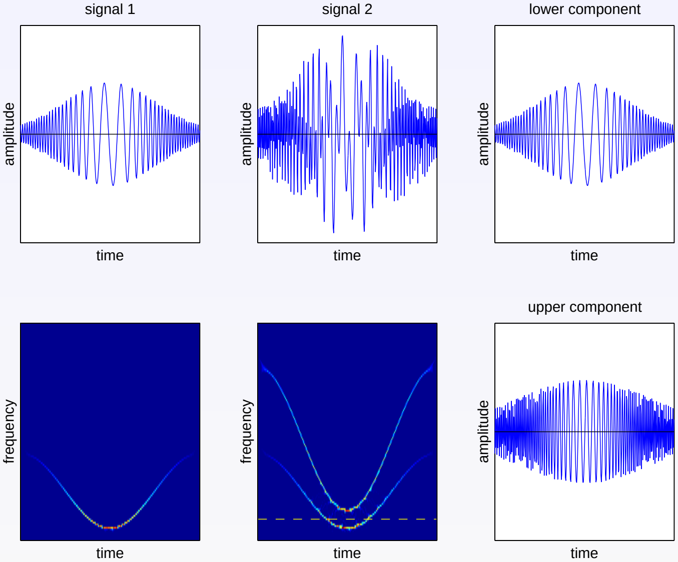

signal 1
signal 2
amplitude
time
amplitude
time
frequency
time

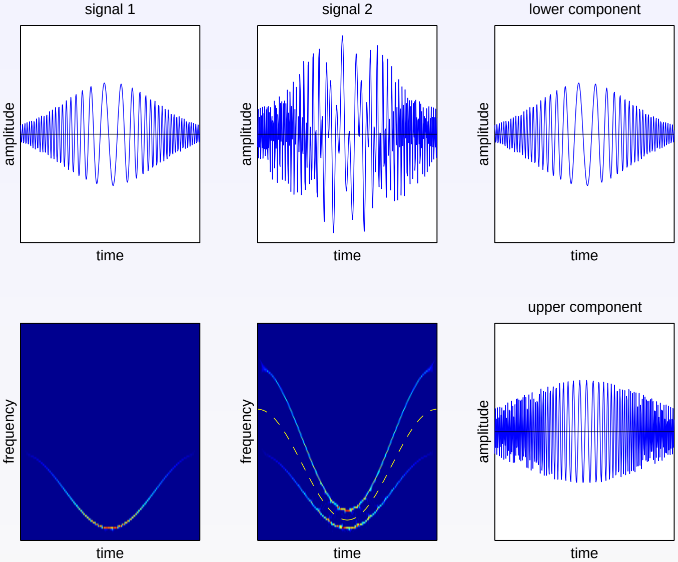

signal 1
signal 2
amplitude
time
frequency


signal 1
signal 2
lower component
amplitude
time
frequency


signal 1
signal 2
lower component
upper component
amplitude
time
frequency


signal 1
signal 2
lower component
amplitude
time
upper component
frequency


signal 1
signal 2
lower component
amplitude
time
frequency
upper component


signal 1
signal 2
lower component
amplitude
time
amplitude
time
amplitude
time
upper component
frequency
time
frequency
time
amplitude
time

Aim

Given an observation $x(t)$, get a representation of the form :

$$x(t) = \sum_{k=1}^{K} a_k(t)\,\psi_k(t),$$

where the $a_k(t)$'s measure "amplitude modulations" and the $\psi_k(t)$'s "oscillations".

$\Rightarrow$ "EMD" (*Empirical Mode Decomposition*)

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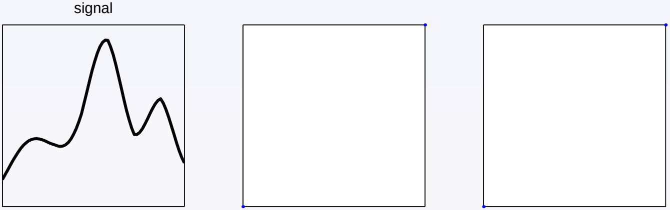
Idea

"signal = fast oscillations superimposed to slow oscillations"



Idea

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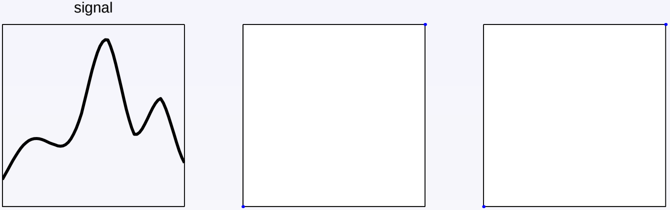

Idea

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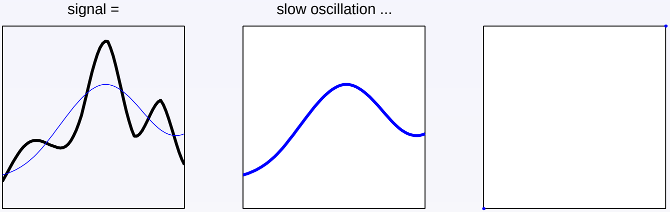

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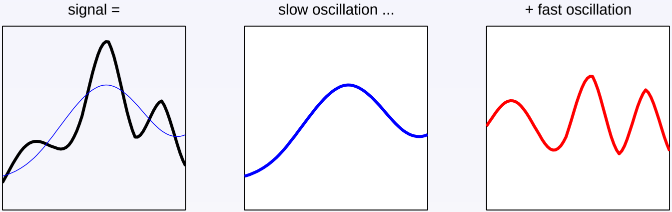

Idea

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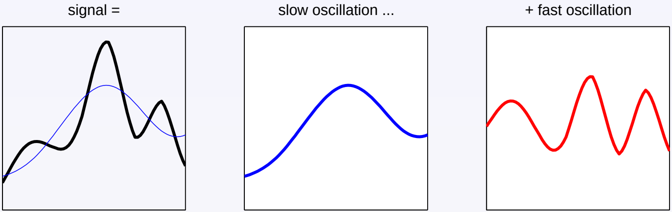

and iteration...

Implementation (Huang *et al.*, '98) — Identify (locally) the fastest oscillation, subtract it to the initial signal and iterate on the residual :

1. identify local maxima and minima in the signal
2. deduce an upper and a lower envelope by interpolation (cubic splines)
   1. subtract the mean envelope from the signal
   2. iterate until #{extrema} = #{zeroes} ± 1
3. subtract the so-obtained *Intrinsic Mode Function* (IMF) from the signal
4. iterate on the residual

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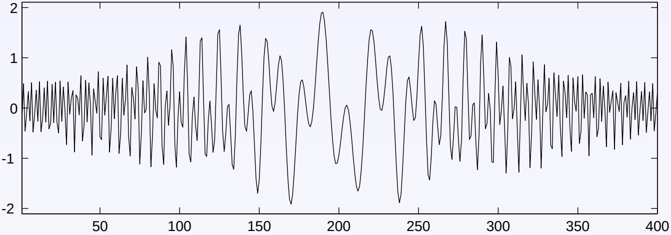
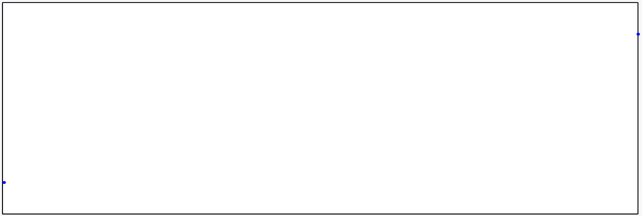
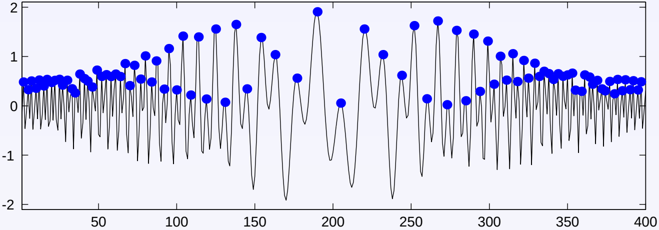
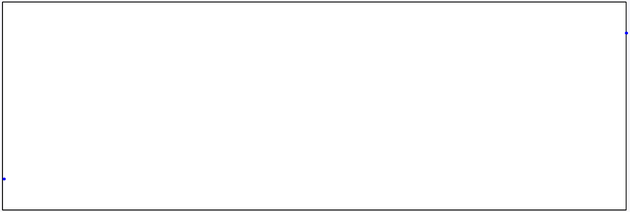
IMF 1; iteration 0

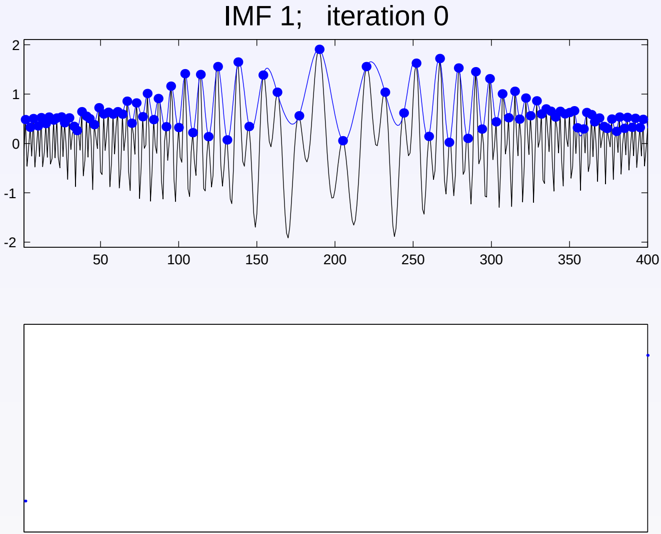


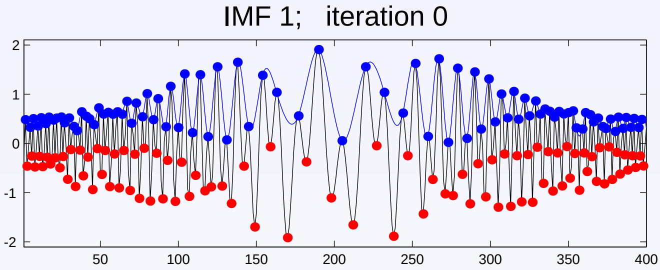


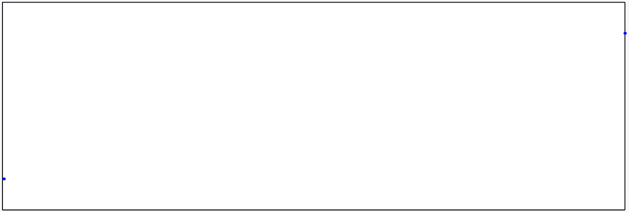
IMF 1; iteration 0

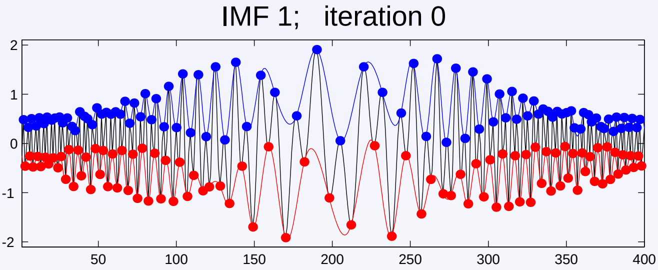


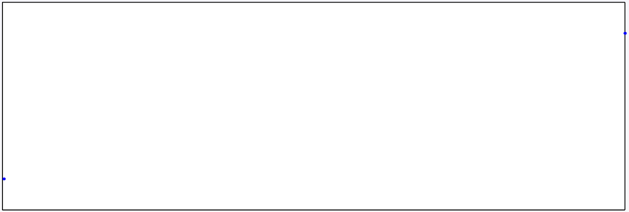



IMF 1; iteration 0
2
1
0
-1
-2
50
100
150
200
250
300
350
400

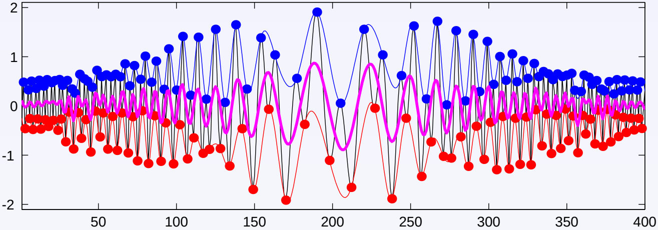

IMF 1; iteration 0
2
1
0
-1
-2
50
100
150
200
250
300
350
400

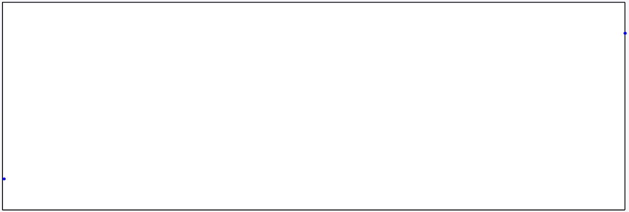


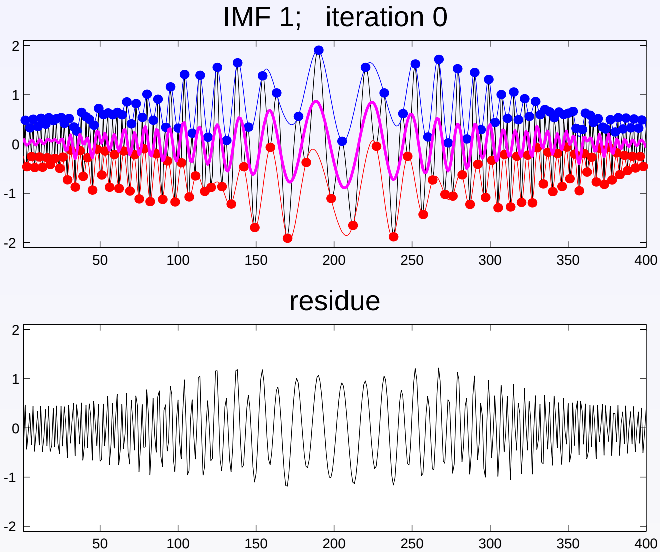

IMF 1; iteration 0
2
1
0
-1
-2
50
100
150
200
250
300
350
400



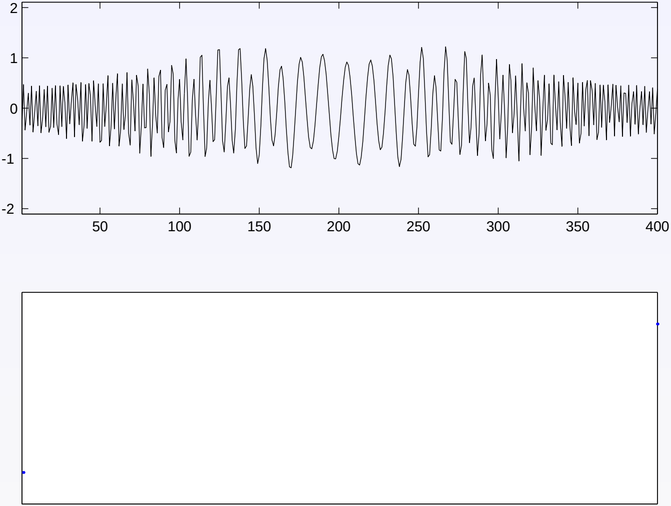
IMF 1; iteration 0



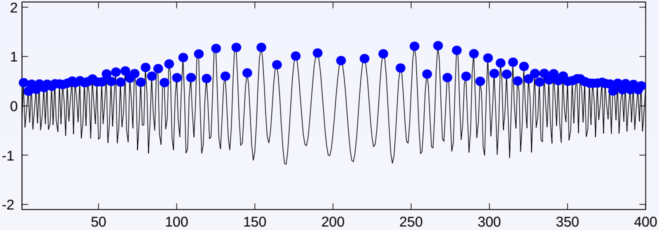


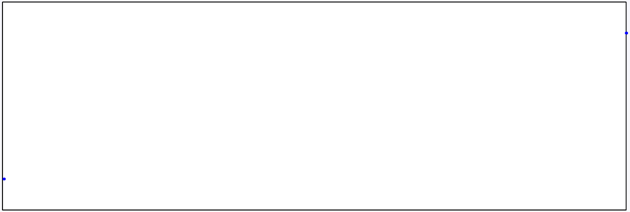

IMF 1; iteration 0
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

IMF 1; iteration 1

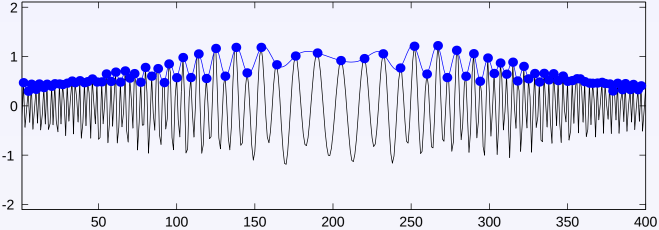


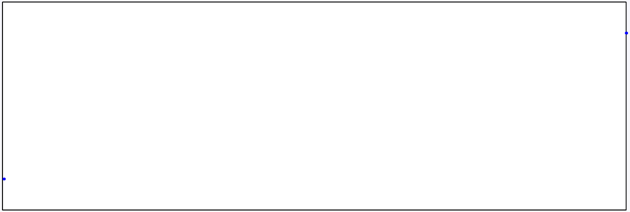
IMF 1; iteration 1



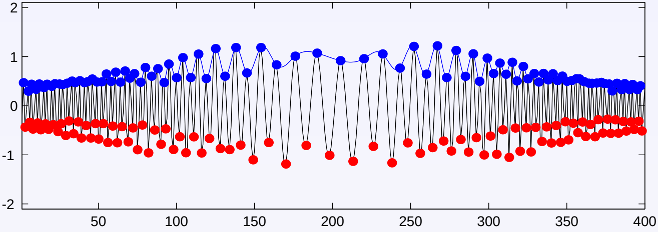


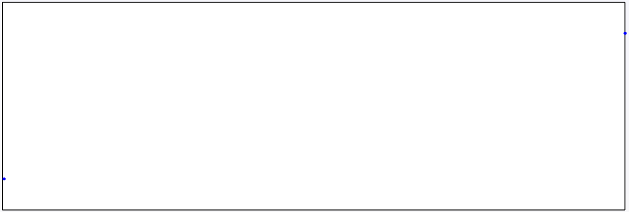
IMF 1; iteration 1

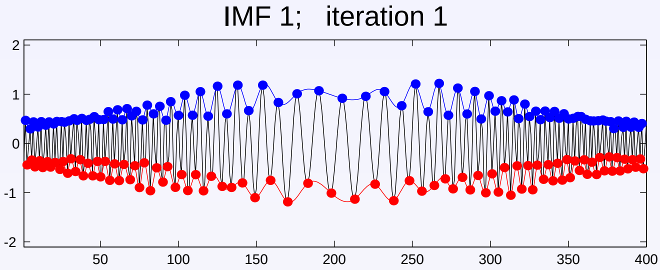


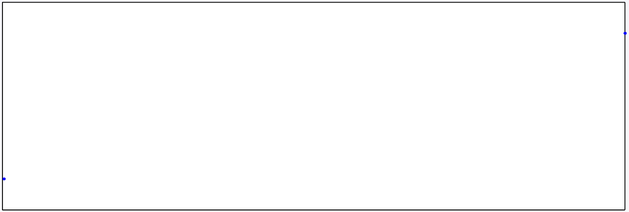


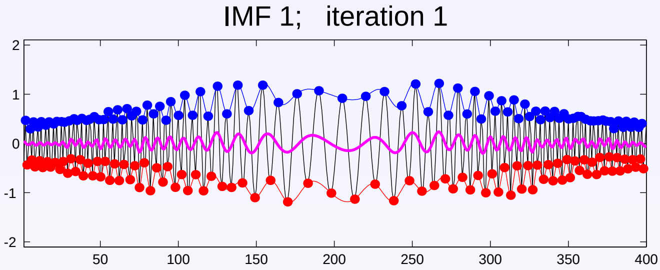
IMF 1; iteration 1

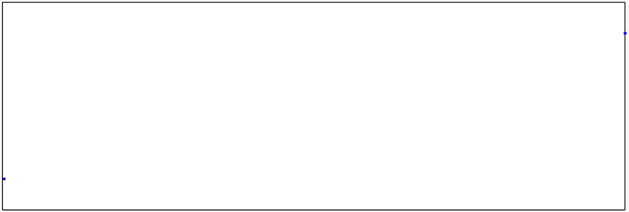


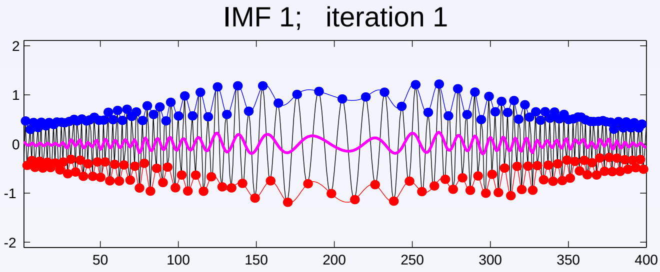


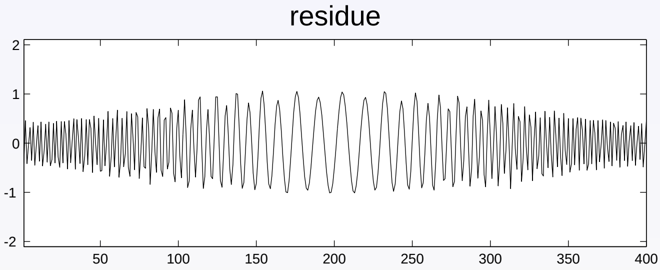

IMF 1; iteration 1
2
1
0
-1
-2
50
100
150
200
250
300
350
400



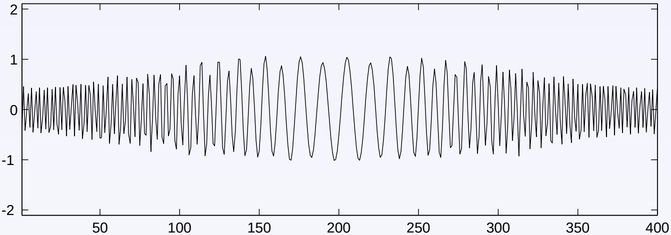

IMF 1; iteration 1
2
1
0
-1
-2
50
100
150
200
250
300
350
400

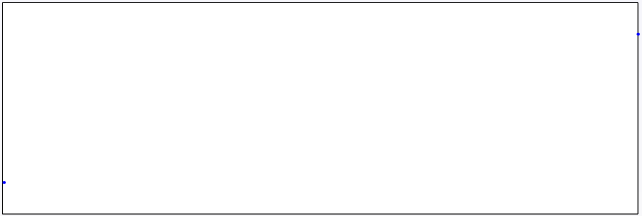



IMF 1; iteration 1
2
1
0
-1
-2
50
100
150
200
250
300
350
400

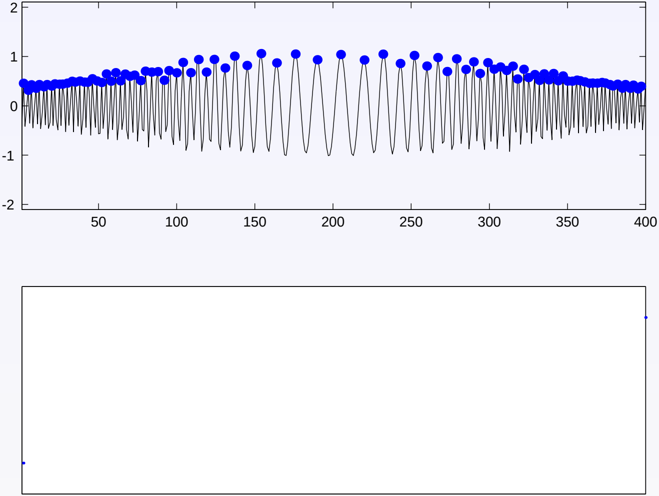

residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

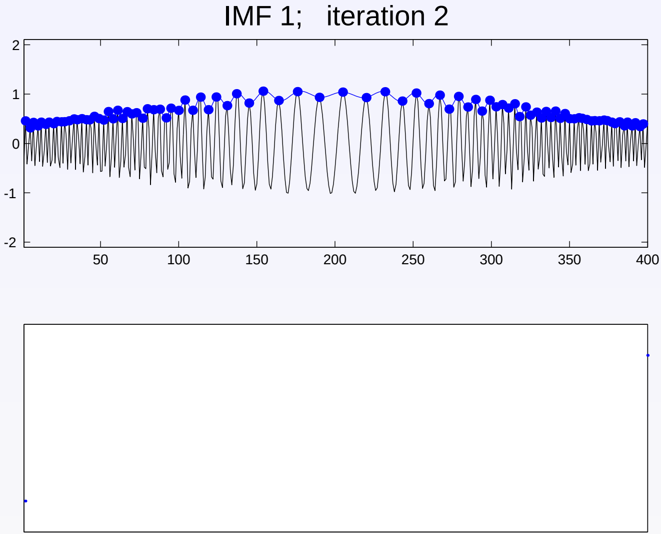
IMF 1;   iteration 2



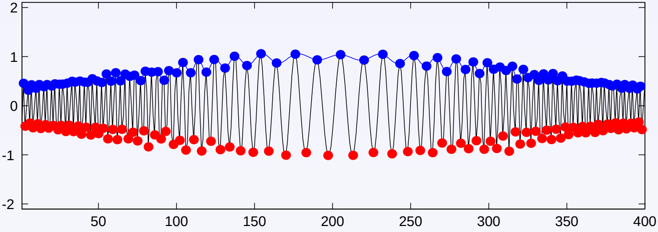


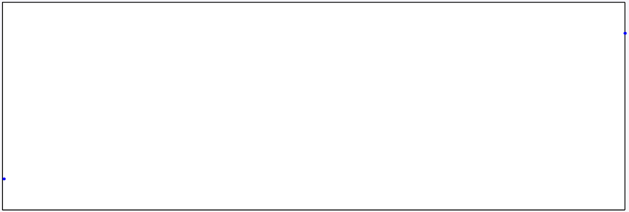
IMF 1; iteration 2

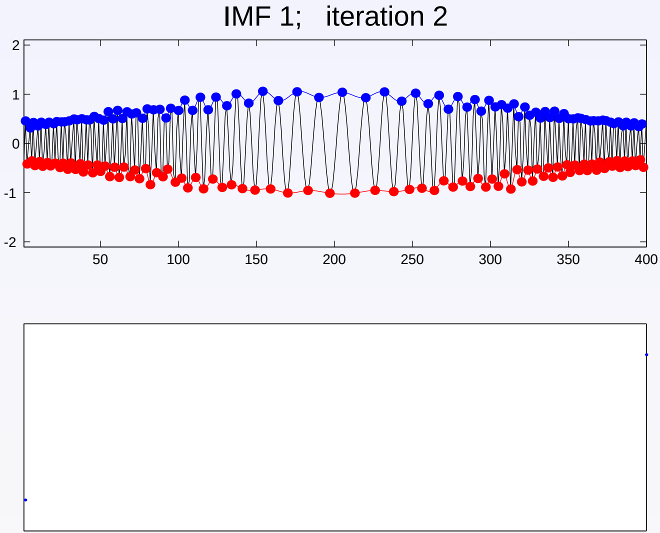



IMF 1; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400

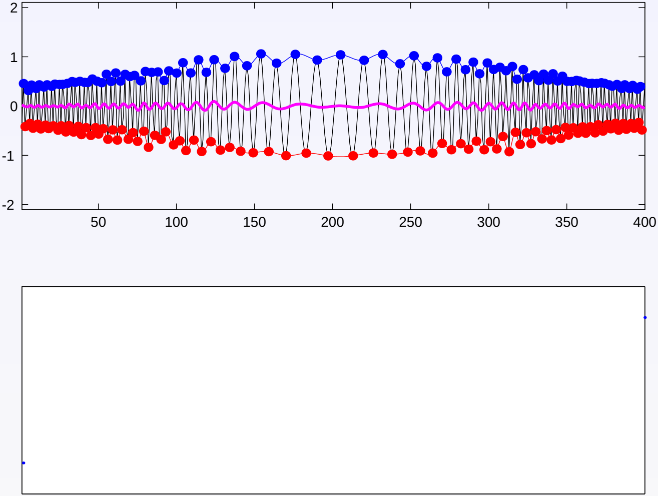
IMF 1; iteration 2



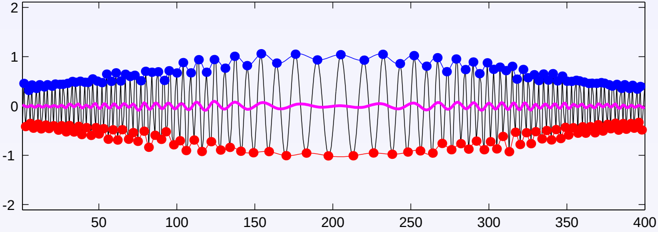



IMF 1; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400

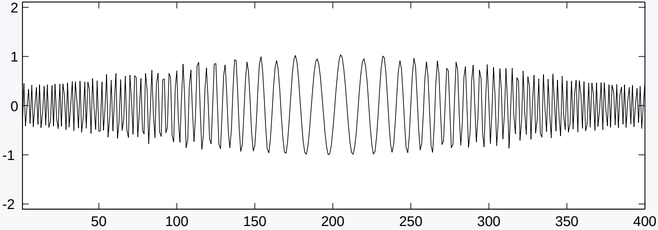
IMF 1; iteration 2

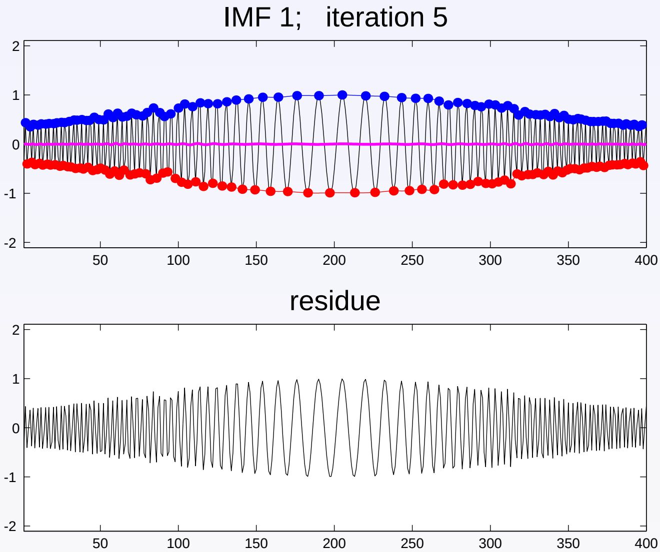


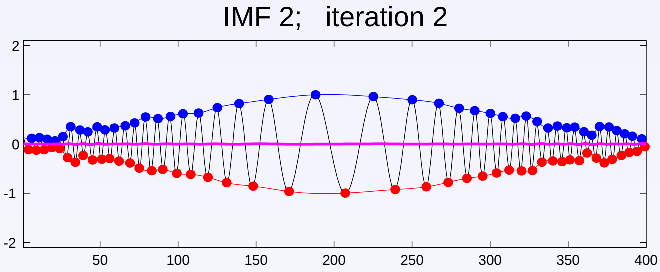
IMF 1; iteration 2



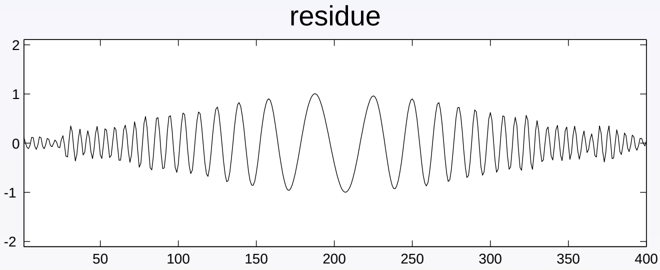
residue



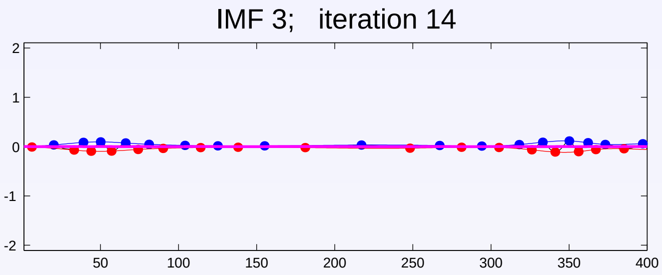

IMF 1; iteration 5
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

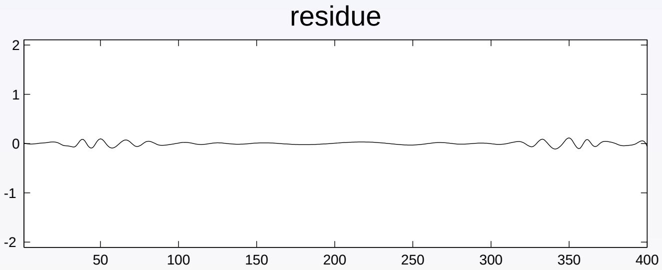


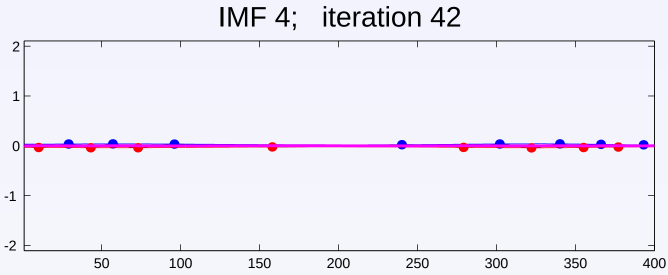
IMF 2; iteration 2
2
1
0
-1
-2
50
100
150
200
250
300
350
400

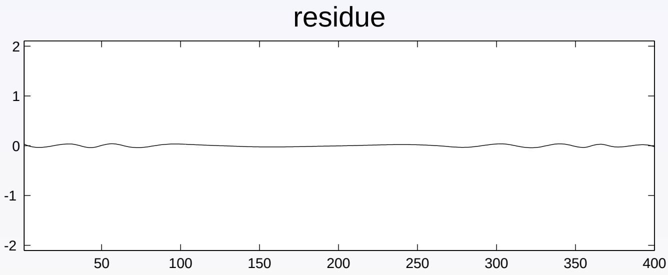


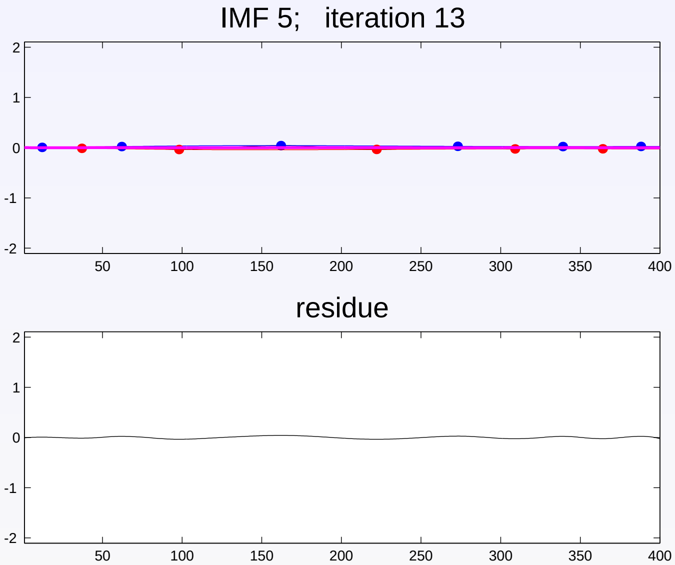
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400


IMF 3; iteration 14
2
1
0
-1
-2
50
100
150
200
250
300
350
400

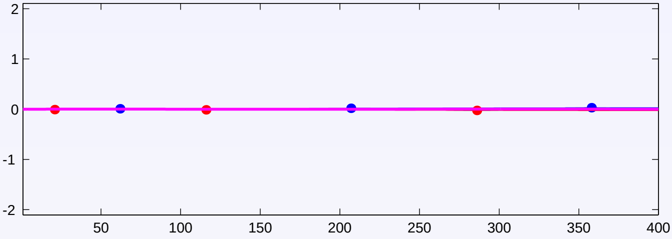

residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400


IMF 4; iteration 42
2
1
0
-1
-2
50
100
150
200
250
300
350
400

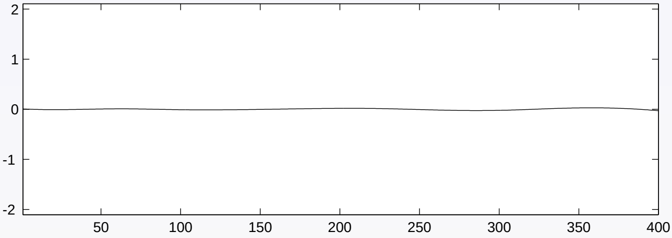

residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

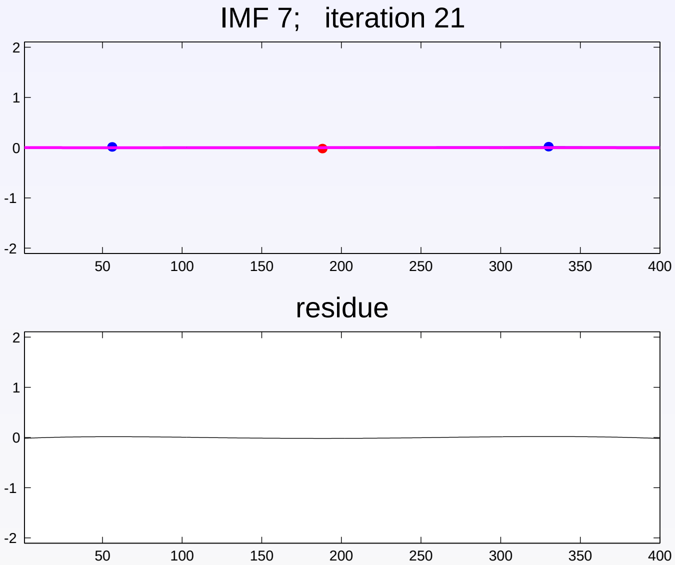

IMF 5; iteration 13
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

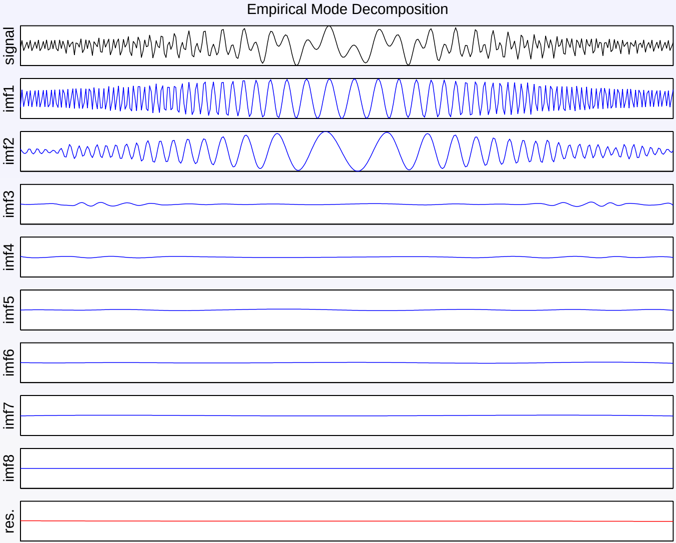
IMF 6; iteration 8

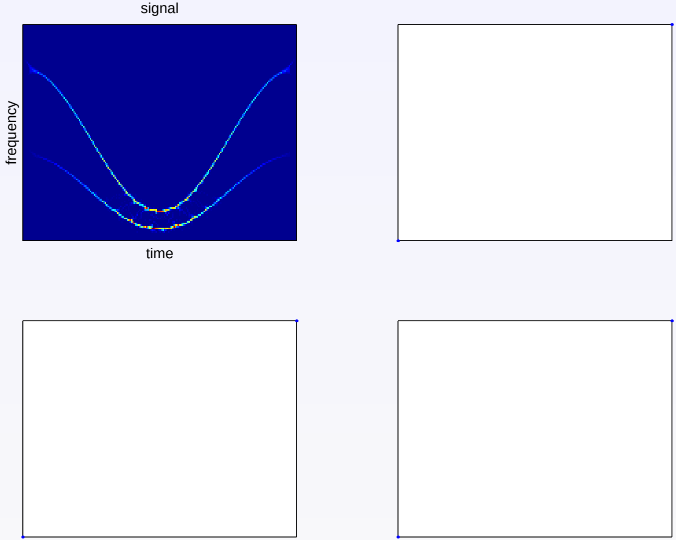


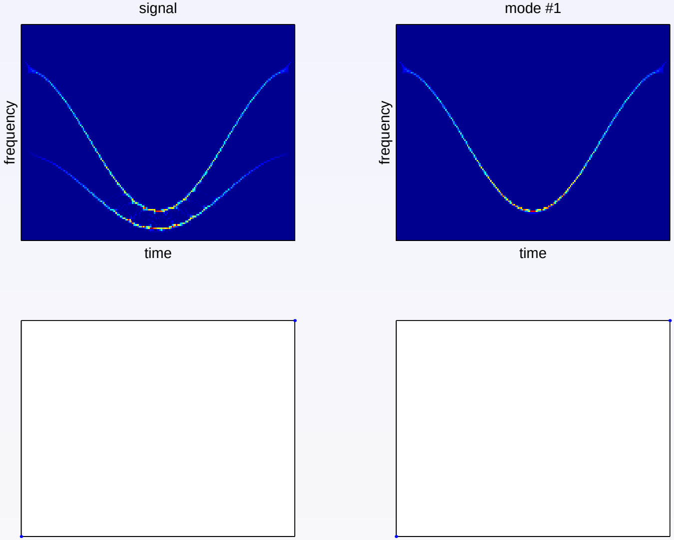
residue

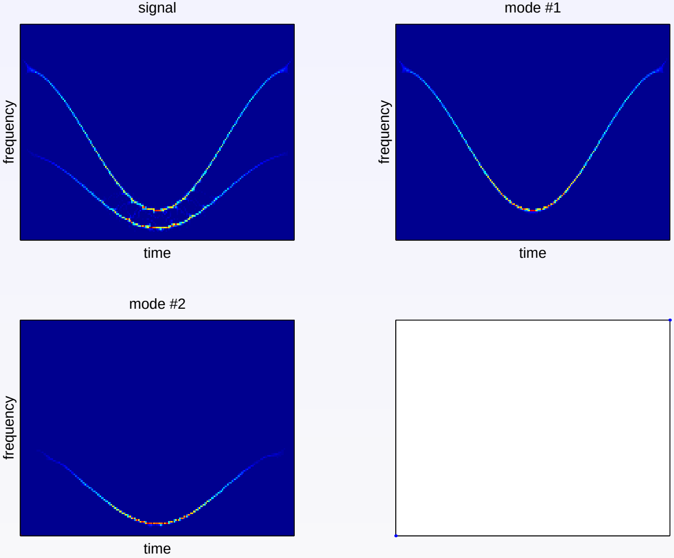


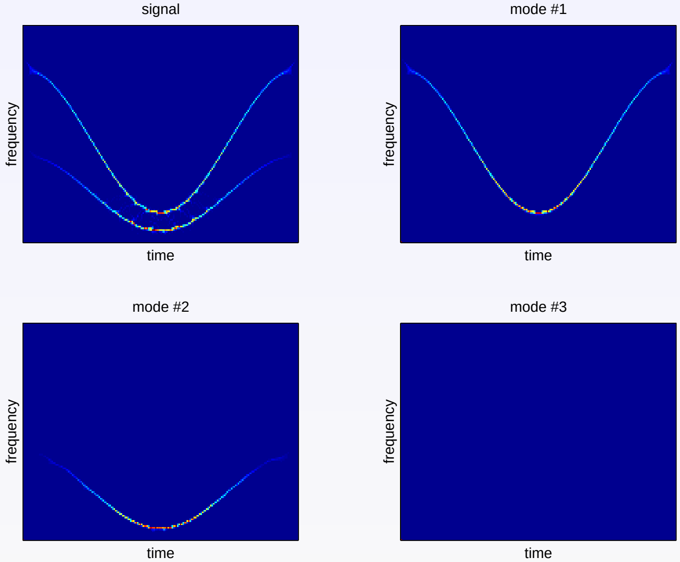

IMF 7; iteration 21
2
1
0
-1
-2
50
100
150
200
250
300
350
400
residue
2
1
0
-1
-2
50
100
150
200
250
300
350
400

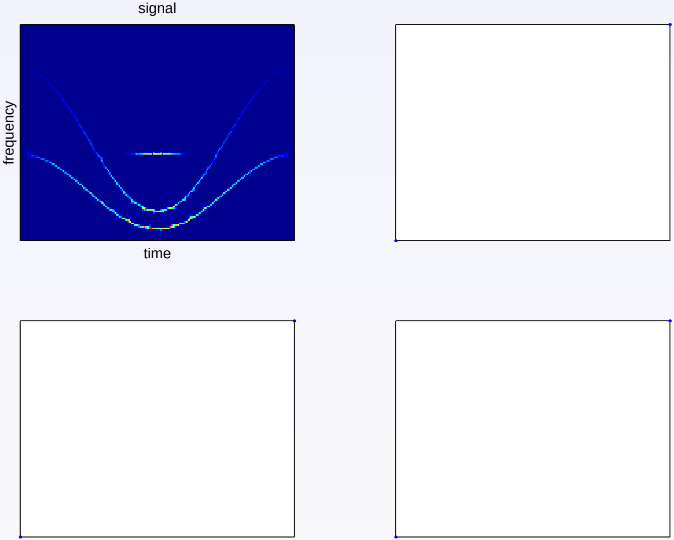

Empirical Mode Decomposition
signal
imf1
imf2
imf3
imf4
imf5
imf6
imf7
imf8
res.

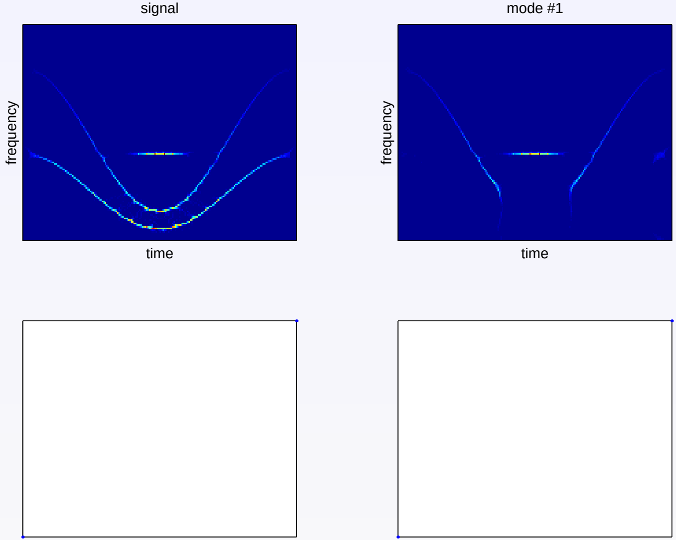

signal
frequency
time

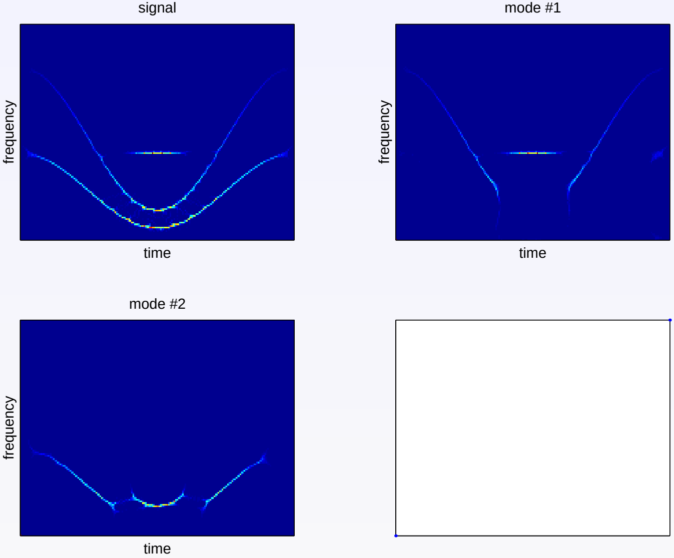

signal
frequency
time
mode #1
frequency
time

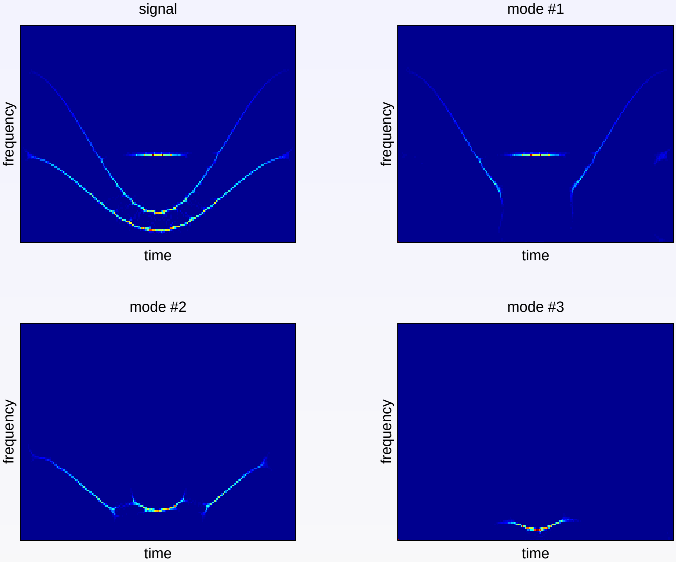

signal
frequency
time
mode #1
frequency
time
mode #2
frequency
time


signal
mode #1
mode #2
mode #3
frequency
time


signal
frequency
time


signal
frequency
time
mode #1
frequency
time


signal
frequency
time
mode #1
frequency
time
mode #2
frequency
time


signal
mode #1
mode #2
mode #3
frequency
time

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- **Locality** — The method operates at the scale of **one** oscillation.
- **Adaptativity** — The decomposition is fully **data-driven.**
- **Arbitrary oscillation** — No assumption on the harmonic structure of oscillations $\Rightarrow$ 1 **non linear** oscillation $=$ 1 mode.
- **Multiresolution** — The iterative process explores **sequentially** the “natural" constitutive scales of a signal.
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EMD vs. wavelets

Empirical Mode Decomposition (EMD)

signal = fast oscillation + slow oscillation & iteration

- separation “fast vs. slow” **data driven**
- **"local"** analysis based on extrema
- *theoretical framework ?*

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Discrete Wavelet Transform (DWT)

signal = approximation + detail
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1. *similarity* : both achieve a decomposition into "**fluctuations**" and "**trend**"

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\begin{aligned}
x(t) &= \sum_k c_k(t) + r_K(t) && \text{(EMD)} \\
&= \sum_k d_k(t) + a_K(t) && \text{(DWT)} \\
\text{with } d_k(t) &= \sum_n \langle x, \psi_{kn} \rangle \, \psi_{kn}(t) \\
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\end{aligned}
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2. *difference* : scales are **pre-determined** for DWT ($\{\varphi, \psi\}_{kn}(t) = 2^{-k/2}\{\varphi, \psi\}(2^{-k}t - n)$) and **adaptive** (data-driven) for EMD

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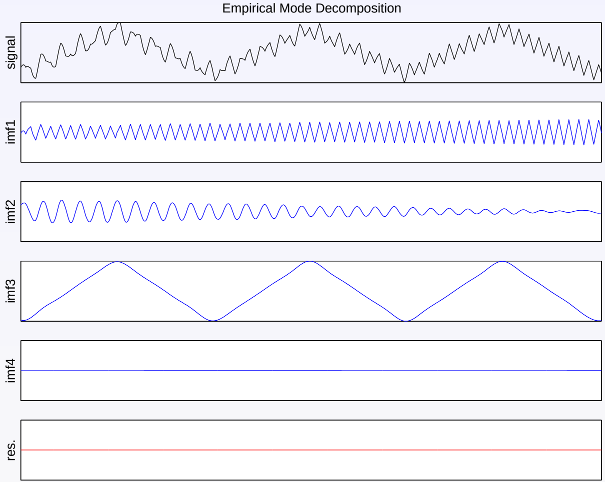
non linear oscillations



multiresolution

- **Stochastic frequency approach** — Decomposition and spectrum analysis, mode by mode, of a wideband noise.
- **Model** — Fractional Gaussian noise (fGn), with spectrum density $S(f) \sim |f|^{1-2H}$, with $0 < H < 1$ (Hurst exponent).

Result

"Spontaneous" emergence of a quasi-dyadic, self-similar, filterbank structure (F., Gonçalvès & Rilling, '03) :

$$S_{k',H}(f) = \rho_H^{\alpha(k'-k)} S_{k,H}(\rho_H^{k'-k} f)$$

for any $k' > k \geq 2$, with $\alpha = 2H - 1$ and $\rho_H \approx 2$.

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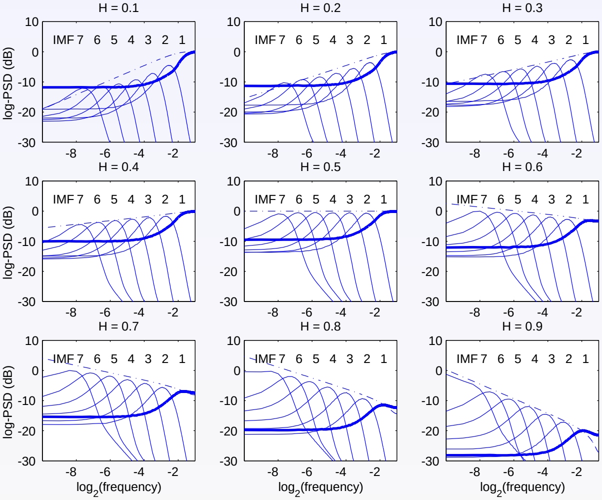
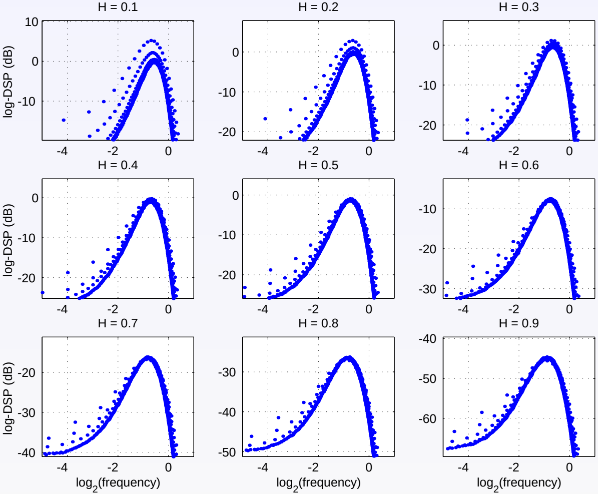
IMF spectra of fGn



renormalized IMF spectra of fGn

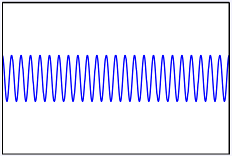


one or two components ?

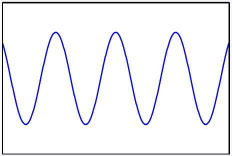
« $\cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1+\omega_2}{2} t\right)\cos\left(\frac{\omega_1-\omega_2}{2} t\right)$ »

mathematics vs. physics

one or two components ?



+



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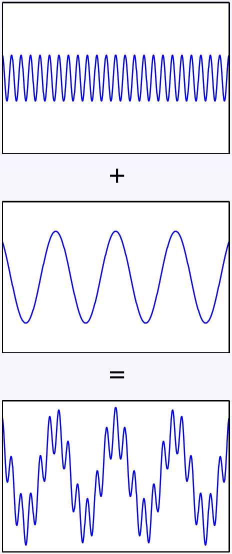

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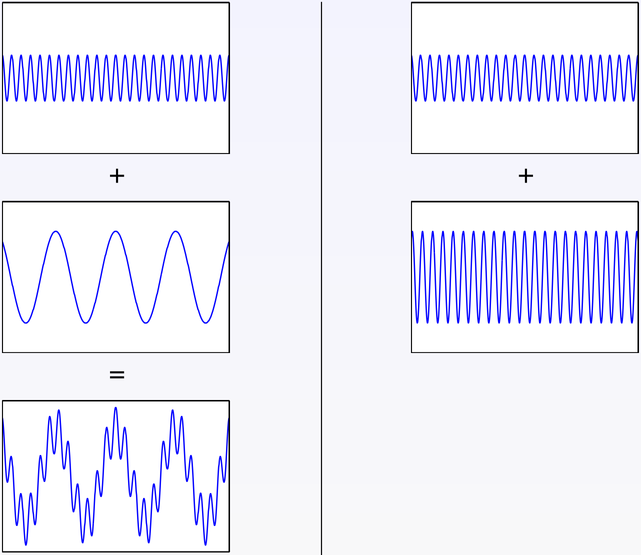

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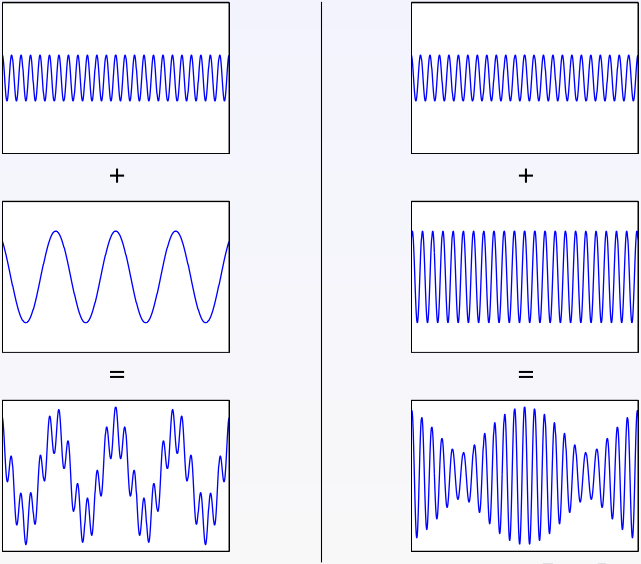

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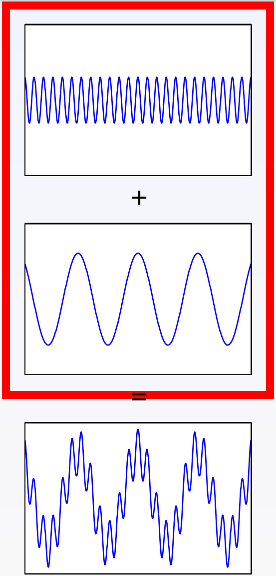

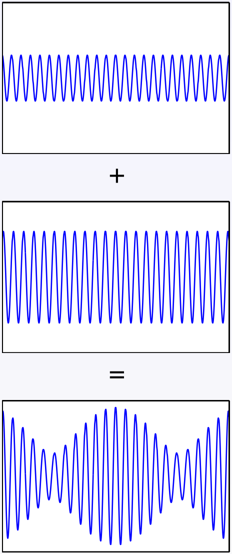

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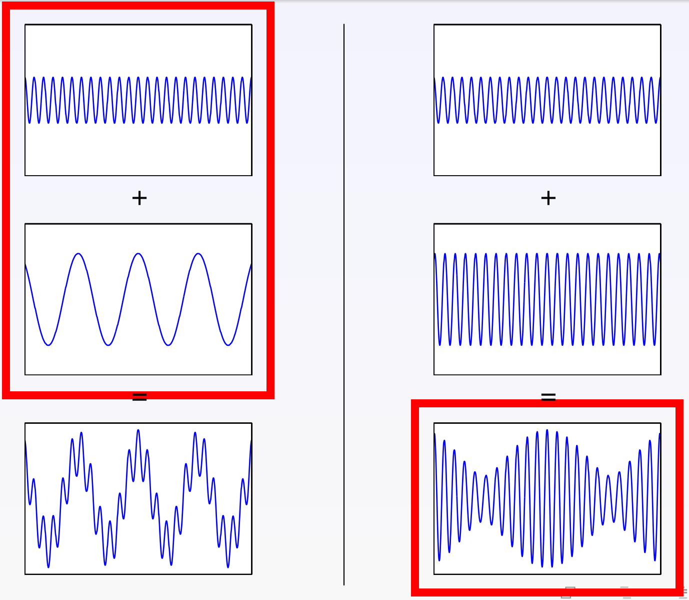

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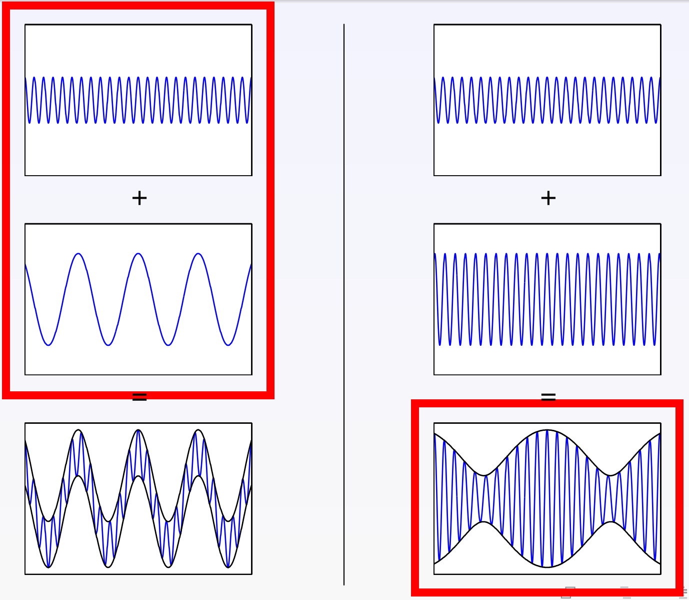

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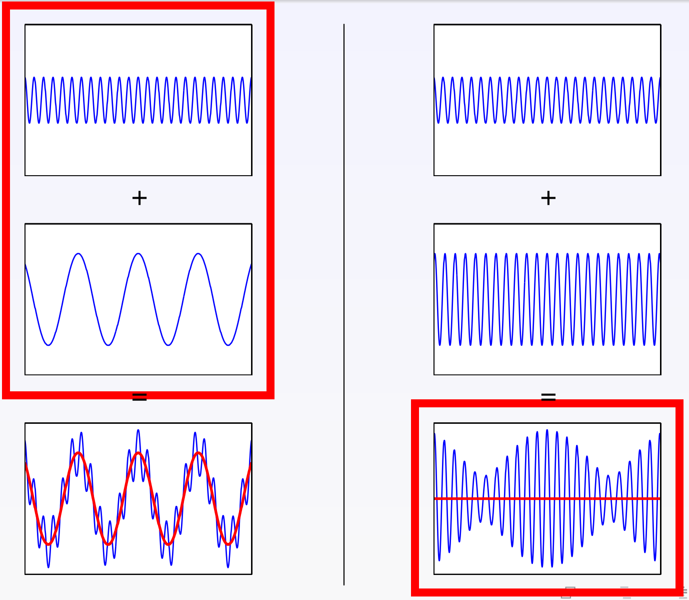

simulations

Signal

$$x(t) = \underbrace{a_1 \cos(2\pi f_1 t)}_{x_1(t)} + \underbrace{a_2 \cos(2\pi f_2 t + \varphi)}_{x_2(t)}, \quad f_1 > f_2$$

Analysis of its EMD

- only the **first IMF** is computed : if separation, it should be equal to the highest frequency component $x_1(t)$
- **criterion** (= 0 if separation) :

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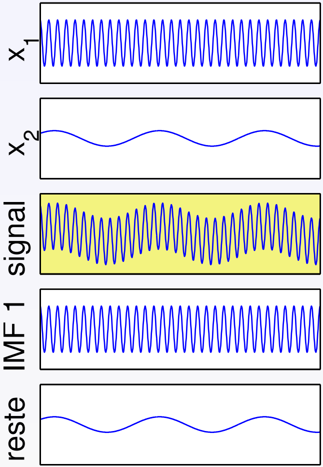
Sum of two tones

f_2/f_1 = 0.08, a_2/a_1 = 0.33



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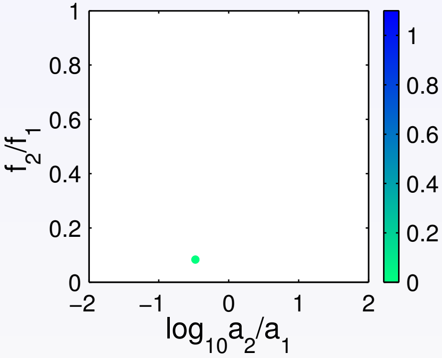

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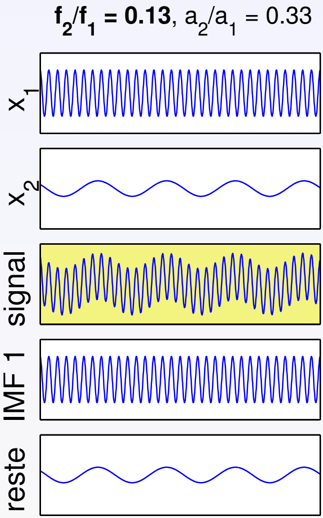

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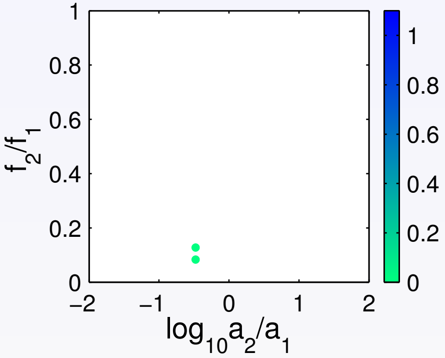

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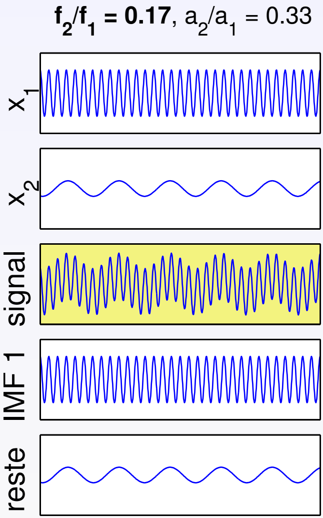

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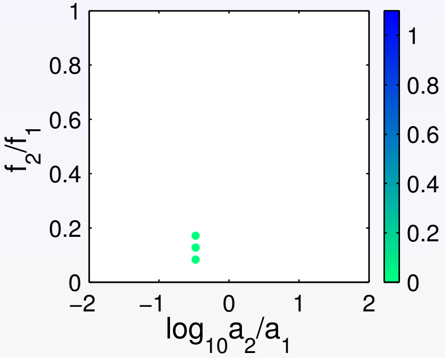

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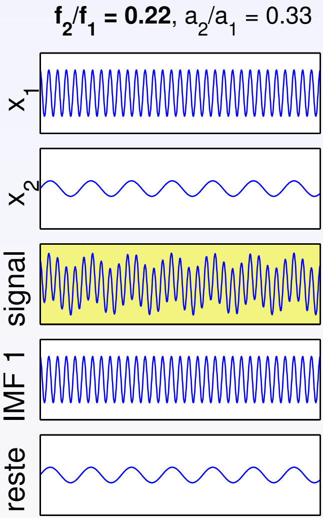

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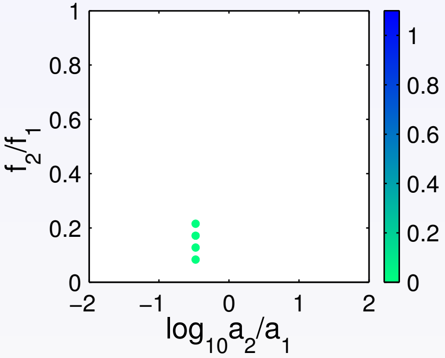
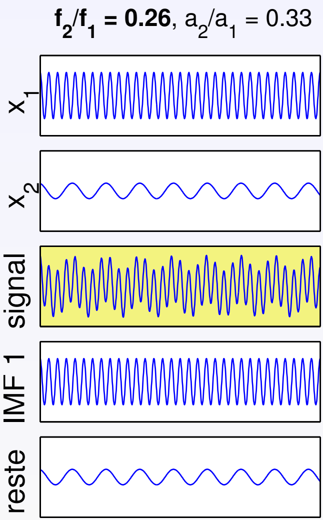
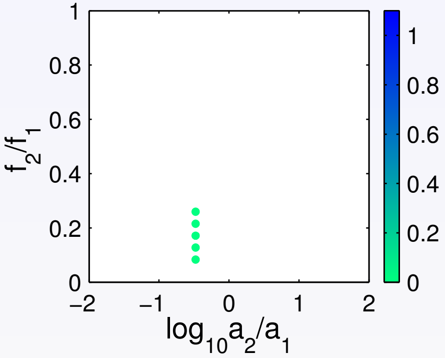


Sum of two tones

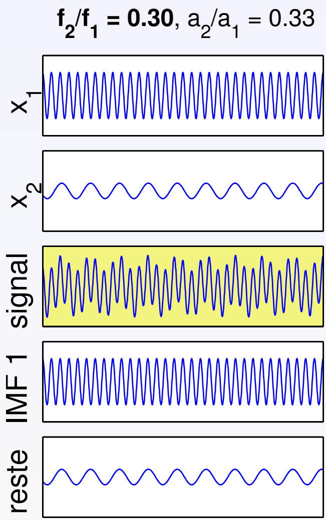


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

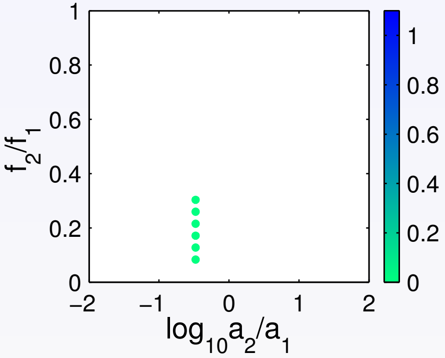


Sum of two tones

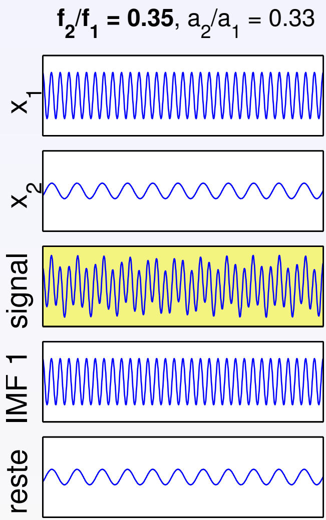


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

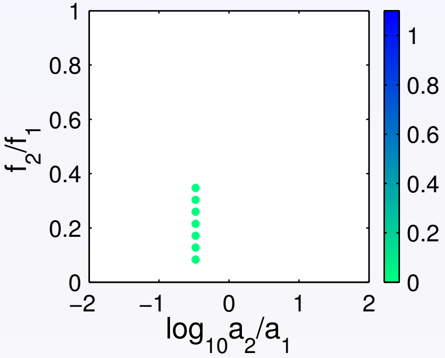


Sum of two tones



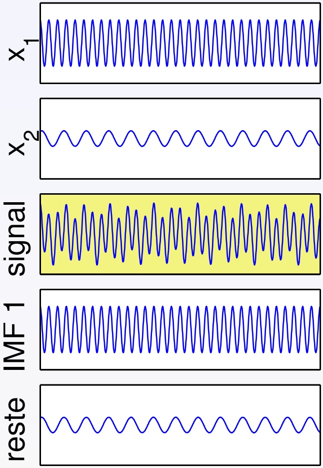
$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}$$

$= 0$ if separation



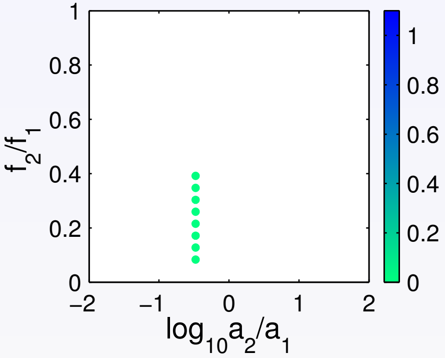
Sum of two tones

f_2/f_1 = 0.39, a_2/a_1 = 0.33



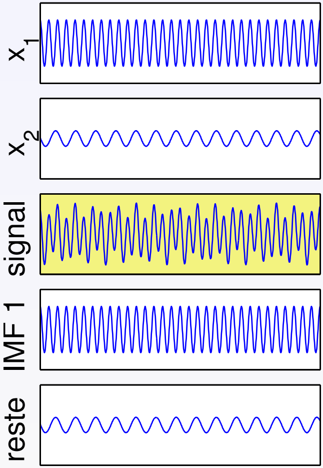
$$c\left(\frac{a_2}{a_1},\frac{f_2}{f_1},\varphi\right)=\frac{\|IMF_1(t)-x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$=0 \quad \text{if separation}$$



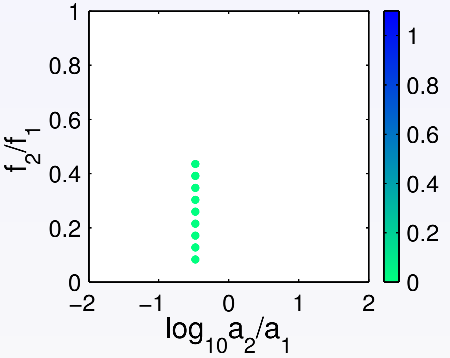
Sum of two tones

$\mathbf{f_2/f_1 = 0.44}$, $a_2/a_1 = 0.33$

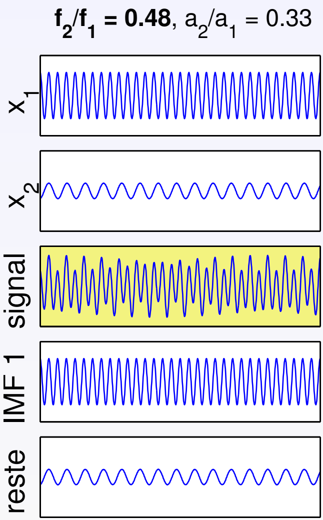


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$

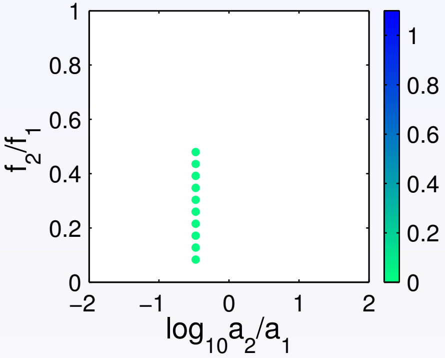


Sum of two tones

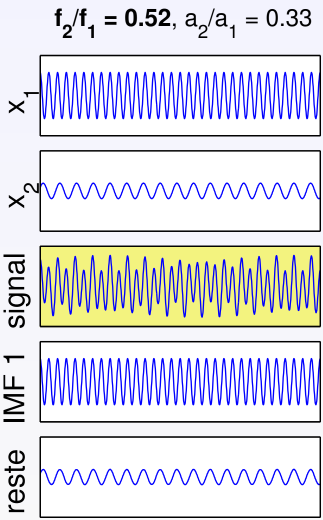


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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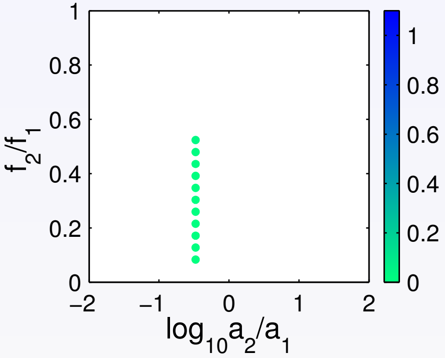
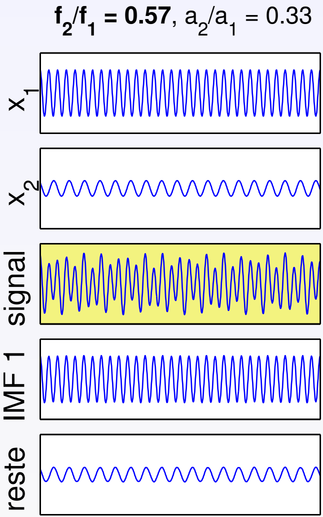


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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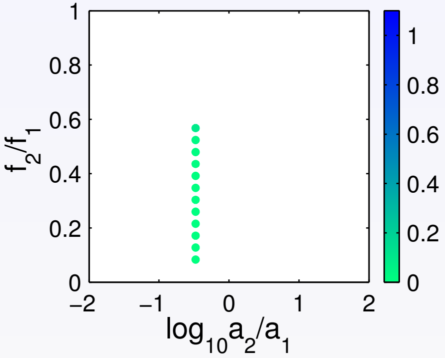
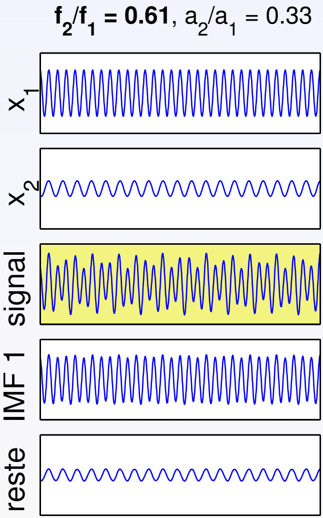


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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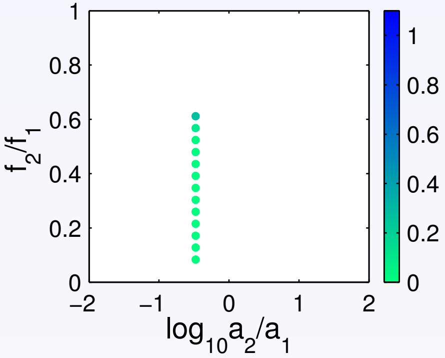
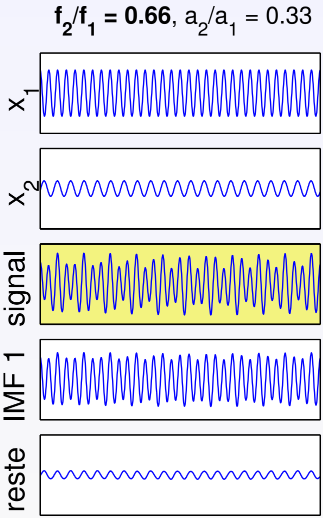


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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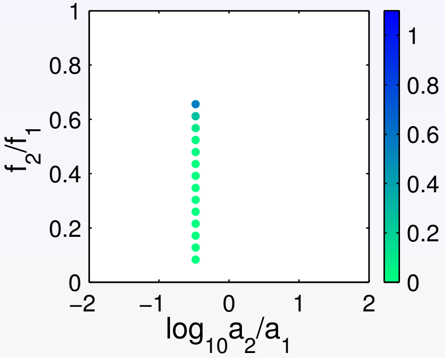
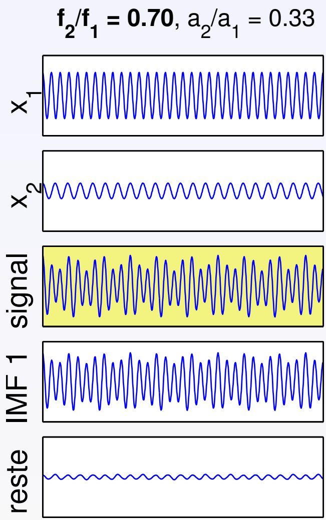
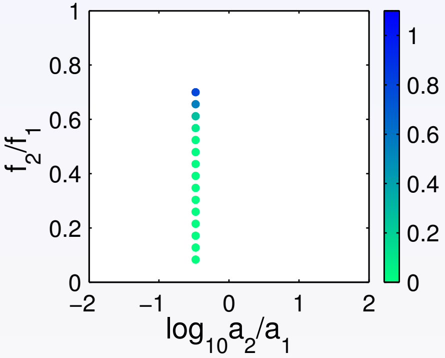


Sum of two tones

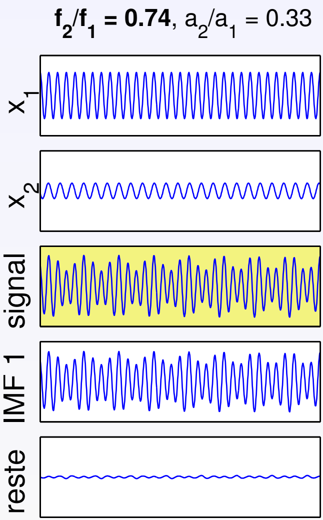


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation



Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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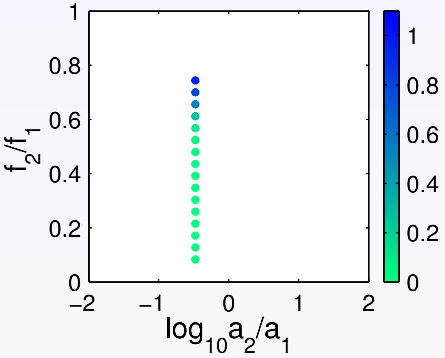
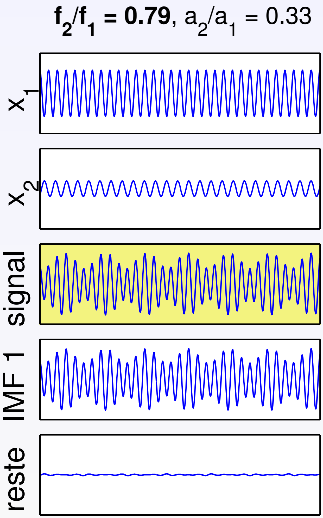


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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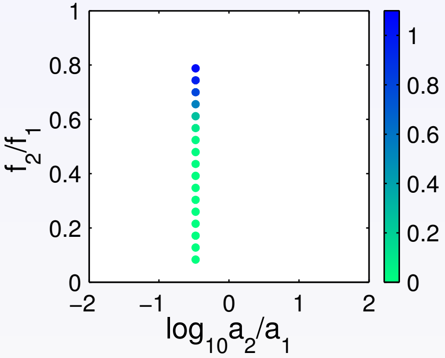
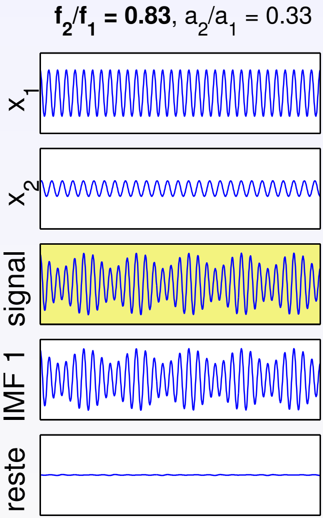


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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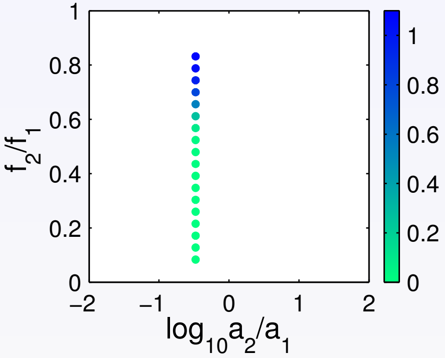
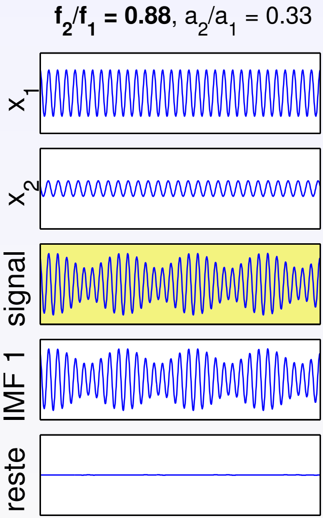


Sum of two tones



$$c\left(\frac{a_2}{a_1},\frac{f_2}{f_1},\varphi\right)=\frac{\|IMF_1(t)-x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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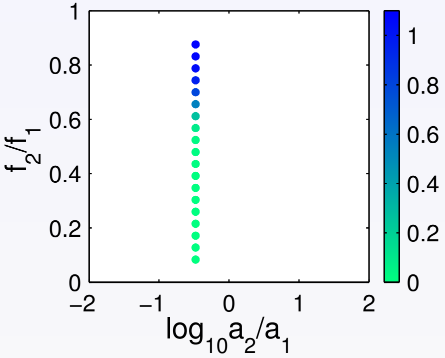
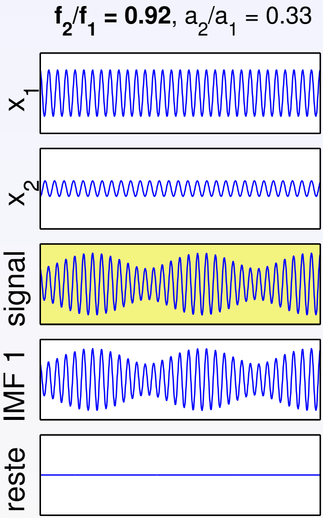


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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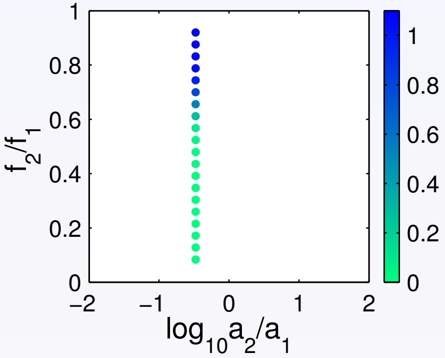

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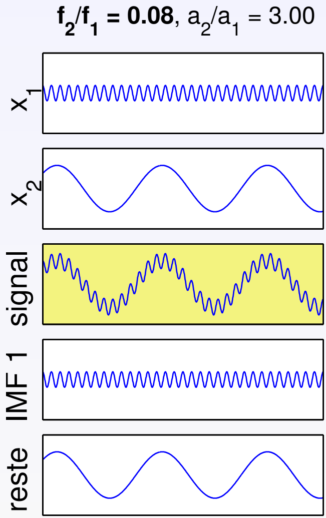

$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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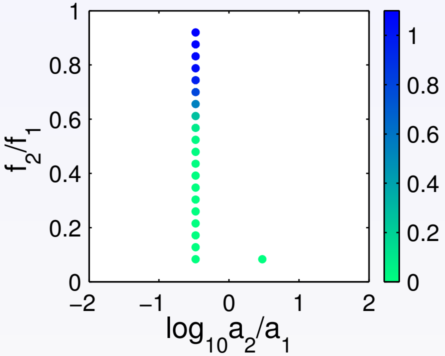
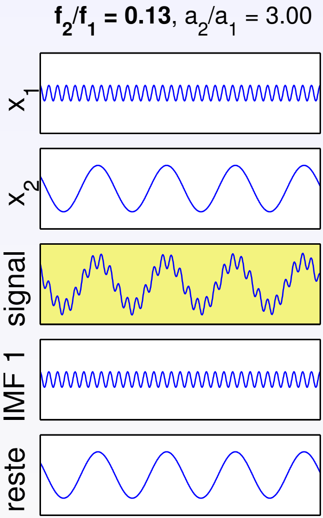


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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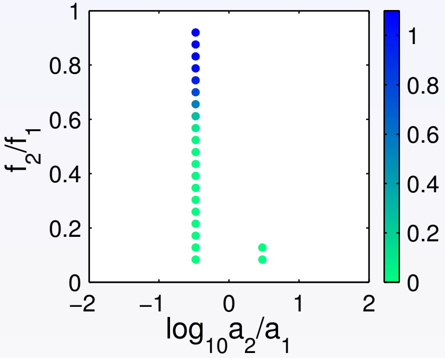
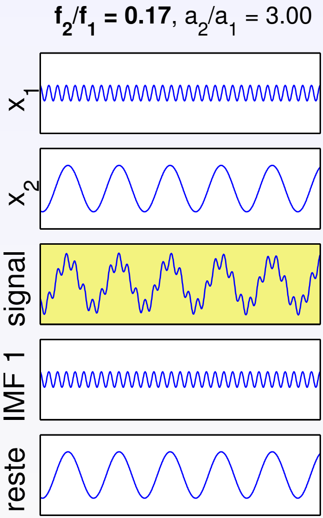


Sum of two tones



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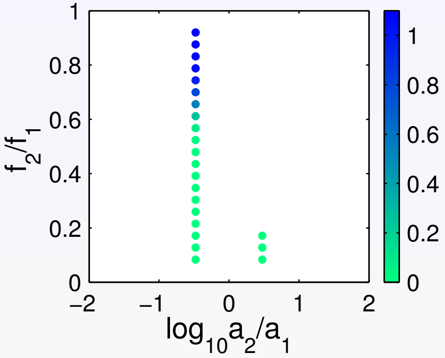
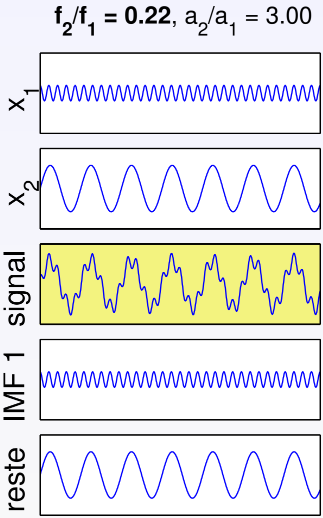


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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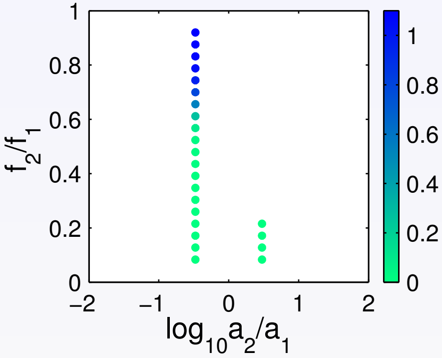
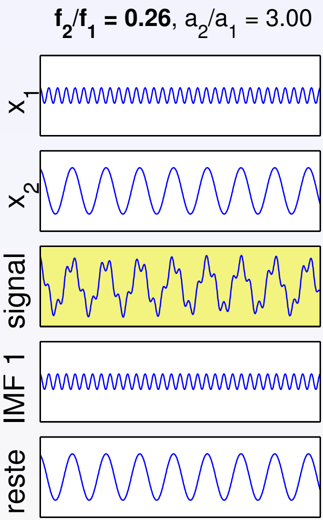


Sum of two tones



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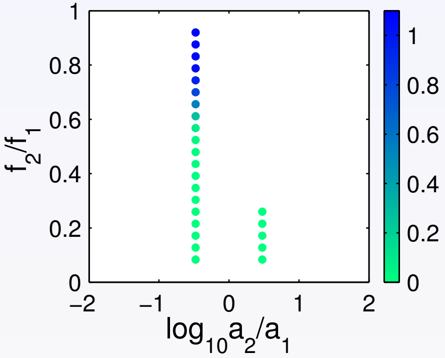
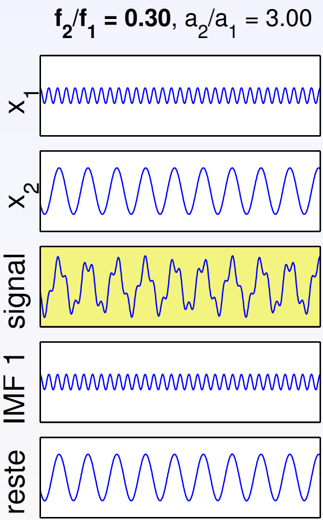


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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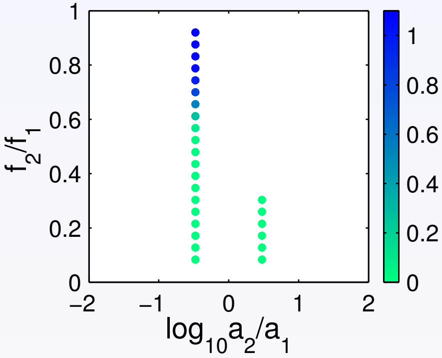
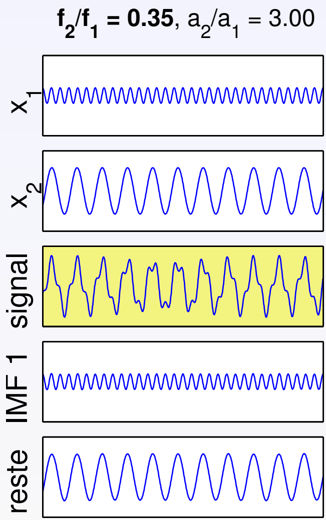


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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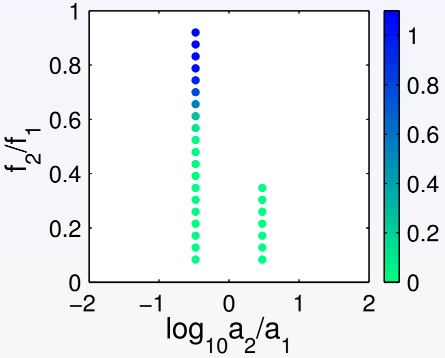
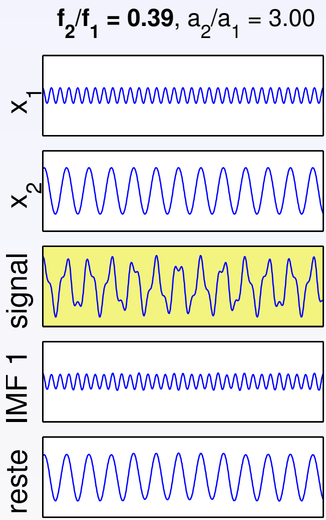


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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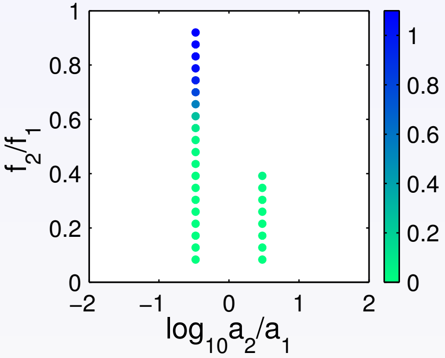
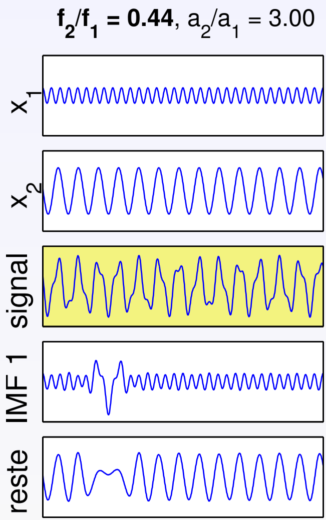


Sum of two tones



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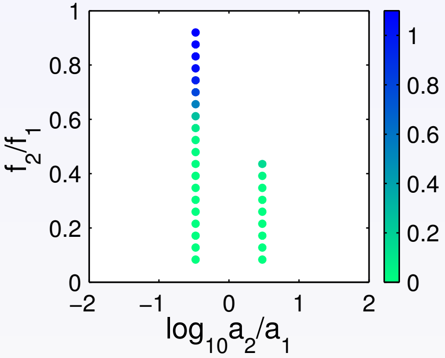
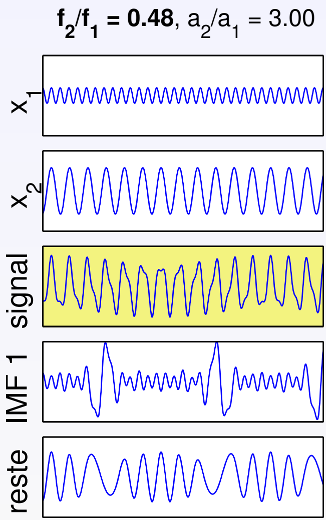


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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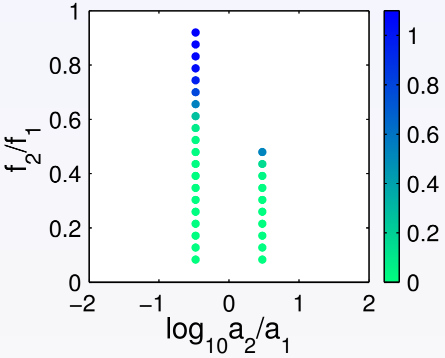
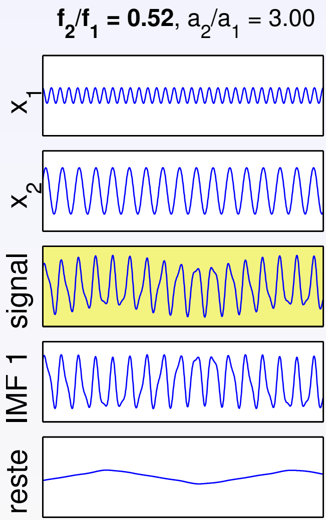


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

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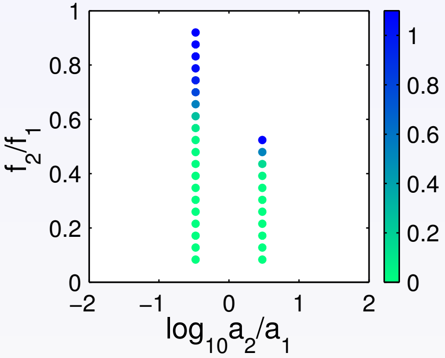
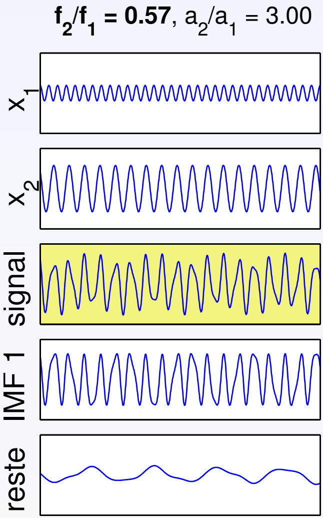
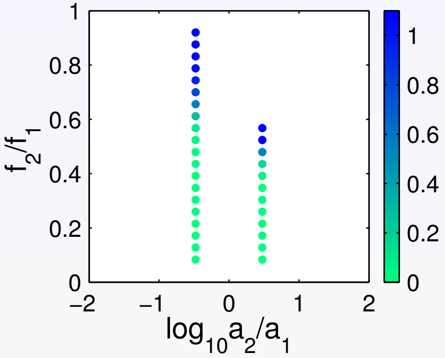


Sum of two tones

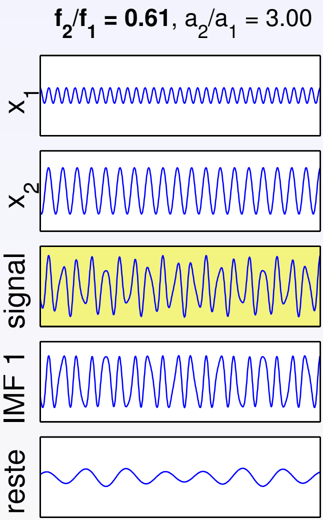


$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



Sum of two tones



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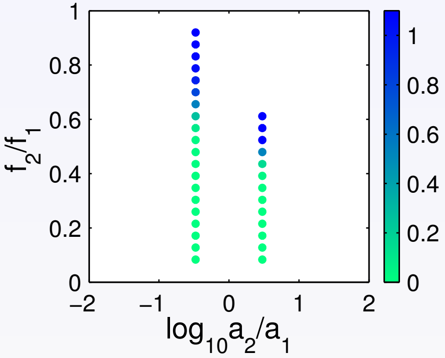
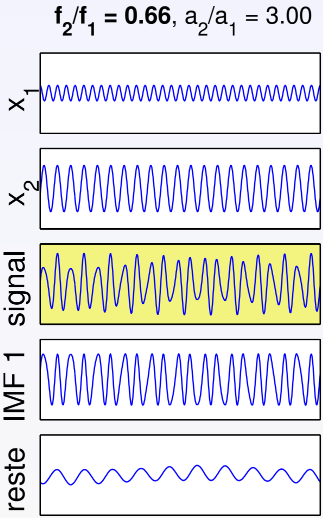


Sum of two tones



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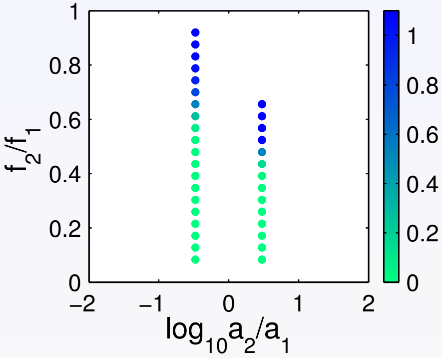
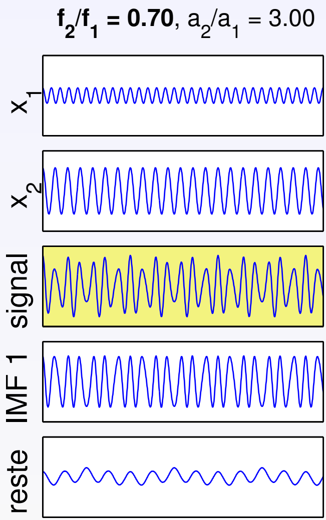


Sum of two tones



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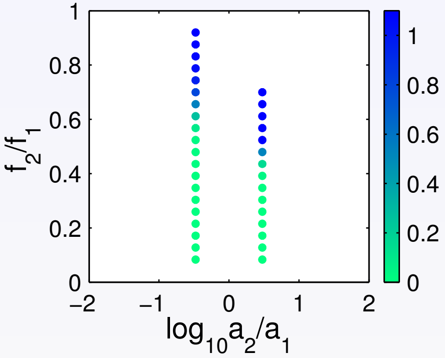
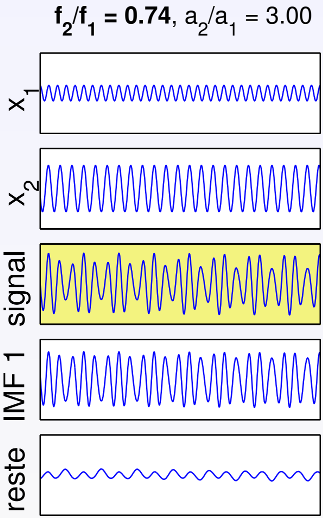
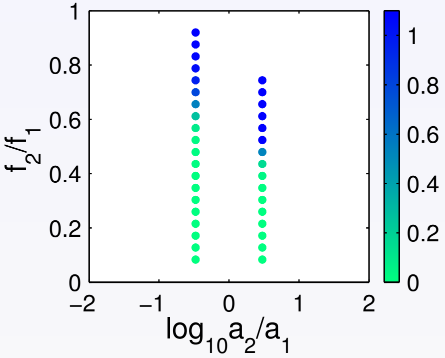


Sum of two tones



$$c\left(\frac{a_2}{a_1},\frac{f_2}{f_1},\varphi\right)=\frac{\|IMF_1(t)-x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$= 0$ if separation



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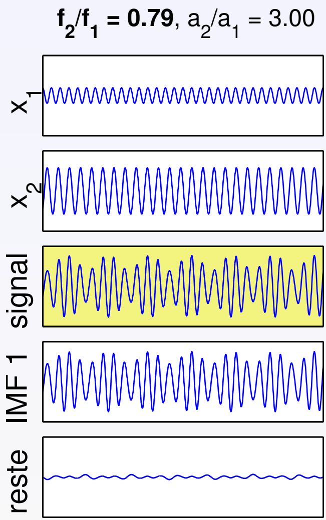

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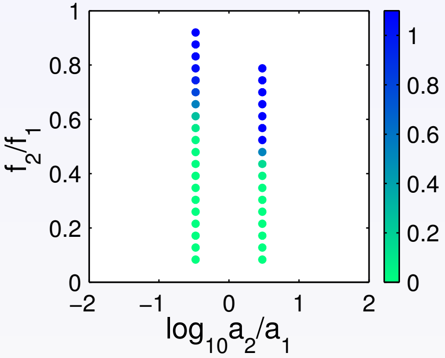
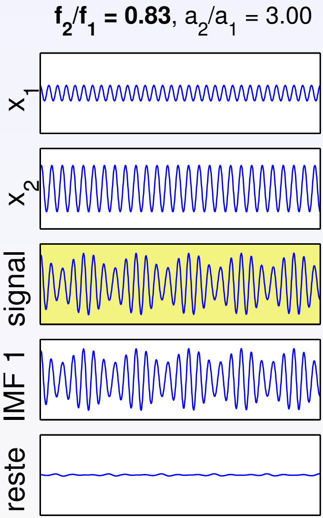


Sum of two tones



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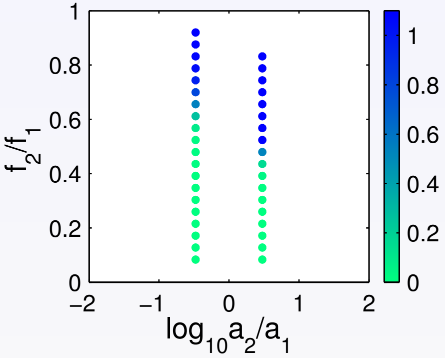

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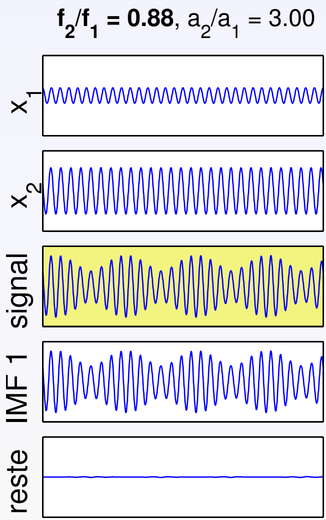

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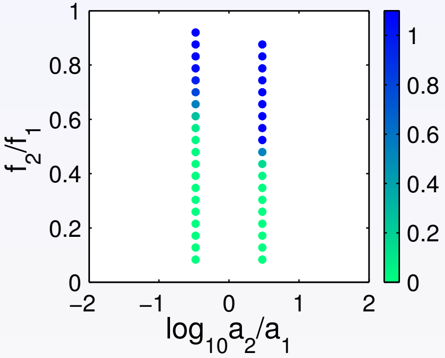
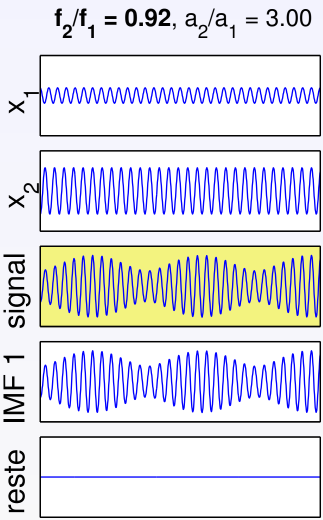


Sum of two tones



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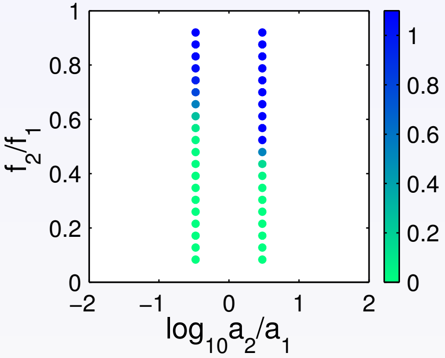
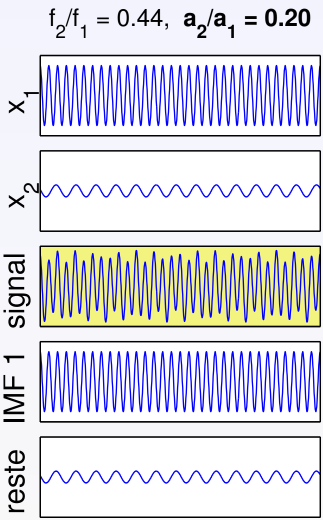


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\| IMF_1(t) - x_1(t) \|_{\ell_2}}{\| x_2(t) \|_{\ell_2}}$$

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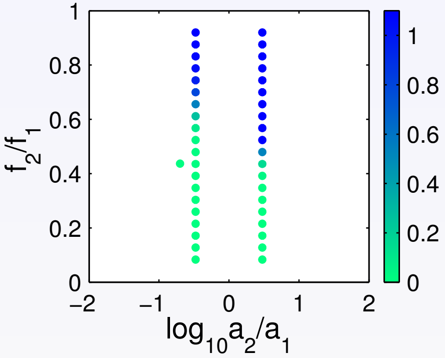

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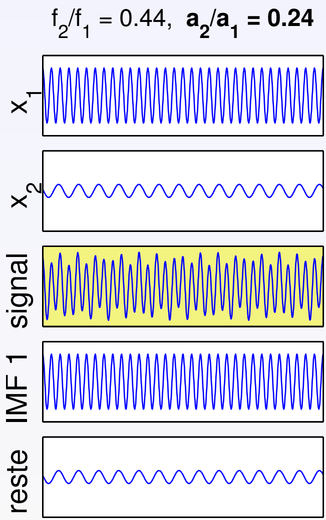

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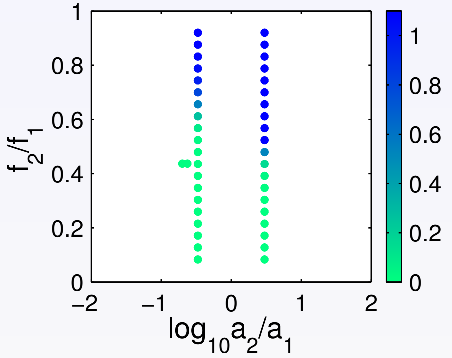

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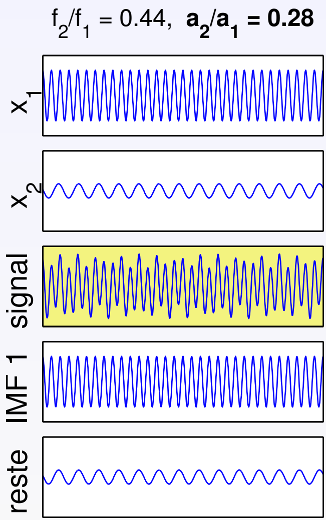

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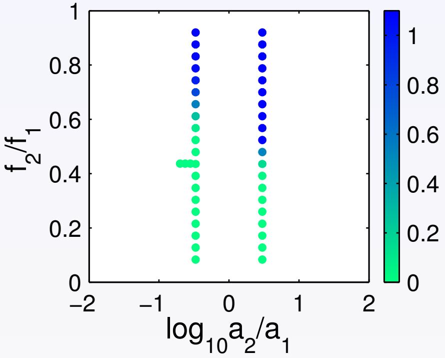

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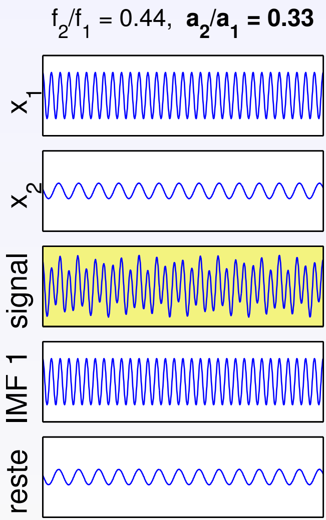

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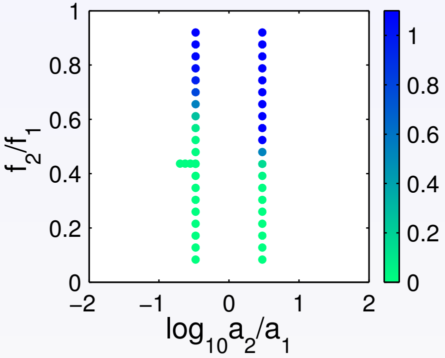
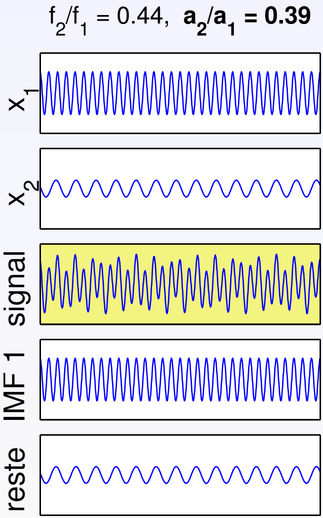


Sum of two tones



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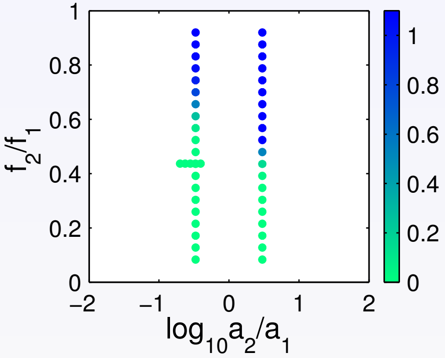
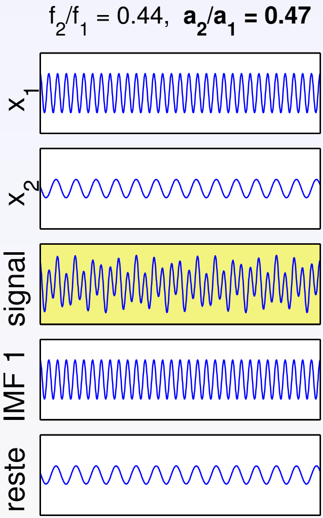


Sum of two tones



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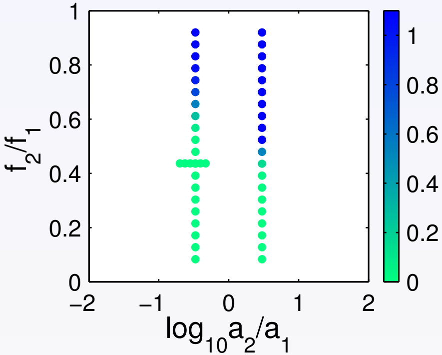
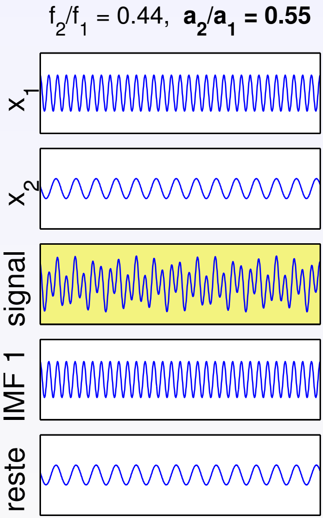


Sum of two tones



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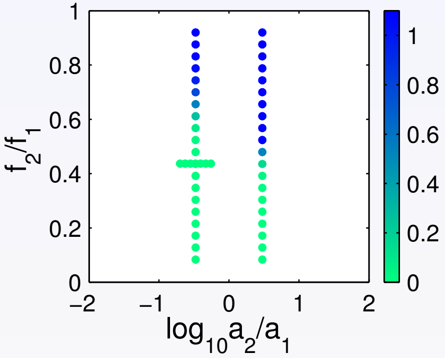
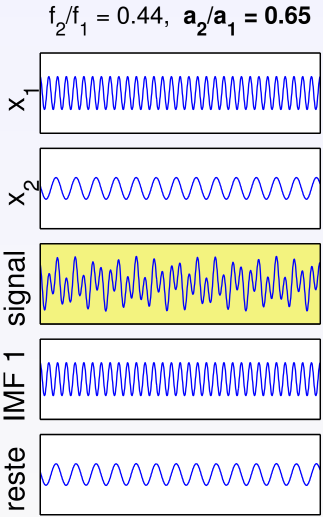


Sum of two tones



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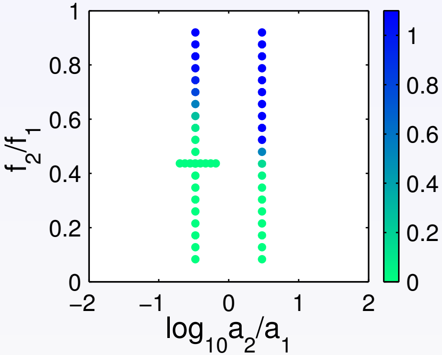
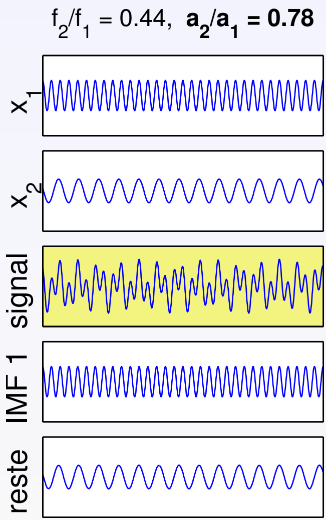


Sum of two tones



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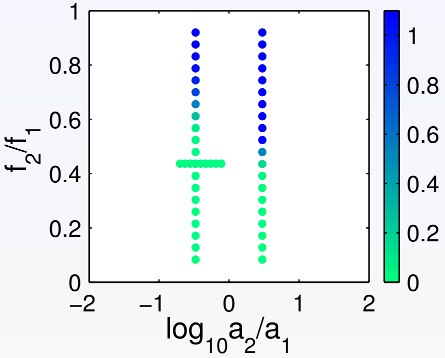

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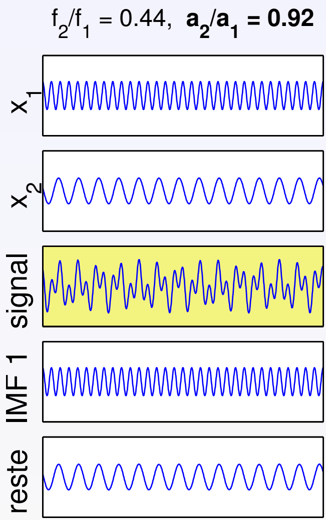

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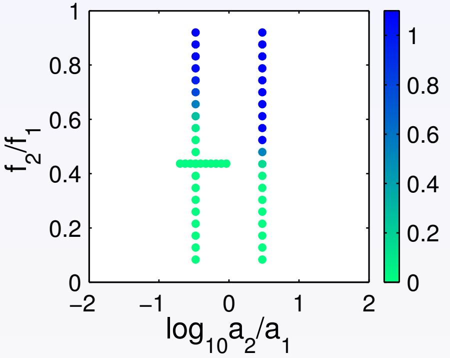

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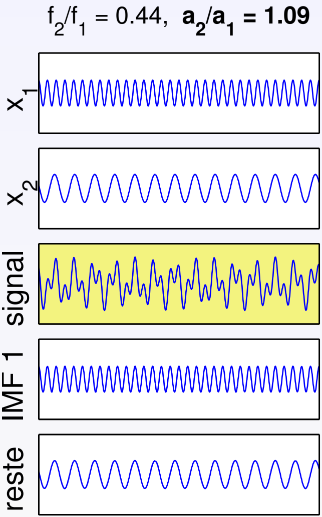

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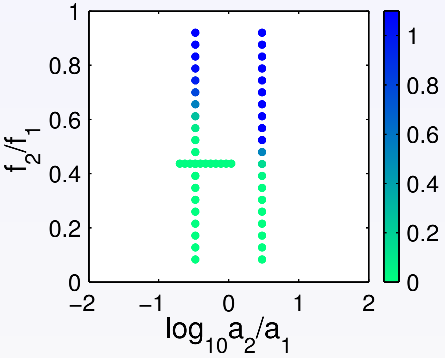

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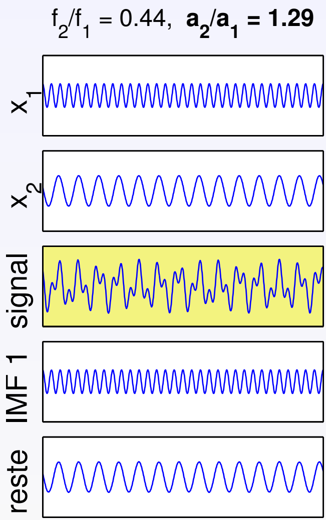

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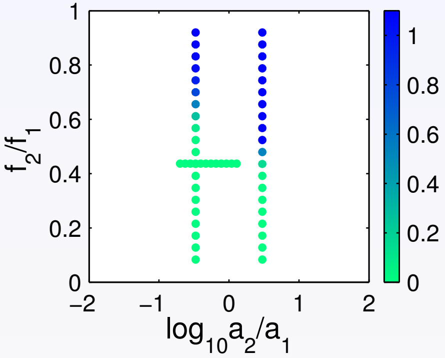
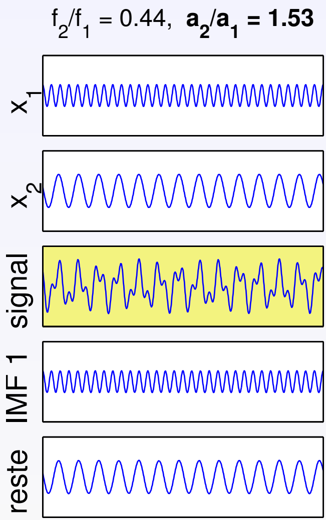


Sum of two tones



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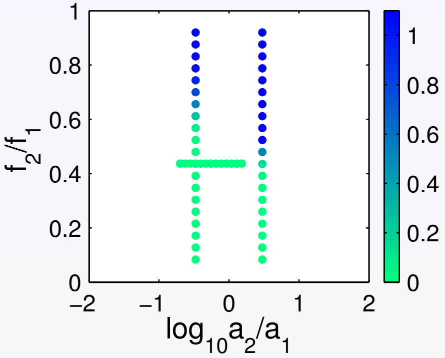

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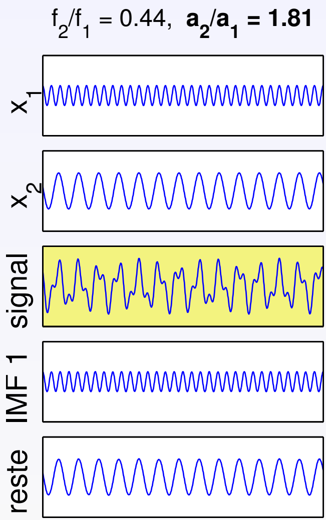

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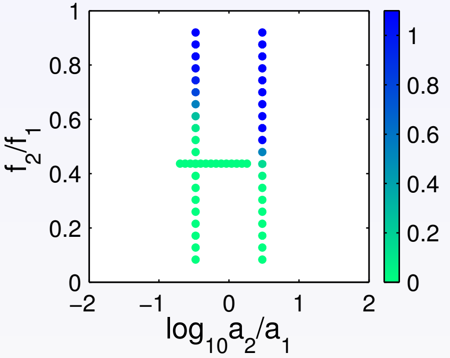
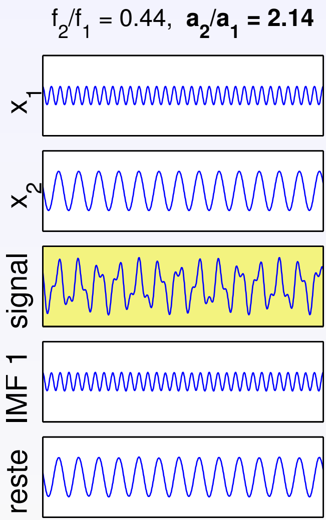


Sum of two tones



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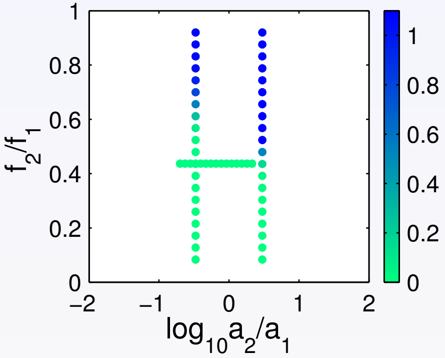

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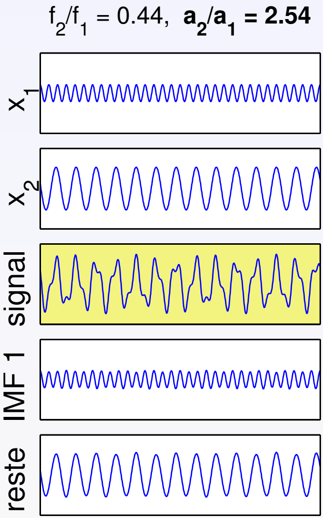

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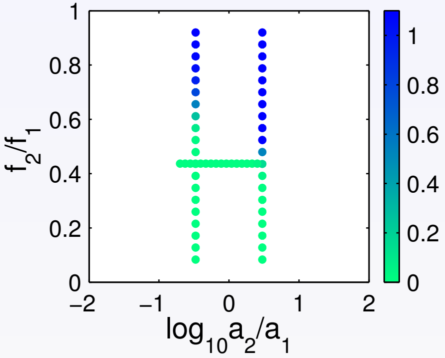

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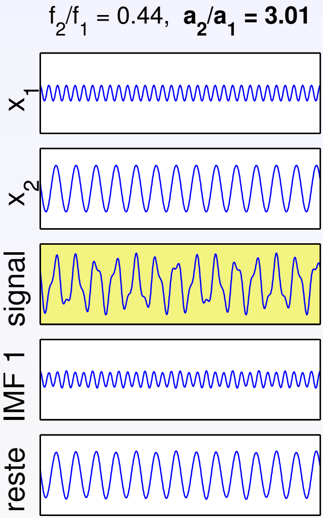

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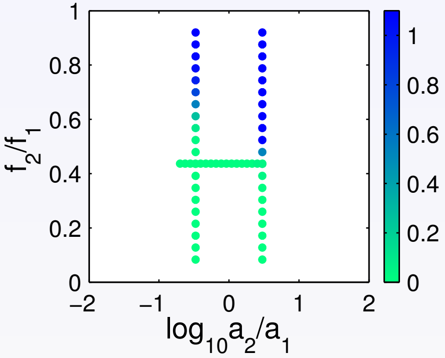

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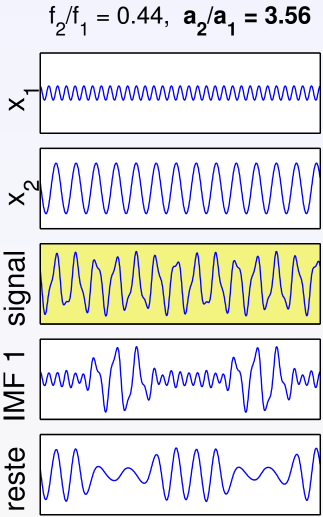

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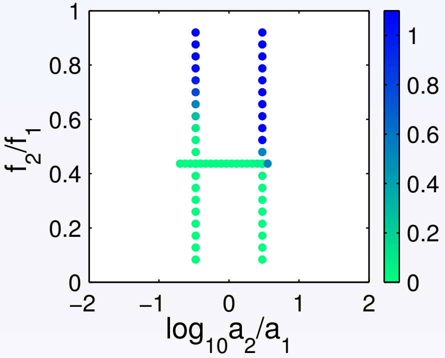
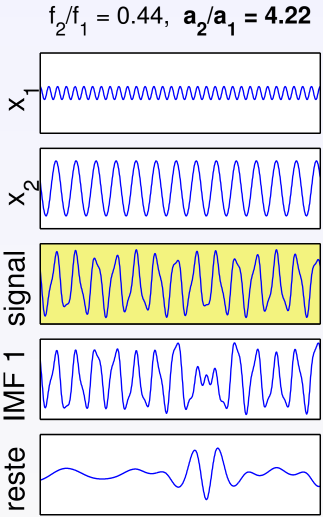
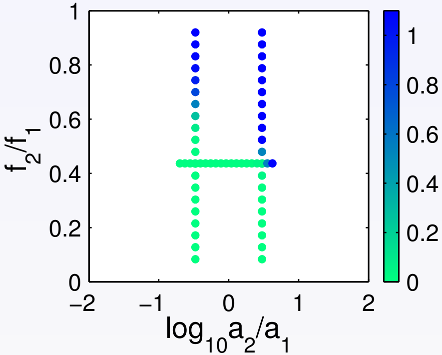


Sum of two tones



$$c\left(\frac{a_2}{a_1}, \frac{f_2}{f_1}, \varphi\right) = \frac{\|IMF_1(t) - x_1(t)\|_{\ell_2}}{\|x_2(t)\|_{\ell_2}}$$

$$= 0 \quad \text{if separation}$$



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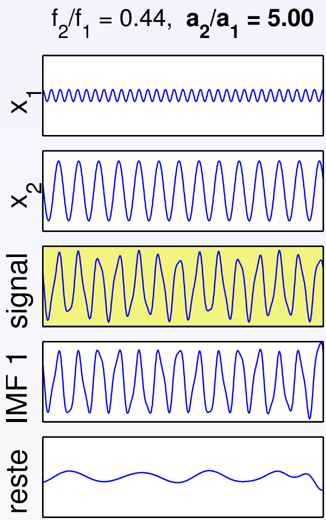

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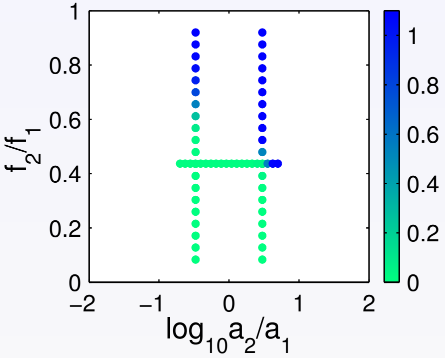

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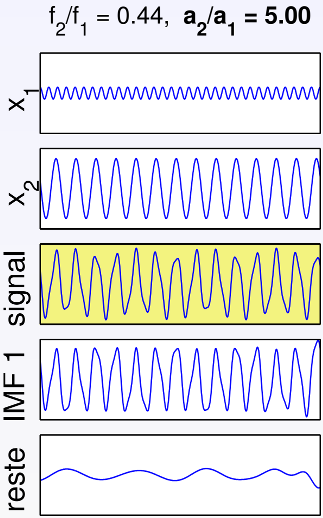

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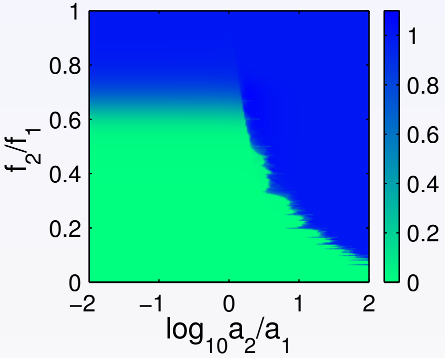

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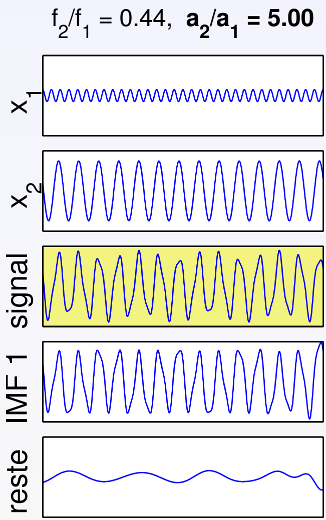

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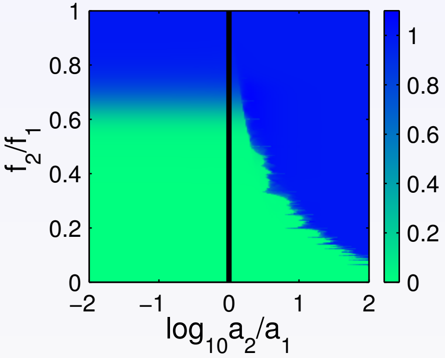

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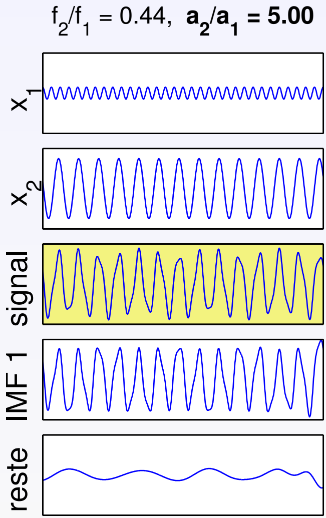

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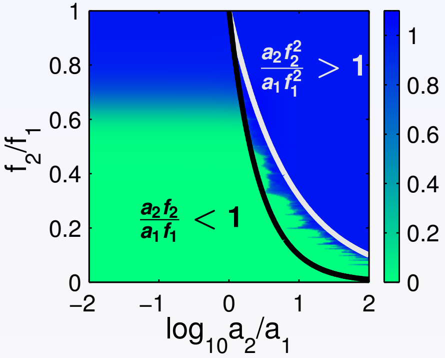

variations on the theme

EMD is based on a general principle that can be extended and connected with other approaches

- Ensemble EMD
- bivariate EMD
- 2D EMD
- wavelets
- synchrosqueezing
- etc.

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Ensemble EMD - principle

Idea

Reduce "mode mixing" by averaging noisy EMDs

In practice (Wu & Huang, '09) :

1. add some controlled noise to data
2. compute EMD
3. reiterate and average

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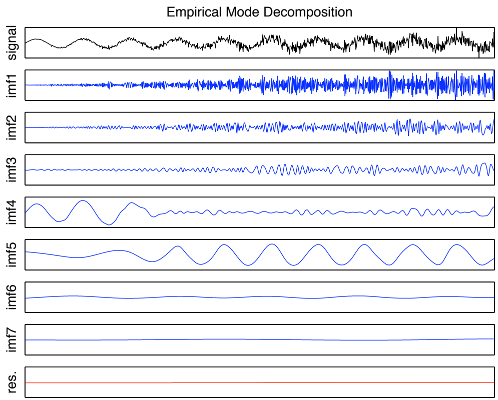

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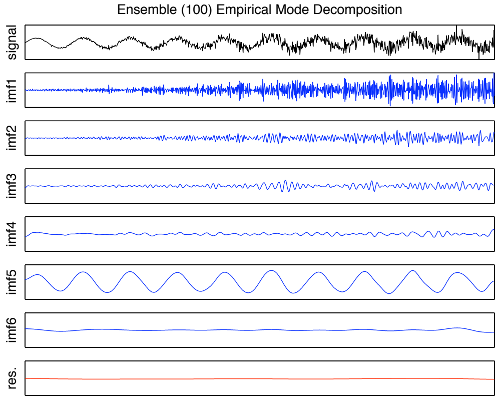


improved EEMD

Problem

1. *Different noise realizations may end up with different numbers of IMFs ⇒ averaging ?*
2. *Noise residual ⇒ approximate reconstruction*

Idea

Add noise "mode by mode" ⇒ "Complete EEMD with Adaptive Noise" (Torres et al., ICASSP'11)

1. meaningful averaging
2. reduced total number of IMFs as compared to EEMD
3. perfect reconstruction

CEEMDAN algorithm 1.

pre-processing step, given a signal $x[n]$

- **generate** J realizations of white Gaussian noise $w^j[n] \in \mathcal{N}(0, 1)$
- **define** $E_k(\cdot)$ as the operator which, given a signal, produces the $k-$th IMF
- **pre-compute** and **store** the $J \times K$ IMFs $E_k(w^j[n])$ for $j = 1, \ldots, J$ and $k = 1, \ldots, K$
- **select** (possibly IMF-dependent) SNRs ε_k, with $k = 1, \ldots, K$

CEEMDAN algorithm 2.

1. generate $x^j[n] = x[n] + \varepsilon_1 \, w^j[n]$ ($j = 1, \ldots, J$) and define

$$\widetilde{IMF}_1[n] = \frac{1}{J} \sum_{j=1}^{J} E_1(x^i[n])$$

2. assign $k = 1$, compute $r_1[n] = x[n] - \widetilde{IMF}_1[n]$. and define

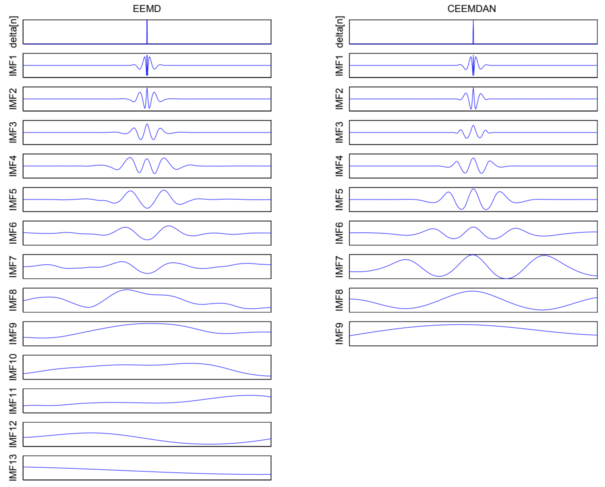
$$\widetilde{IMF}_2[n] = \frac{1}{J} \sum_{j=1}^{J} E_1 \left(r_1[n] + \varepsilon_2 \, E_1(w^j[n]) \right)$$

3. for $k = 2, \ldots, K$, compute $r_k[n] = r_{(k-1)}[n] - \widetilde{IMF}_k[n]$ and define

$$\widetilde{IMF}_{(k+1)}[n] = \frac{1}{J} \sum_{j=1}^{J} E_1(r_k[n] + \varepsilon_{k+1} \, E_k(w^j[n]))$$

4. go to step 3 until no further residue

CEEMDAN — Dirac pulse example



bivariate EMD

Idea

Decompose "coherently" the two components of bivariate or complex-valued signals

In practice (Rilling *et al.*, '07 + Rehman & Mandic, '10) :

1. switch from oscillations to rotations
2. replace envelopes by tubes
3. apply the usual EMD machinery

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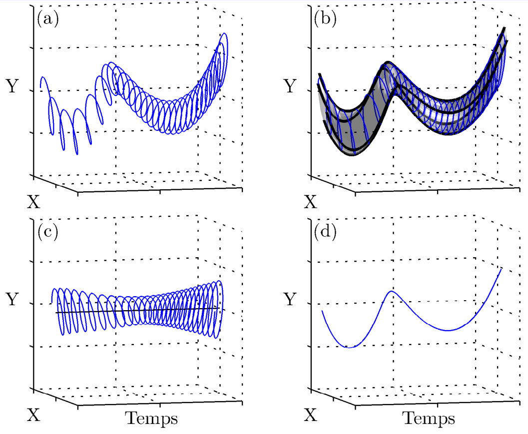
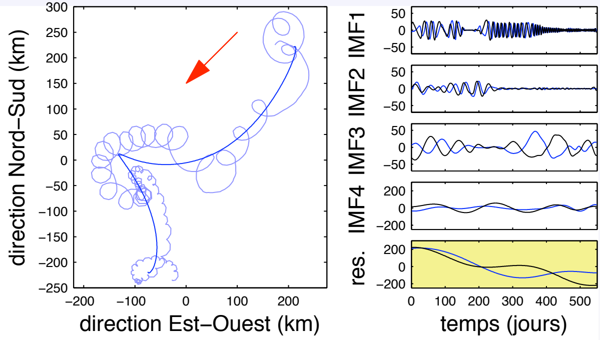


bivariate EMD - example



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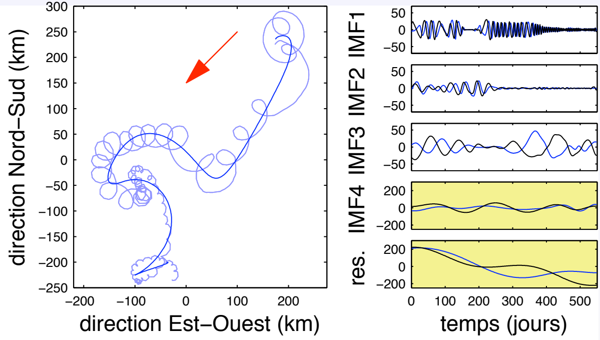

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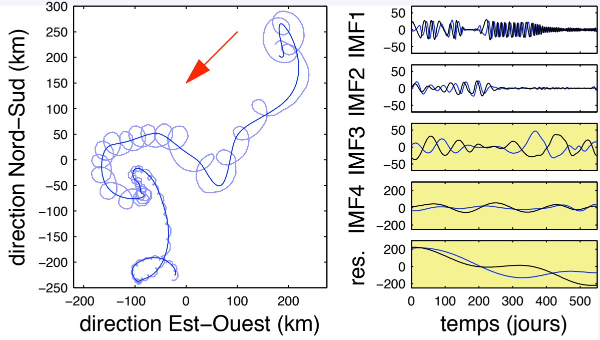

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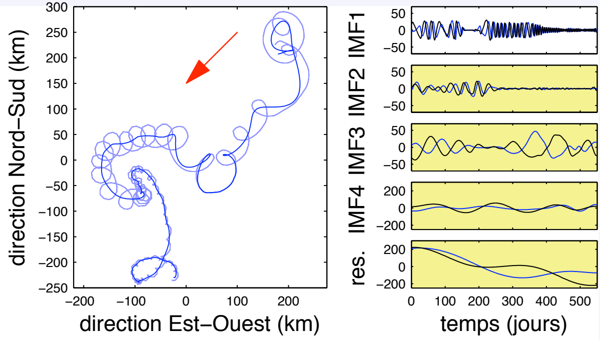

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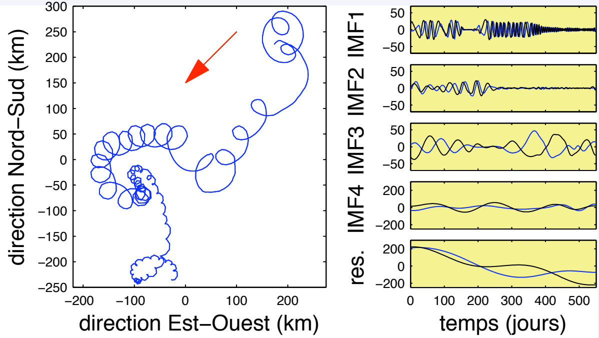

2D EMD

Idea

Decompose images in 2D oscillating patterns

In practice (Linderhed, '02 + Nunes *et al.*, '03 + Damerval *et al.*, '05 + Xu *et al.*, '06) :

1. 2x1D vs. 1x2D schemes
2. extrema ?
3. interpolation ?

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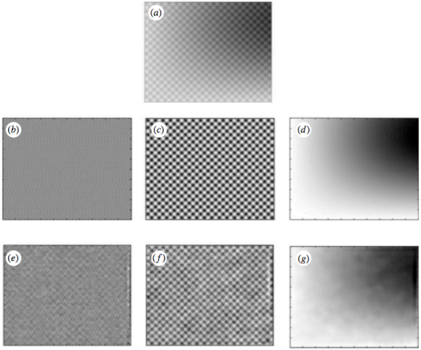

Figure 4. Decomposition of a simulated texture image: (a) the original image, (b–d) the three original components that form the original image, (e and f) the IMFs and (g) the residue.

from (Xu *et al.*, '07)

some uses

- **Pre-processing**
  - baseline removal
  - signal disentanglement
  - selection of significant IMFs
- **Post-processing**
  - Hilbert transform of IMFs
  - grouping of significant IMFs
  - (local) trend removal
  - denoising from partial coarse-to-fine reconstruction
  - gap filling
  - scaling analysis

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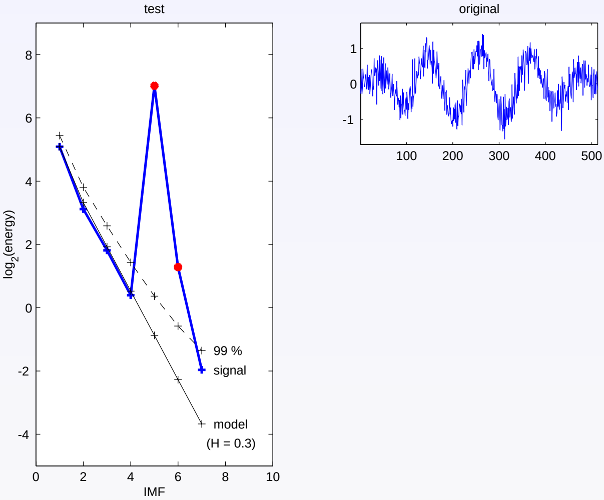
a model-based toy example



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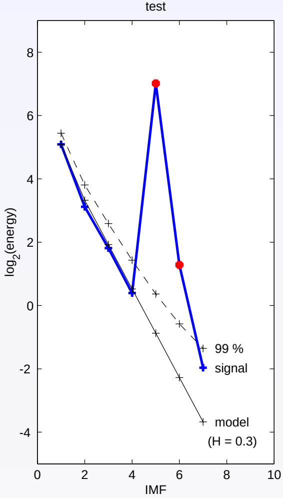

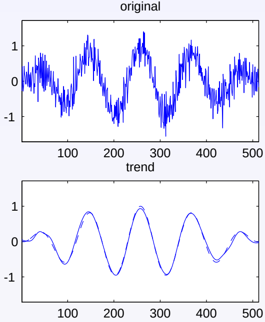

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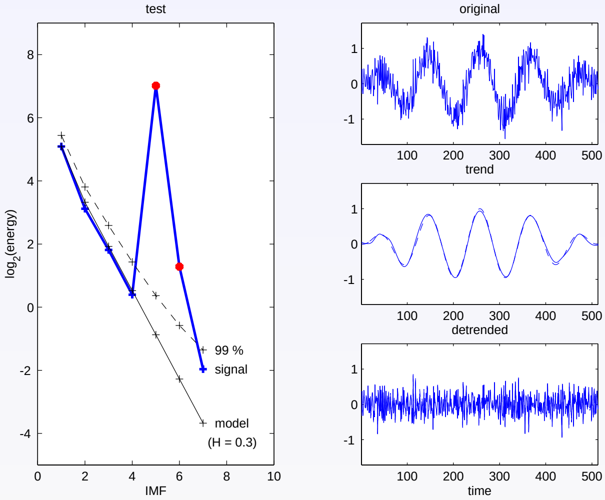


more on “detrending”

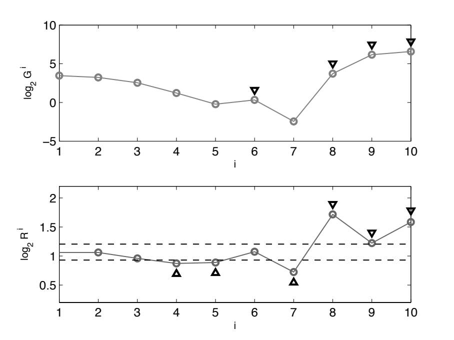
Observation

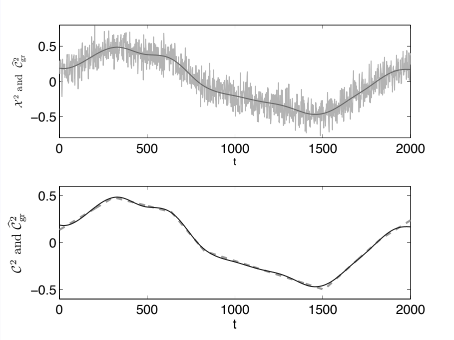
A “trend” is a loosely defined object, e.g., a “long-term change in the mean” (Chatfield, ’96)

- as opposed to “fluctuations”, an EMD-based definition of a “trend” may correspond to (some of) the last IMF(s)
- a possible strategy (Moghtaderi *et al.*, ’11) for selecting those relevant modes combine ratios of
  1. **zero-crossings**
  2. **energy**

  between successive adjacent modes

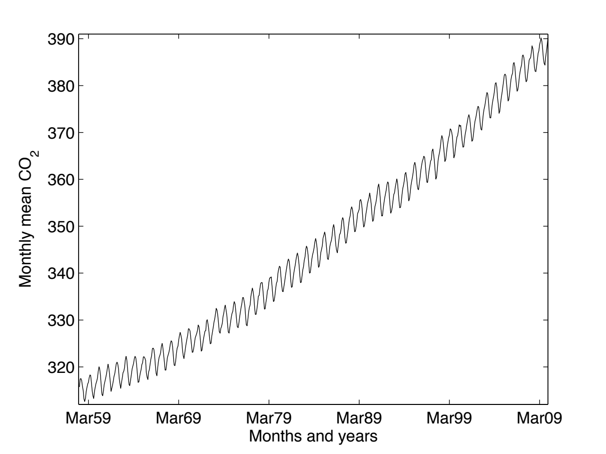
a model-free toy example





from (Moghtaderi *et al.*, '11)

monthly mean CO_2



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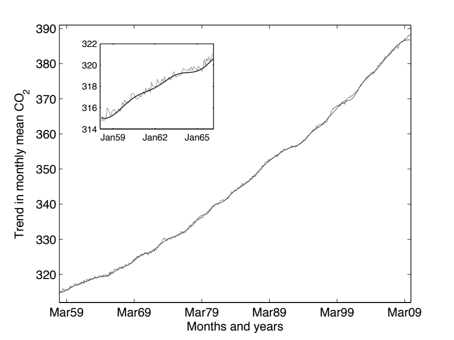

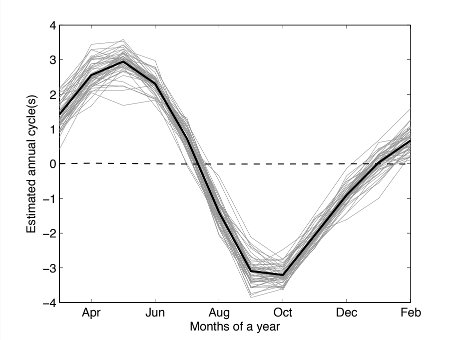

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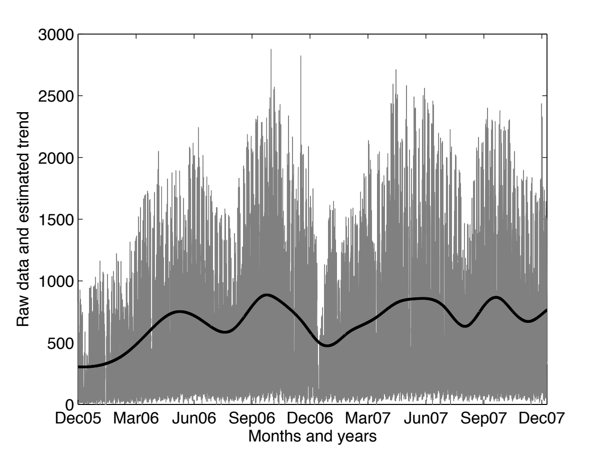
shared bicycles (Lyon Vélo'v system)



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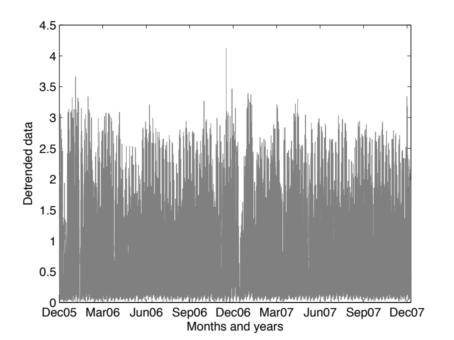
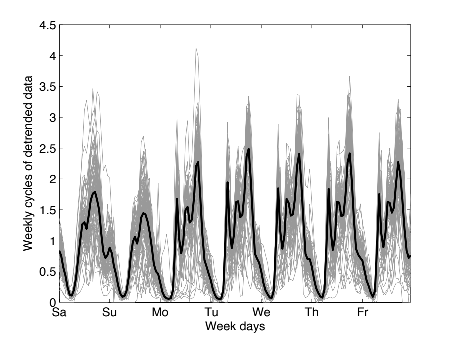
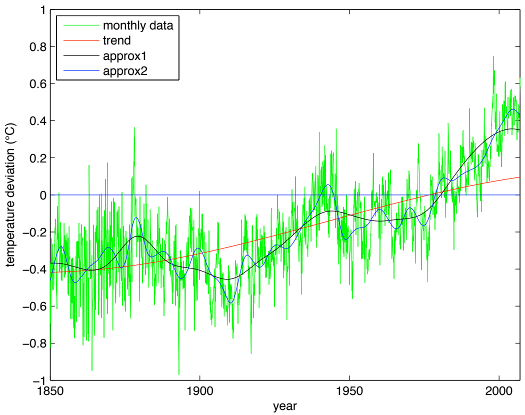




from (Moghtaderi *et al.*, '11)

global Earth surface temperature



gap filling

Problem

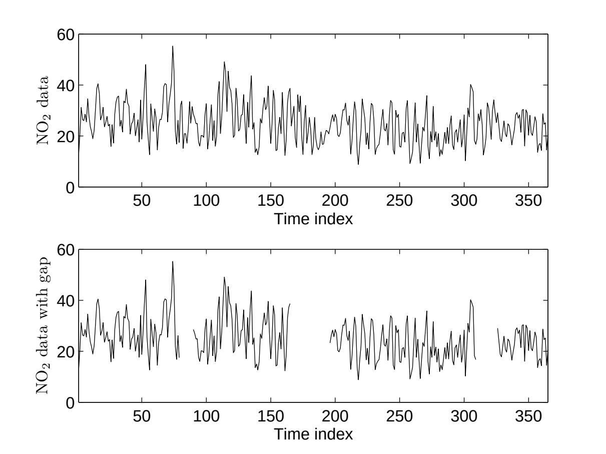
Analyze and/or reconstruct data with gaps, due to unavailable and/or corrupted measurements

Idea

Carry over the problem to IMFs

1. **construct** gapped IMFs
2. **fill in** gaps in each mode, based on geometrical constraints
3. **add up** all gap-filled IMFs

gap filling



from (Moghtaderi *et al.*, '12)

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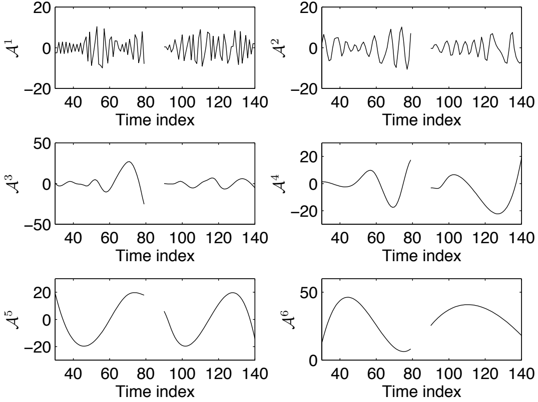

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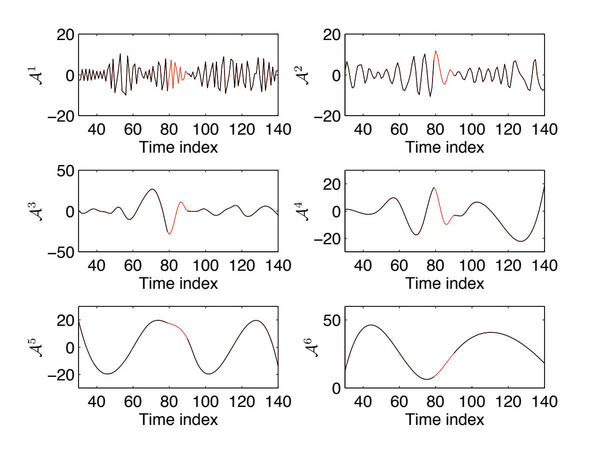

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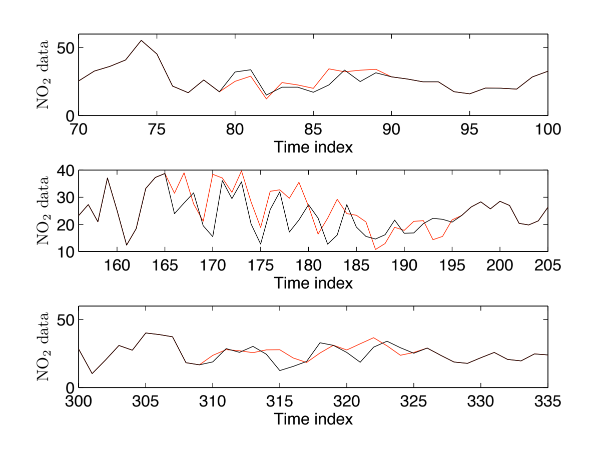

from (Moghtaderi *et al.*, ’12)

heart rate variability

- **Objective** — Assessment of cardiovascular autonomic control
- **Methods** — Spectrum analysis of RR intervals, LF (0.04–0.15 Hz) vs. HF (0.15–0.4 Hz) contributions quantifying the sympatho-vagal balance
- **Issues** — VLF trends and/or non steady-state measurements

Idea

Switch from fixed, time-invariant, LF/HF filters to data-adaptive, time-varying, slow/fast modes

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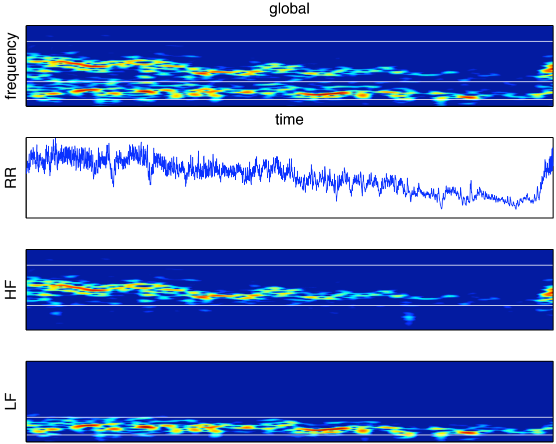
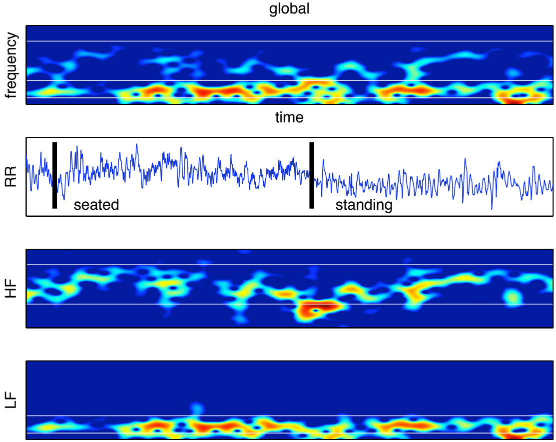
Switch from fixed, time-invariant, LF/HF filters to data-adaptive, time-varying, slow/fast modes

HRV analysis. Example 1



HRV analysis. Example 2 (Souza Neto *et al.*, ’02)



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- . . . is **intuitive** but still **lacks from general theory** : current on-going work for possible ways out include
  - **modifications** of the original algorithm, e.g., by constrained variational approaches (Meignen and Perrier, ’11 ; Pustelnik *et al.*, ’12) in place of sifting
  - alternative transforms via **EMD-like** decompositions, e.g., “synchrosqueezing” (WU, Daubechies *et al.*, ’10-11)

much work to be done but **worth investigating** !

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preprints, Matlab codes & contact

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