Time-frequency energy distributions, old and new

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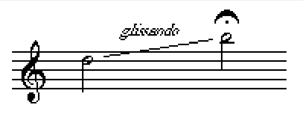
thanks to François Auger, Pierre Borgnat, Éric Chassande-Mottin, Paulo Gonçalves, Cédric Richard & Jun Xiao

intuition

Idea



Give a mathematical sense to musical notation

Aim

Write the "musical score" of a signal with multiple, evolutive components with that additional constraint of getting, in the case of an isolated chirp $x(t) = a(t) \exp\{i\varphi(t)\}$, *a* **localized** *representation*

$$\rho(t, f) \sim a^2(t)\, \delta\left(f - \dot{\varphi}(t)/2\pi\right).$$

local methods and localization

- **The example of the short-time FT** — One defines the **local** quantity

$$F_x^{(h)}(t,f) = \int x(s)\, \overline{h(s-t)}\, e^{-i2\pi fs}\, ds,$$

where $h(t)$ is some short-time observation window.

- **Measurement** — The representation results from an interaction between the signal and a **measurement device** (the window $h(t)$).
- **Trade-off** — A short window favors the "resolution" in time at the expense of the "resolution" in frequency, and vice-versa.

adaptation

- **Chirps** — Adaptation to **pulses** if $h(t) \to \delta(t)$ and to **tones** if $h(t) \to 1 \Rightarrow$ adapting the analysis to arbitrary **chirps** suggests to make $h(t)$ **(locally) depending on the signal**.
- **Linear chirp** — In the linear case $f_x(t) = f_0 + \alpha t$, the equivalent frequency width δf_S of the **spectrogram** $S_x^{(h)}(t, f) := |F_x^{(h)}(t, f)|^2$ behaves as:

$$\delta f_S \approx \sqrt{\frac{1}{\delta t_h^2} + \alpha^2 \, \delta t_h^2}$$

for a window $h(t)$ with an equivalent time width $\delta t_h \Rightarrow$ minimum for $\delta t_h \approx 1/\sqrt{\alpha}$ (but α **unknown**. . .).

self-adaptation and Wigner-Ville distribution

- **Matched filtering** — If one takes for the window $h(t)$ the **time-reversed** signal $x_-(t) := x(-t)$, one readily gets that $F_x^{(x_-)}(t,f) = W_x(t/2, f/2)/2$, where

$$W_x(t,f) := \int x(t+\tau/2)\,\overline{x(t-\tau/2)}\,e^{-i2\pi f\tau}\,d\tau$$

  is the **Wigner-Ville Distribution** (Wigner, '32; Ville, '48).
- **Linear chirps** — The WVD **perfectly** localizes on **straight lines** of the plane:

$$x(t) = \exp\{i2\pi(f_0 t + \alpha t^2/2)\} \Rightarrow W_x(t,f) = \delta\left(f - (f_0 + \alpha t)\right).$$

- **Remark** — Localization via self-adaptation leads to a **quadratic** transformation (energy distribution).

interferences

- **Quadratic superposition** — For any pair of signals $\{x(t), y(t)\}$ and coefficients (a, b), one gets

$$W_{ax+by}(t,f) = |a|^2\, W_x(t,f) + |b|^2\, W_y(t,f) + 2\, Re\left\{a\,\overline{b}\,W_{x,y}(t,f)\right\},$$

with

$$W_{x,y}(t,f) := \int x(t+\tau/2)\,\overline{y(t-\tau/2)}\,e^{-i2\pi f\tau}\,d\tau$$

- **Drawback** — Interferences between **disjoint** component reduce readability.
- **Advantage** — Inner interferences between **coherent** components guarantee localization.

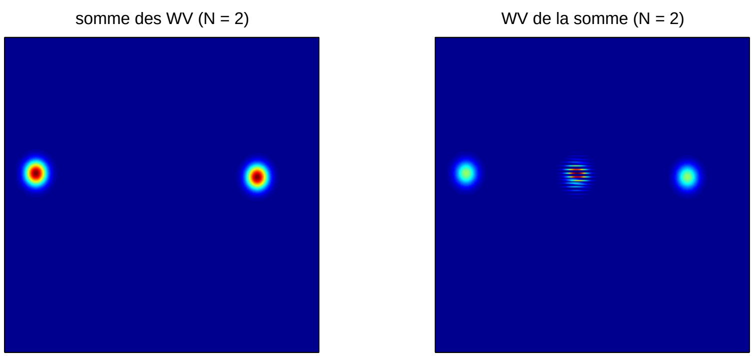
interferences

- **Janssen's formula (Janssen, '81)** — It follows from the **unitarity** of $W_x(t,f)$ that:

$$|W_x(t,f)|^2 = \iint W_x\left(t+\frac{\tau}{2}, f+\frac{\xi}{2}\right) W_x\left(t-\frac{\tau}{2}, f-\frac{\xi}{2}\right) d\tau\, d\xi$$

- **Geometry (Hlawatsch & F., '85)** — Contributions located in any two points of the plane plan interfere to create a third contribution
  1. midway of the segment joining the two components
  2. oscillating (positive and negative values) in a direction perpendicular to this segment
  3. with a "frequency" proportional to their "time-frequency distance".

interferences and readability



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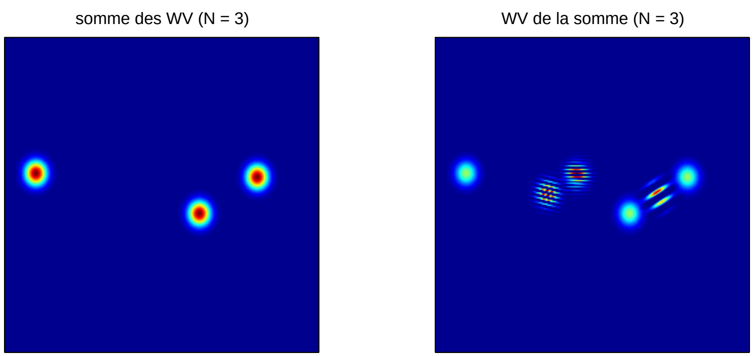

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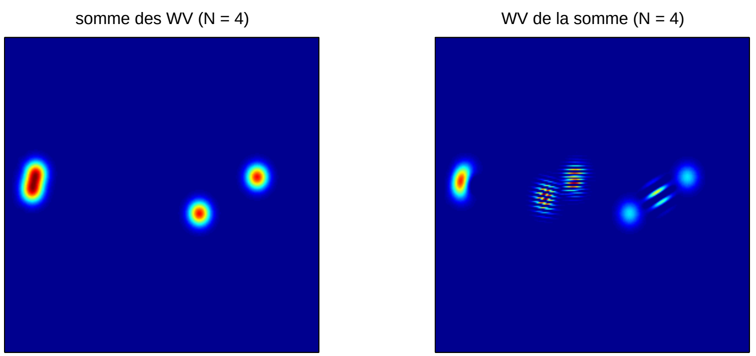

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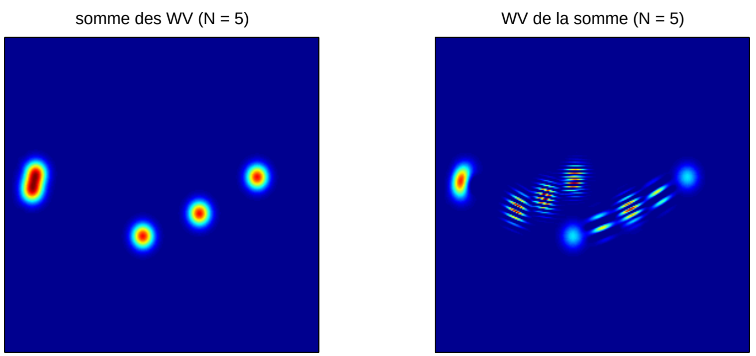

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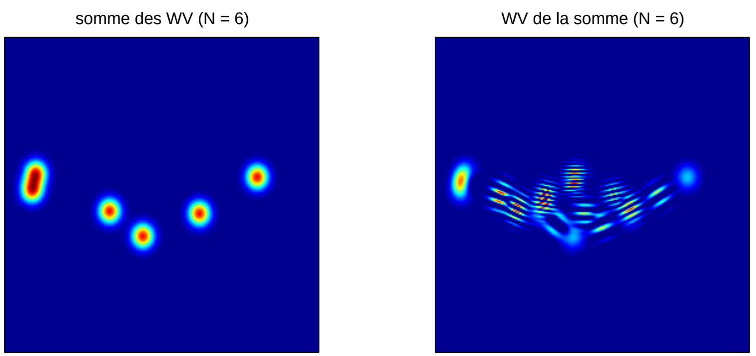

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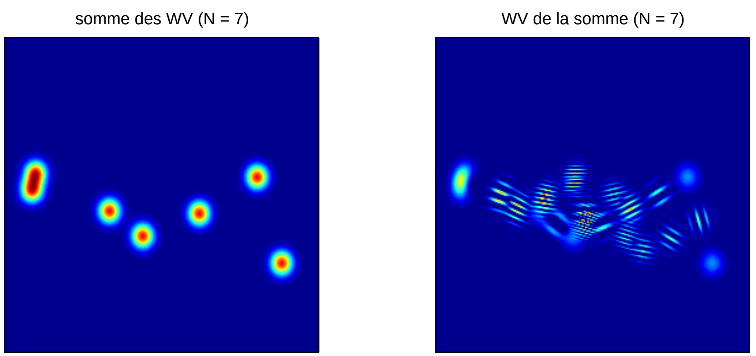

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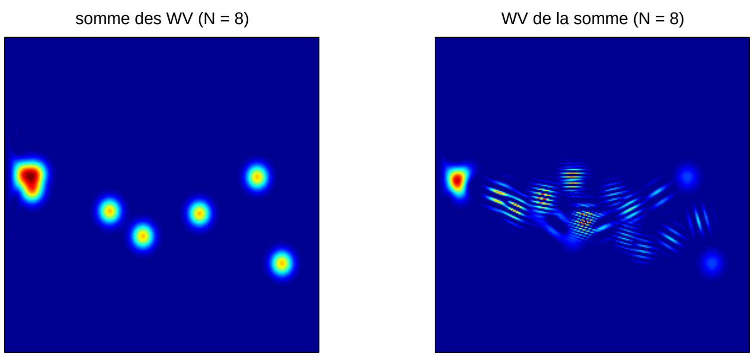

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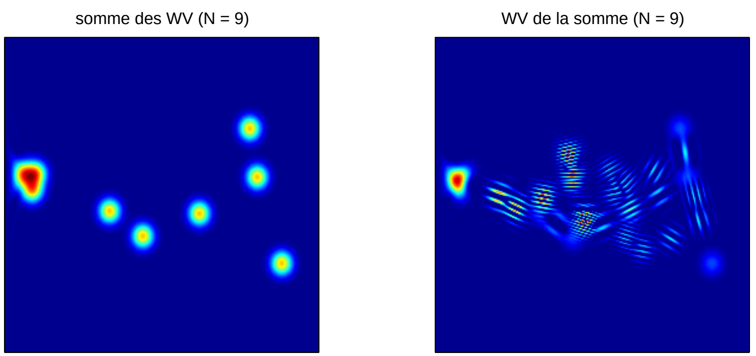

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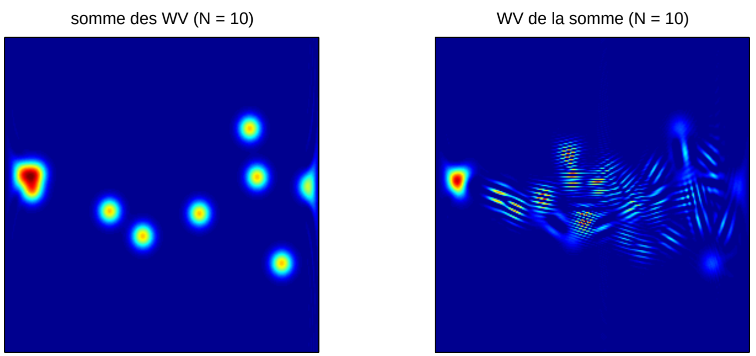

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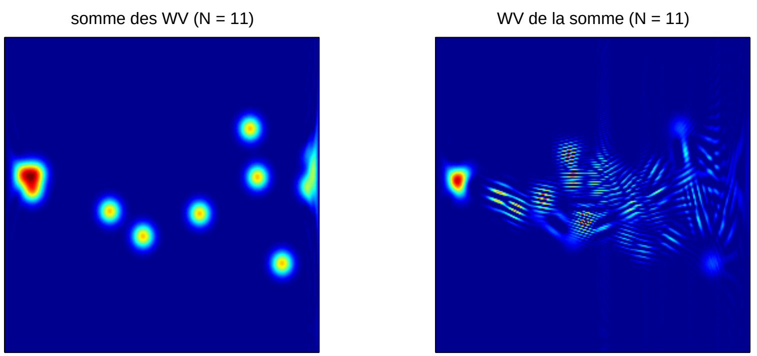

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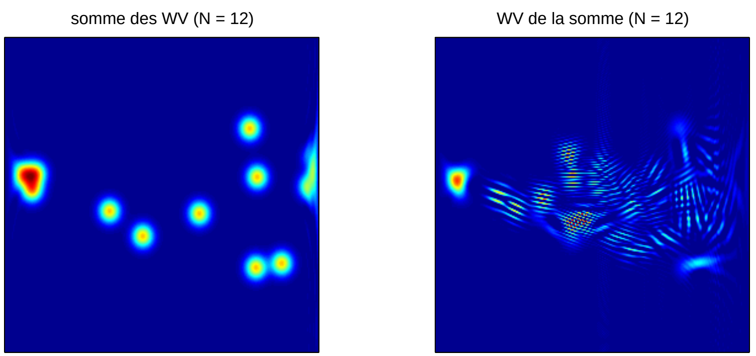

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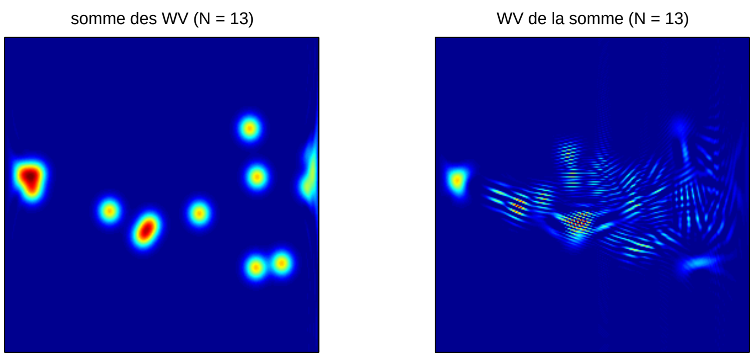

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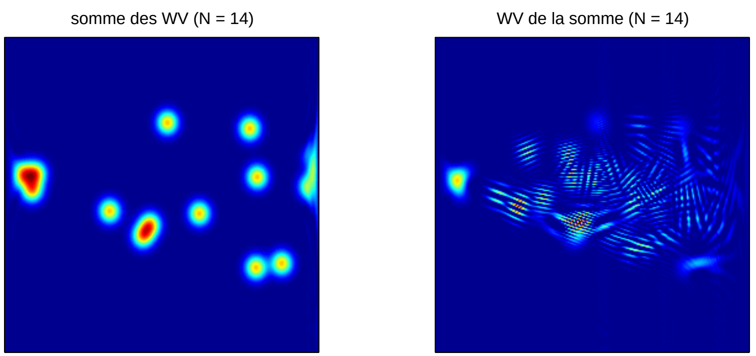

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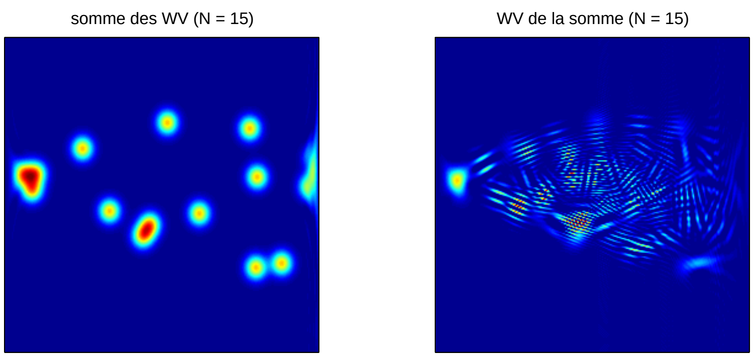

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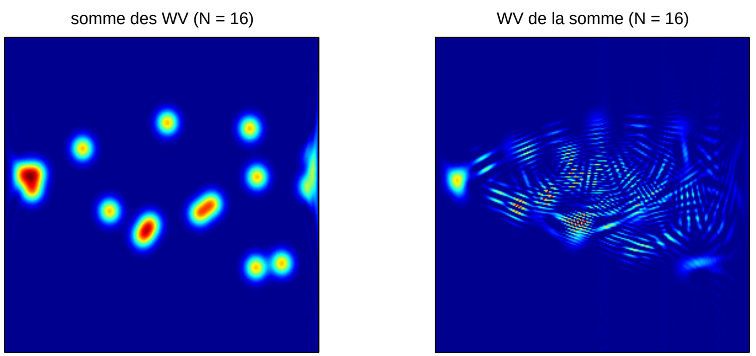
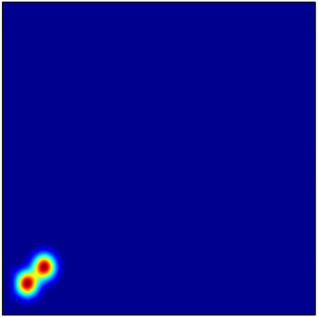
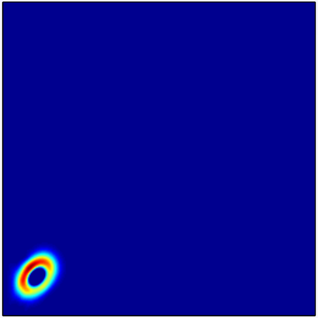


interferences and localization

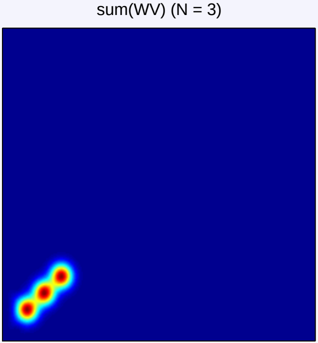
sum(WV) (N = 2)

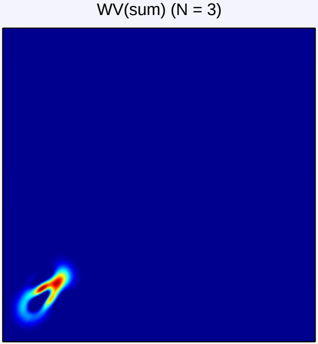


WV(sum) (N = 2)



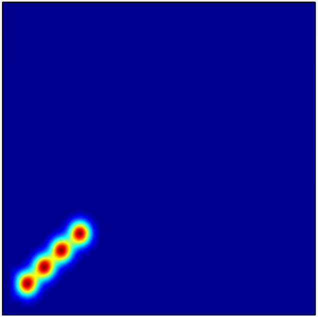
interferences and localization



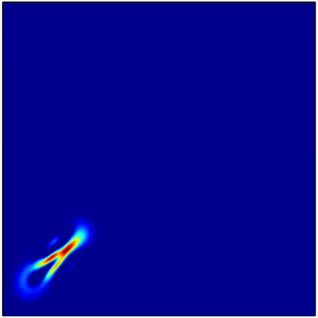


interferences and localization

sum(WV) (N = 4)

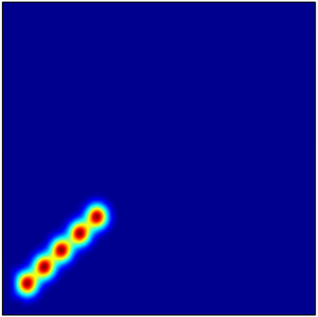


WV(sum) (N = 4)

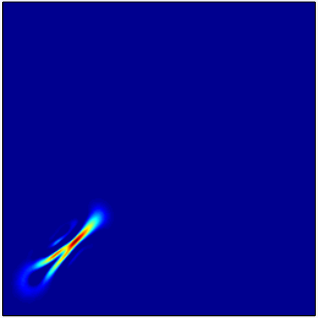


interferences and localization

sum(WV) (N = 5)



WV(sum) (N = 5)



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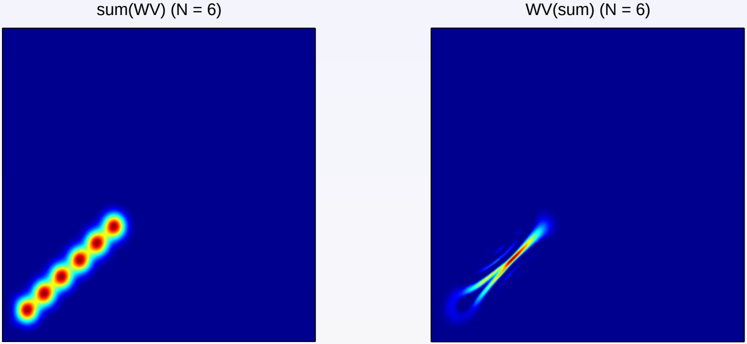

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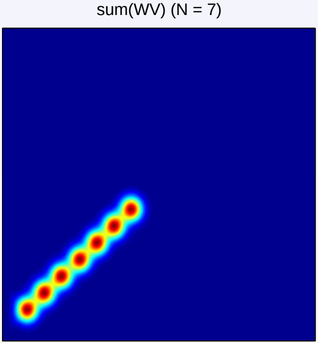
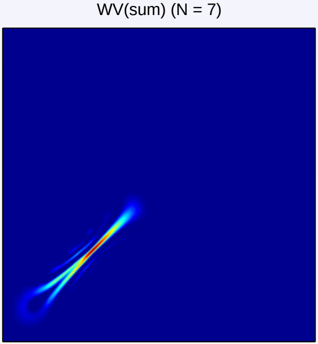
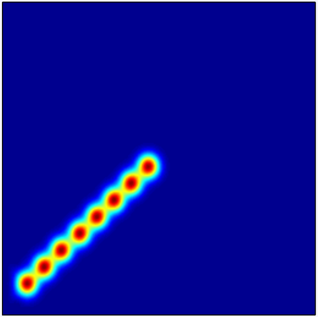


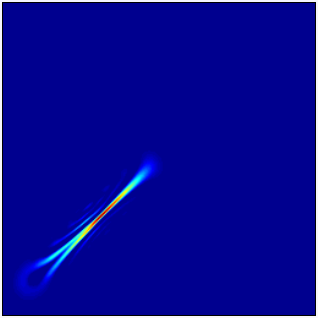


interferences and localization

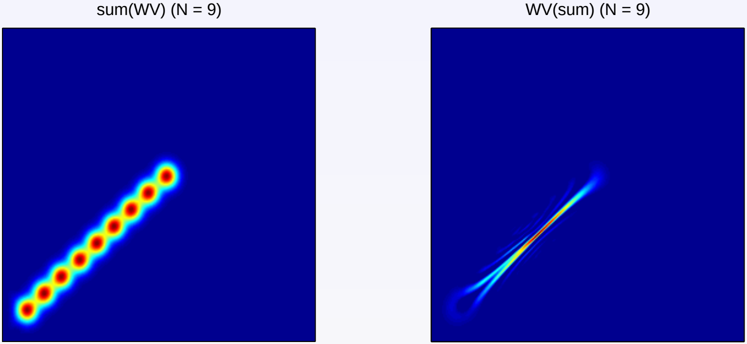
sum(WV) (N = 8)



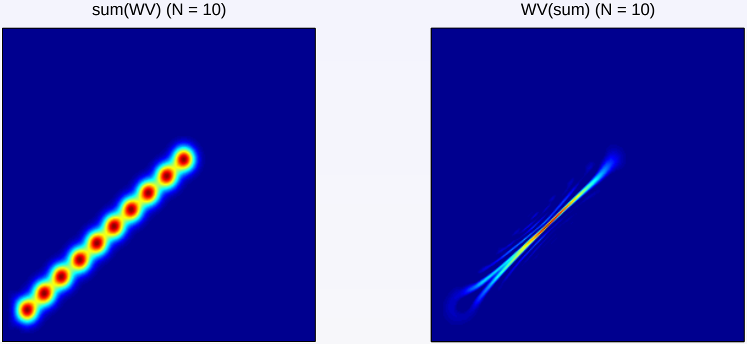
WV(sum) (N = 8)



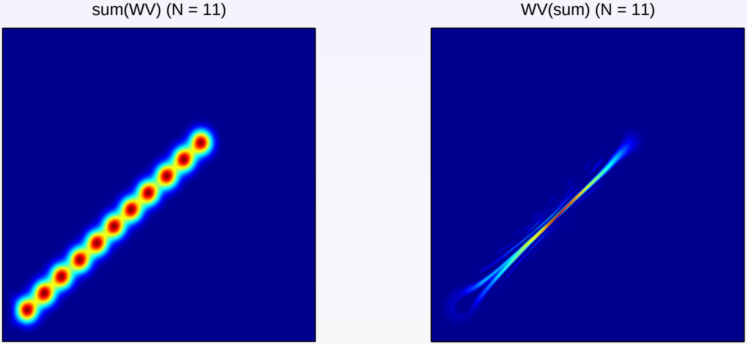
interferences and localization



interferences and localization

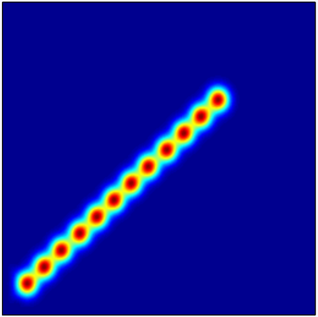


interferences and localization

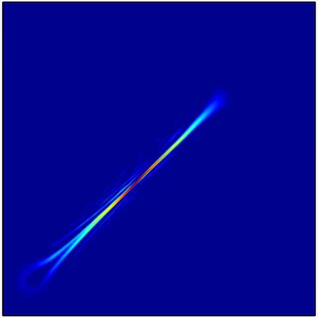


interferences and localization

sum(WV) (N = 12)

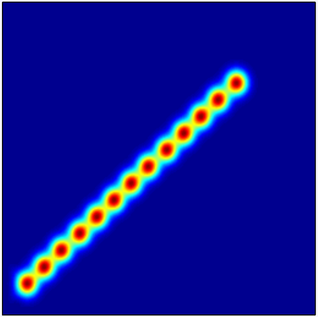


WV(sum) (N = 12)

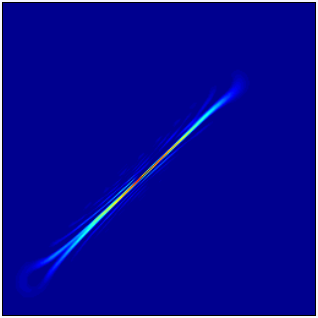


interferences and localization

sum(WV) (N = 13)



WV(sum) (N = 13)



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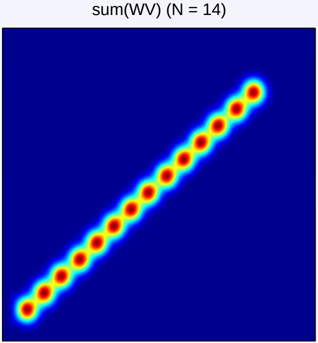
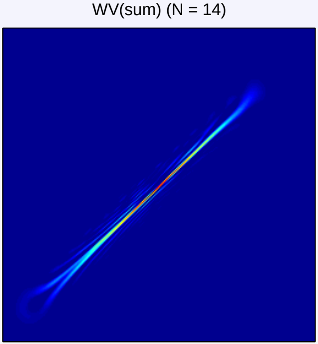
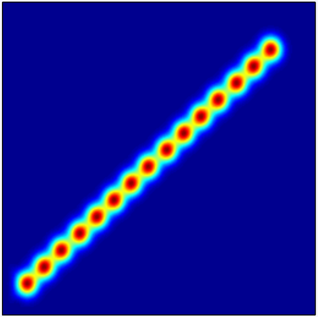


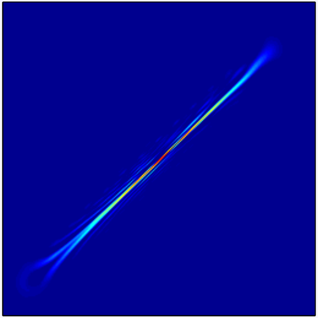


interferences and localization

sum(WV) (N = 15)



WV(sum) (N = 15)



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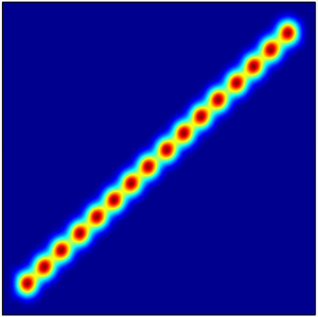
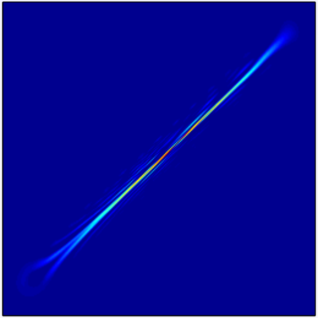




classes of quadratic distributions

Observation

Many quadratic distributions have been proposed in the literature since more than half a century (e.g., spectrogram and DWV): **none fully extends the notion of spectrum density to the nonstationary case**.

Principle of conditional unicity — Classes of quadratic distributions of the form $\rho_x(t,f) = \langle x, \mathbf{K}_{t,f} x \rangle$ can be constructed based on **covariance requirements** :

$$\begin{array}{ccc} x(t) & \rightarrow & \rho_x(t,f) \\ \downarrow & & \downarrow \\ (\mathbf{T}x)(t) & \rightarrow & \rho_{\mathbf{T}x}(t,f) = (\tilde{\mathbf{T}}\rho_x)(t,f) \end{array}$$

classes of quadratic distributions

- **Cohen's class** — Covariance wrt **shifts** $(\mathbf{T}_{t_0,f_0}x)(t) = x(t-t_0)\exp\{i2\pi f_0 t\}$ leads to **Cohen's class** (Cohen, '66) :

$$C_x(t,f) := \iint W_x(s,\xi)\,\Pi(s-t,\xi-f)\,ds\,d\xi,$$

  with $\Pi(t,f)$ "arbitrary" (and to be specified via additional constraints).

- **Variations** — Other choices possibles, e.g., $(\mathbf{T}_{t_0,f_0}x)(t) = (f/f_0)^{1/2}x(f(t-t_0)/f_0) \rightarrow$ **affine class** (Rioul & F, '92), etc.

Cohen's class and smoothing

- **Spectrogram** — Given a low-pass window $h(t)$, one gets the **smoothing** relation:

$$S_x^{(h)}(t,f) := |F_x^{(h)}(t,f)|^2 = \iint W_x(s,\xi)\, W_h(s-t,\xi-f)\, ds\, d\xi$$

- **From Wigner-Ville to spectrograms** — A generalization amounts to choose a smoothing function $\Pi(t,f)$ allowing for a **continuous** and **separable** transition between Wigner-Ville and a spectrogram (**smoothed pseudo-Wigner-Ville** distributions) :

$$\begin{array}{ccccc} Wigner-Ville & \ldots \rightarrow & PWVL & \ldots \rightarrow & spectrogram \\ \delta(t)\,\delta(f) & & g(t)\,H(f) & & W_h(t,f) \end{array}$$

from Wigner-Ville to spectrogram, and back

time-frequency spectrum

Definition (Martin, '82)

One of the most "natural" extensions of the power spectrum density is given by the **Wigner-Ville Spectrum** *:*

$$\mathbf{W}_x(t,f) := \int r_x\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-i2\pi f\tau}\, d\tau$$

- **Interpretation** — FT of a **local** correlation.
- **Properties** — PSD if $x(t)$ stationary, marginals, etc.
- **Relation with the WVD** — Under simple conditions, one has $\mathbf{W}_x(t,f) = \mathbb{E}\{W_x(t,f)\}$.

estimation of the Wigner-Ville spectrum

Aim

Approach $\mathbb{E}\{W_x(t,f)\}$ on the basis of only one realization.

- **Assumption** — **Local** stationarity (in time and in frequency).
- **Estimators** — Smoothing of the DWV :

$$\hat{\mathbf{W}}_x(t,f) = (\Pi ** W_x)(t,f)$$

  i.e., Cohen's class.
- **Properties** — **Statistical** (bias-variance) and **geometrical** (localization) trade-offs, both controlled by $\Pi(t,f)$.

global vs. local

- **Global approach** — The Wigner-Ville Distribution localizes perfectly on **straight lines** of the plane (linear chirps). One can construct other distributions localizing on more general **curves** (ex.: **Bertrand**'s distributions adapted to hyperbolic chirps).
- **Local approach** — A different possibility consists in revisiting the smoothing relation defining the spectrogram and in considering localization wrt the instantaneous frequency as it can be measured **locally**, at the scale of the short-time window $\Rightarrow$ **reassignment**.

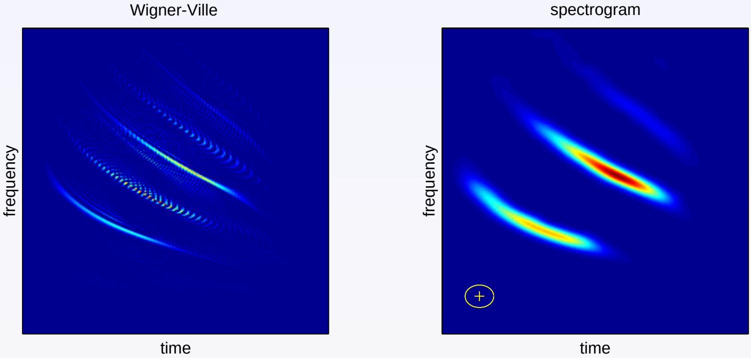
reassignment

- **Principle** — The key idea is (1) to replace the **geometrical** center of the smoothing time-frequency domain by the **center of mass** of the WVD over this domain, and (2) to **reassign** the value of the smoothed distribution to this local centroïd:

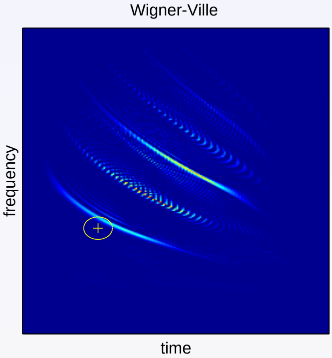
$$S_x^{(h)}(t,f) \mapsto \iint S_x^{(h)}(s,\xi)\,\delta\left(t-\hat{t}_x(s,\xi), f-\hat{f}_x(s,\xi)\right)\,ds\,d\xi.$$

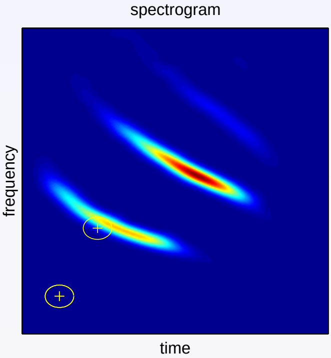
- **Remark** — Reassignment has been first introduced for the only spectrogram (Kodera *et al.*, '76), but its principle has been further generalized to **any** distribution resulting from the smoothing of a localizable mother-distribution (Auger & F., '95).

reassignment

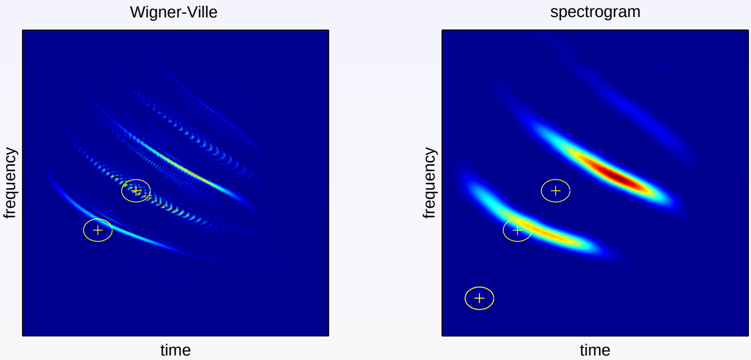


reassignment

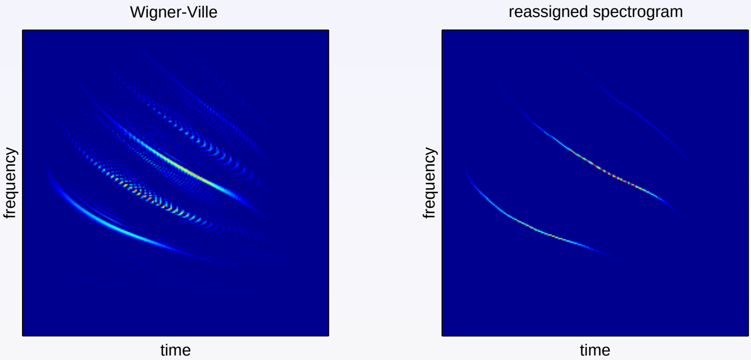




reassignment



reassignment



reassignment in action

- **Spectrogram — Implicit** computation of the local centroïds (Auger & F., '95) :

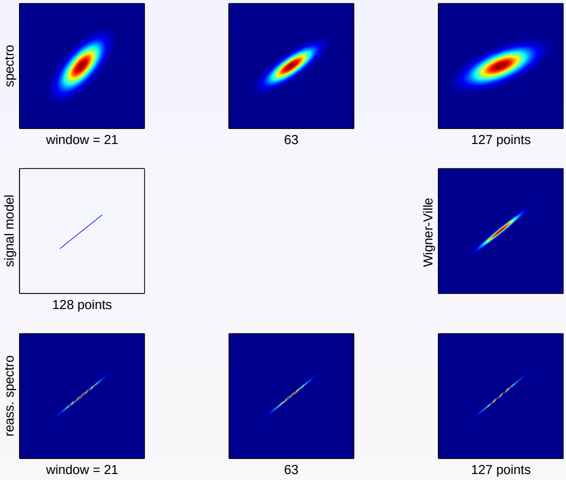
$$\hat{t}_x(t,f) = t + Re\left\{\frac{F_x^{(\mathcal{T}h)}}{F_x^{(h)}}\right\}(t,f)$$

$$\hat{f}_x(t,f) = f - Im\left\{\frac{F_x^{(\mathcal{D}h)}}{F_x^{(h)}}\right\}(t,f),$$

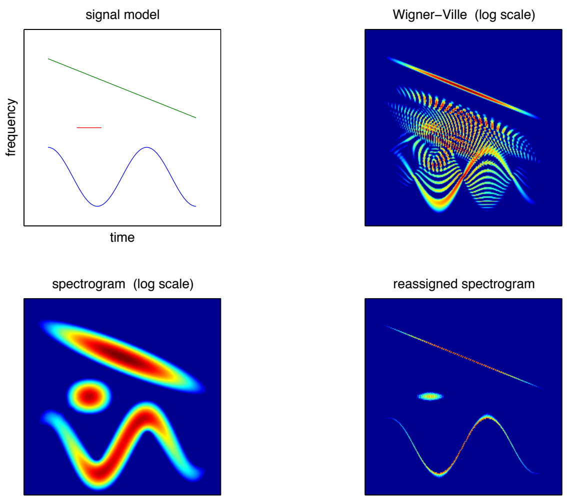
with $(\mathcal{T}h)(t) = t\,h(t)$ and $(\mathcal{D}h)(t) = (dh/dt)(t)/2\pi$.

- **Beyond spectrograms —** Possible generalizations to other smoothings (smoothed pseudo-Wigner-Ville, scalogram, etc.).

independence wrt window size



an example of comparison



reassignment and estimation

- **Advantage** — Very good properties of **localization** for chirps ($>$ spectrogram).
- **Limitation** — High **sensitivity to noise** ($<$ spectrogram).

Aim

Reduce fluctuations while preserving localization.

Idea (Xiao & F., '06)

Adopt a **multiple windows** *approach.*

back to spectrum estimation

- **Stationary processes** — The **power spectrum density** can be viewed as:

$$\mathbf{S}_x(f) = \lim_{T\to\infty} \mathbb{E}\left\{ \frac{1}{T} \left| \int_{-T/2}^{+T/2} x(t)\, e^{-i2\pi f t}\, dt \right|^2 \right\}$$

- **In practice** — Only one, finite duration, realization $\Rightarrow$ crude periodogram (squared FT) = **non consistent estimator with large variance**

classical way out (Welch, '67)

- **Principle** — Method of **averaged periodograms**

$$\hat{\mathbf{S}}_{x,K}^{(W)}(f) = \frac{1}{K}\sum_{k=1}^{K} S_x^{(h)}(t_k, f)$$

  with $t_{k+1} - t_k$ of the order of the width of the window $h(t)$.

- **Bias-variance trade-off** — Given T (finite), increasing $K \Rightarrow$ **reduces variance**, but **increases bias**

multitaper solution (Thomson, '82)

- **Principle** — Computing

$$\hat{\mathbf{S}}_{x,K}^{(T)}(f) = \frac{1}{K} \sum_{k=1}^{K} S_x^{(h_k)}(0, f)$$

with $\{h_k(t), k \in \mathbb{N}\}$ a family of orthonormal windows extending over the whole support of the observation $\Rightarrow$ **reduced variance, without sacrifying bias**

- **Nonstationary extension — Multitaper spectrogram**

$$\hat{\mathbf{S}}_{x,K}^{(T)}(f) \to S_{x,K}(t, f) := \frac{1}{K} \sum_{k=1}^{K} S_x^{(h_k)}(t, f)$$

- **Limitation** — Localization controlled by most spread spectrogram.

multitaper reassignment

Idea

Combining the advantages of reassignment (wrt localization) with those of multitapering (wrt fluctuations) :

$$S_{x,K}(t,f) \rightarrow RS_{x,K}(t,f) := \frac{1}{K}\sum_{k=1}^{K} RS_x^{(h_k)}(t,f)$$

1. **coherent averaging of chirps** (localization independent of the window)
2. **incoherent averaging of noise** (different TF distributions for different windows)

in practice

- **Choice of windows — Hermite functions**

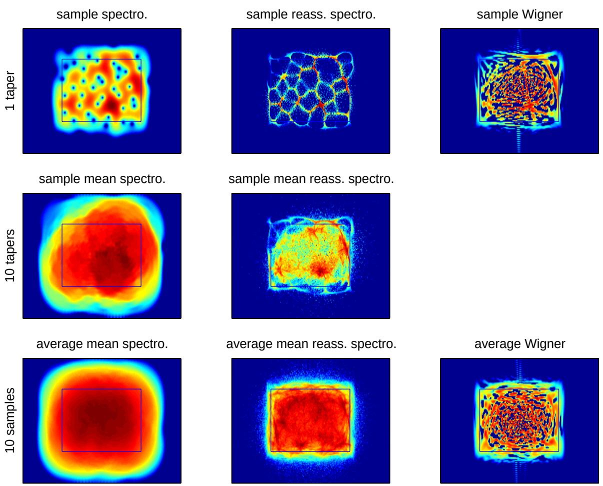
$$h_k(t) = (-1)^k \frac{e^{-t^2/2}}{\sqrt{\pi^{1/2} 2^k k!}} (\mathcal{D}^k \gamma)(t); \gamma(t) = e^{t^2}$$

rather than **Prolate Spheroidal Wave functions**

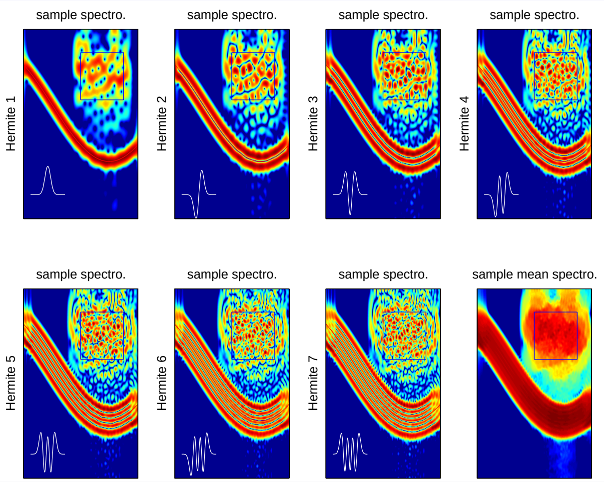
- **Two main reasons**
  1. WVD with **elliptic symmetry** and **maximum concentration** in the plane.
  2. **recursive** computation of $h_k(t)$, $(\mathcal{T} h_k)(t)$ and $(\mathcal{D} h_k)(t) \Rightarrow$ better implementation in **discrete-time**. In particular:

$$(\mathcal{D} h_k)(t) = (\mathcal{T} h_k)(t) - \sqrt{2(k+1)}\, h_{k+1}(t)$$

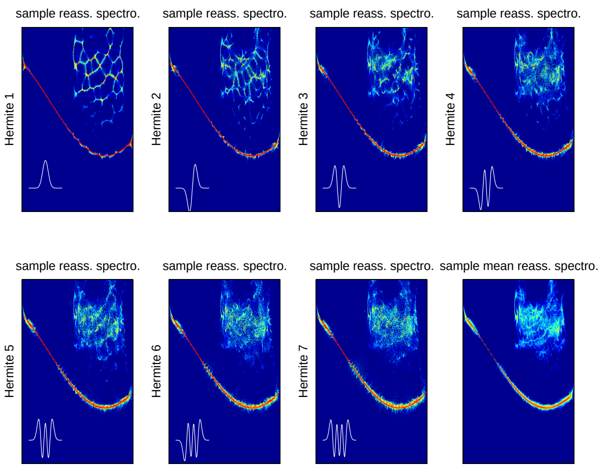
example 1


sample spectro.
sample reass. spectro.
sample Wigner
1 taper
sample mean spectro.
sample mean reass. spectro.
10 tapers
average mean spectro.
average mean reass. spectro.
average Wigner
10 samples

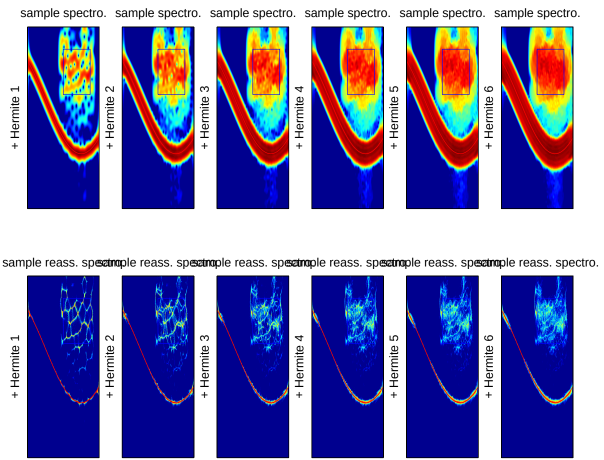
example 2


sample spectro.
sample spectro.
sample spectro.
sample spectro.
Hermite 1
Hermite 2
Hermite 3
Hermite 4
sample spectro.
sample spectro.
sample spectro.
sample mean spectro.
Hermite 5
Hermite 6
Hermite 7

example 2


sample reass. spectro.
sample reass. spectro.
sample reass. spectro.
sample reass. spectro.
Hermite 1
Hermite 2
Hermite 3
Hermite 4
sample reass. spectro.
sample reass. spectro.
sample reass. spectro.
sample mean reass. spectro.
Hermite 5
Hermite 6
Hermite 7

example 2


sample spectro.
sample spectro.
sample spectro.
sample spectro.
sample spectro.
sample spectro.
+ Hermite 1
+ Hermite 2
+ Hermite 3
+ Hermite 4
+ Hermite 5
+ Hermite 6
sample reass. spectro.
sample reass. spectro.
sample reass. spectro.
sample reass. spectro.
sample reass. spectro.
sample reass. spectro.
+ Hermite 1
+ Hermite 2
+ Hermite 3
+ Hermite 4
+ Hermite 5
+ Hermite 6

chirp enhancement

Idea

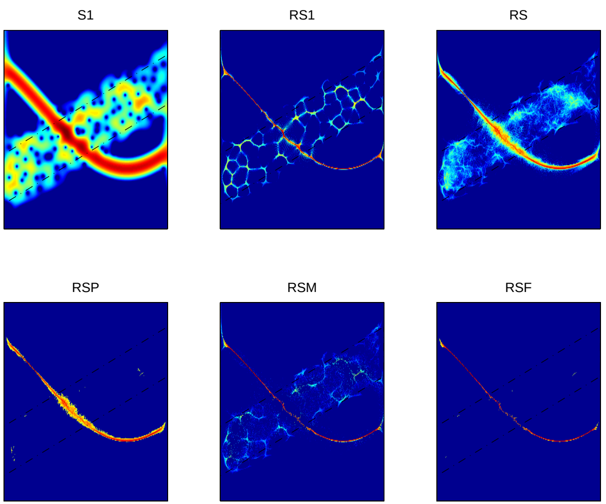
Reassigned chirps **globally** *invariant wrt tapers ⇒ "differences" wrt successive tapers mostly non-zero in noisy regions*

1. **average ratios (∼ log-differences)**

$$RSD_{x,K}(t,f) = \frac{1}{K-1} \sum_{k=1}^{K-1} \frac{RS_x^{(h_{k+1})}(t,f)}{RS_x^{(h_k)}(t,f)}.$$

2. **threshold and mask**

example 3



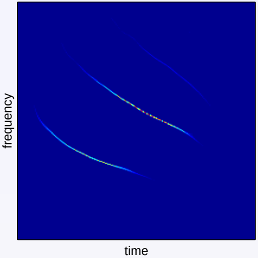
a "compressed sensing" approach

Discrete time

signal of dimension $N \Rightarrow$ TF distribution of dimension $\approx N^2$

Few components

$K \ll N \Rightarrow$ at most $KN \ll N^2$ non zero values in the TF plane



Sparsity

minimizing the ℓ_0 "norm" not feasible, but almost optimal solution by minimizing the ℓ_1 norm

a "compressed sensing" approach

Idea (F. & Borgnat, 2008-2010)

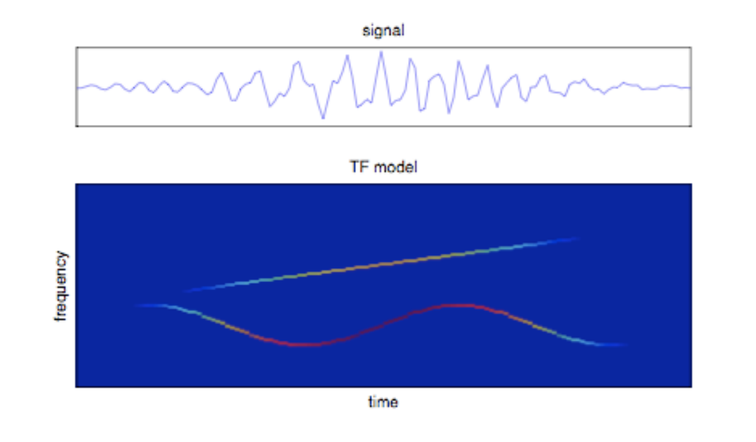
1. *choose a domain Ω neighnouring the origin of the AF plane*
2. *solve the program*

$$\min_{\rho} \|\rho\|_1 \, ; \, \mathcal{F}\{\rho\} - A_x = 0|_{(\xi,\tau)\in\Omega}$$

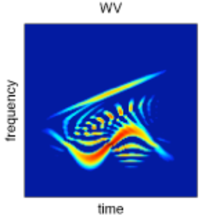
3. *the exact equality over Ω can be relaxed to*

$$\min_{\rho} \|\rho\|_1 \, ; \, \|\mathcal{F}\{\rho\} - A_x\|_2 \leq \epsilon|_{(\xi,\tau)\in\Omega}$$

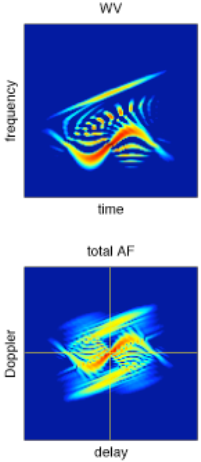
a toy example



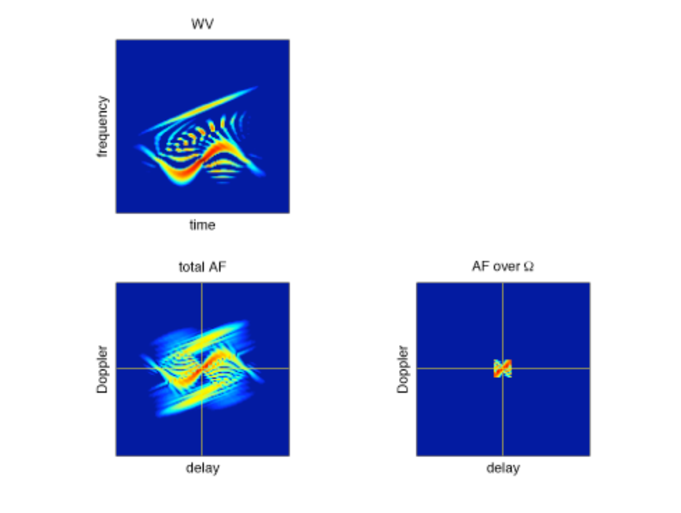
Wigner



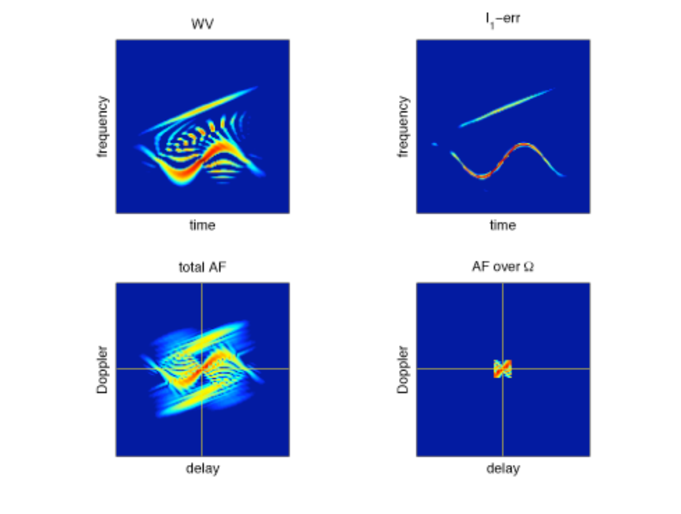
ambiguity



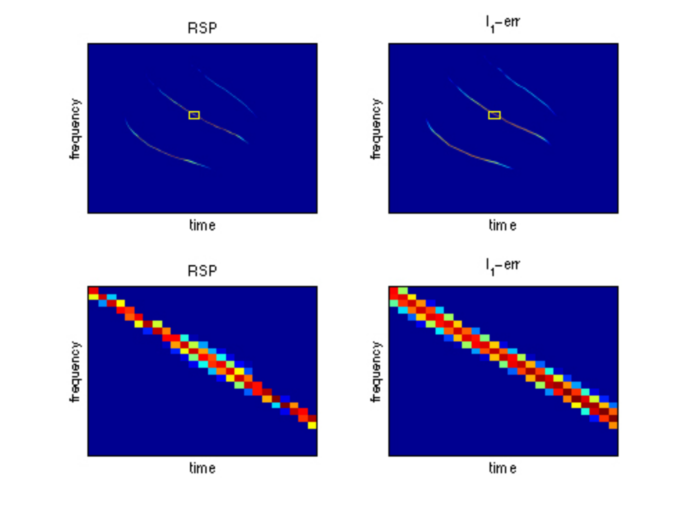
selection


WV
frequency
time
total AF
Doppler
delay
AF over Ω
Doppler
delay

sparse solution



comparison sparsity vs. reassignment



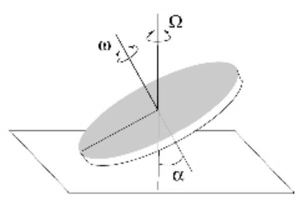
detection/estimation of chirps

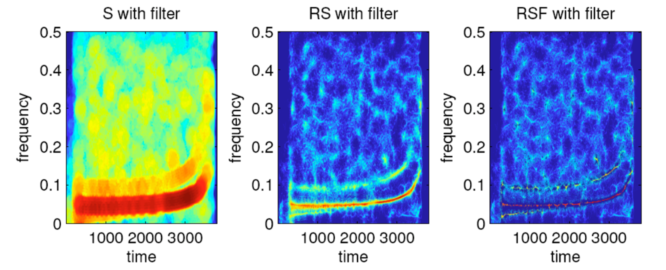
- **Optimality** — Matched filtering, maximum likelihood, contrast,. . . : basic ingredient = **correlation** "received signal — copy of emitted signal".
- **Time-frequency interpretation** — **Unitarity** of a time-frequency distribution $\rho_x(t, f)$ guarantees the equivalence:

$$|\langle x, y\rangle|^2 = \langle\langle \rho_x, \rho_y \rangle\rangle.$$

- **Chirps** — Unitarity + localization ⇒ detection/estimation via **path integration** in the plane (e.g., Wigner-Ville and linear chirps). Approximation = reassigned spectrogram and "any" chirp.

Euler's disk

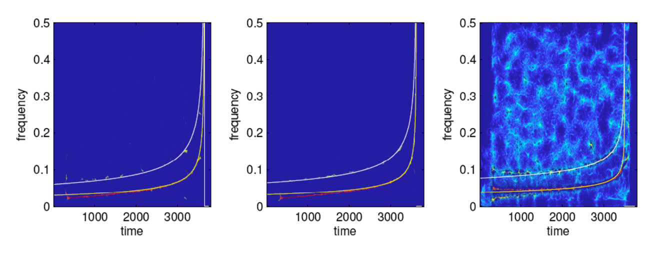

Ω
ω
α


S with filter
RS with filter
RSF with filter
frequency
time
0.5
0.4
0.3
0.2
0.1
0
1000 2000 3000

Euler's disk — Hough 1

Idea

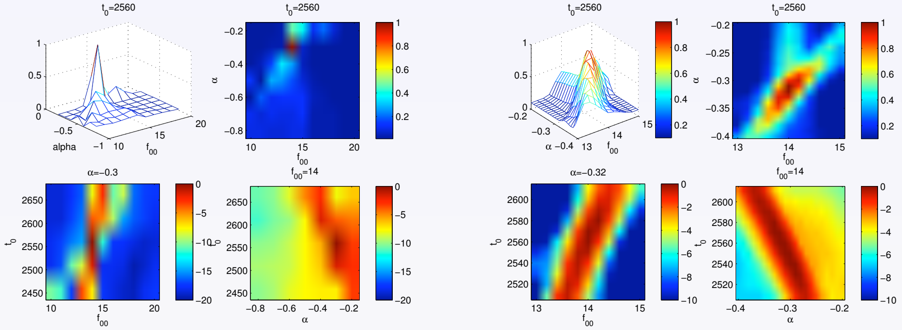
Path integration along power-law trajectories $f = f_0(t_0 - t)^\alpha$



Euler's disk — Hough 2

Result

Coarse and refined estimation of parameters f_0, t_0 and α



revisiting stationarity

Observation

Discrepancy between **theory** *(invariance over all times, stochastic framework) and* **practice** *(observation scale, deterministic signals)*

Aim

Get an **operational** *(i.e., equipped with interpretation + test) definition of stationarity*

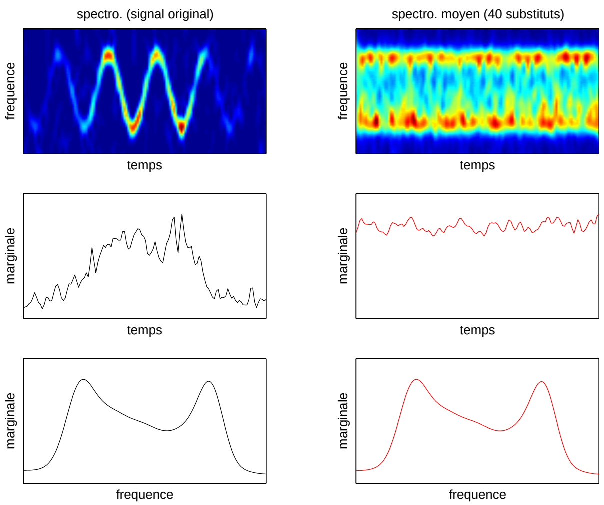
Idea

Operate in the time-frequency plane and compare with a **stationarized** *reference*

time-frequency stationarity

- **Principle**
  - 2nd-order stationarity $\Rightarrow \mathbf{W}_x(t,f) = \Gamma_x(f)$ (PSD) for **any** t
  - adaptation to sub-regions of the TF plane, in **time** (observation scale) and/or in **frequency** (subbands)
  - test by comparing **local** vs. **global** spectral features
- **Significance**
  - nonstationarity = **structured** organization of spectral content over time: local "$\neq$" global
  - null hypothesis of stationarity = **stationarized** (i.e., unstructured in time) reference: local "=" global
  - phase randomization in the Fourier domain (method of **surrogate** data)

stationarization via surrogates



a distance-based test

- **Dissimilarity measures**
  - $\kappa_{\mathrm{KL}}(G,H) := \int_\Omega (G(f) - H(f)) \log(G(f)/H(f))\, df$ (Kullback-Leibler divergence)
  - $\kappa_{\mathrm{LS}}(G,H) := \int_\Omega |\log(G(f)/H(f))|\, df$ (log-spectral deviation)
  - $\kappa(G,H) := \kappa_{\mathrm{KL}}(G,H).\,(1 + \lambda\, \kappa_{\mathrm{LS}}(G,H))$ (combined)
- **Comparison local vs. global**

$$\{c_n^{(y)} := \kappa\left(S_{y,K}(t_n,.), \langle S_{y,K}(t_n,.)\rangle_{n=1,\ldots N}\right), n = 1, \ldots N\}$$

with either

- $y(t) = x(t)$ (observed signal)
- $\{y(t) = s_j(t); j = 1, \ldots J\}$ (stationarized surrogates)

proposed test

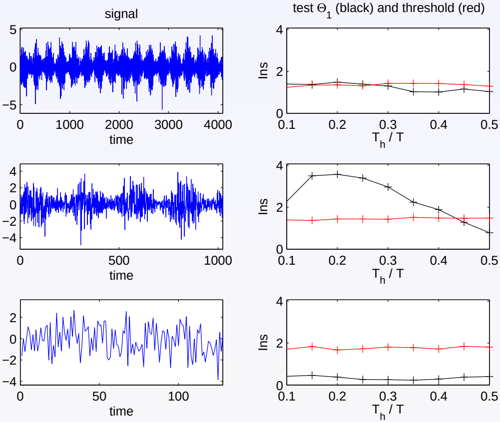
- **Fluctuations** — "Nonstationarity" assessed by the l_2-norm $L(g,h) := \sum_{n=1}^{N} (g_n - h_n)^2 / N$
  - $\Theta_1 = L\left(c^{(x)}, \langle c^{(x)} \rangle_{n=1,\dots N}\right)$ (signal)
  - $\Theta_0(j) = L\left(c^{(s_j)}, \langle c^{(s_j)} \rangle_{n=1,\dots N}\right), j = 1, \dots J$ (surrogates)
- **Test**

$$\begin{cases} \Theta_1 > \gamma & : \text{"}\textit{nonstationarity}\text{"};\\ \Theta_1 < \gamma & : \text{"}\textit{stationarity}\text{"}. \end{cases}$$

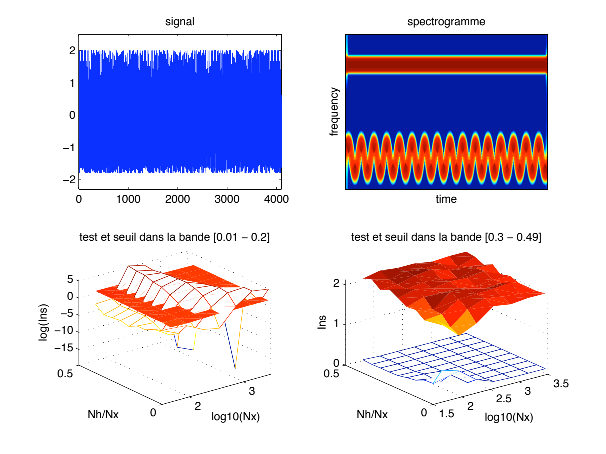
  with threshold γ deduced from the distribution of $\Theta_0(j)$ for a given level of significance (probability of rejecting the null hypothesis of stationarity)

- **From detection to estimation**
  - **index** of nonstationarity: $\mathrm{INS} := \sqrt{\frac{\Theta_1}{\frac{1}{J}\sum_{j=1}^{J}\Theta_0(j)}}$
  - **scale** of nonstationarity: $\mathrm{SNS} := \frac{1}{N_x} \arg\max_{N_h} \{\mathrm{INS}(N_h)\}$

test in action (stochastic case)



test in action (deterministic case)



a kernel-based test

Idea

Surrogates = learning set ⇒ kernel methods machinery

- **Nonlinear** mapping $\varphi(.)$ from input space to **feature** space, with a kernel such that $K(x_i, x_j) = \langle \varphi(x_i), \varphi(x_j) \rangle$
- **One-class SVM** (Support Vector Machines)
  - **implicit** density estimation for outlier rejection
  - optimal **hyperplane** solution of the quadratic program

$$\min_{w,\rho,\xi} \tfrac{1}{2}\|w\|^2 + \tfrac{1}{\nu J}\sum_{j=1}^{J} \xi_j - \rho$$
$$\text{subject to } \langle w, \varphi(x_j) \rangle \geq \rho - \xi_j,\ \xi_j \geq 0$$

  - **decision** function given by $d(z) = \mathrm{sgn}(\langle w, \varphi(z) \rangle - \rho)$

feature space

- **Normalized** spectral "slices" of multitaper spectrograms

$$\tilde{S}_n(f) = S_{x,K}(t_n, f) / \int_0^\infty S_{x,K}(t_n, f) df; n = 1, ..., N$$

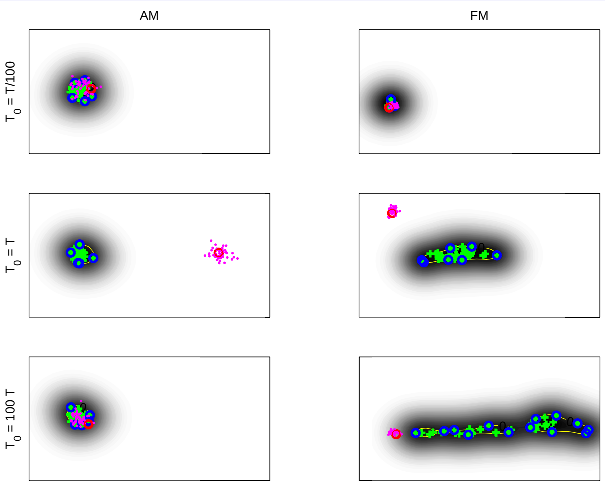
- Time evolution of local **power** P_n and **frequency** F_n

$$P_n =< 1 >_{\tilde{S}_n};\ F_n =< f >_{\tilde{S}_n};\ F_n^2 =< f^2 >_{\tilde{S}_n}$$

- **Local** vs. **global** features

$$\begin{cases} P = \text{std}(\{P_n\}_{n=1..N}) / \text{mean}(\{P_n\}_n) \\ F = \text{std}(\{F_n\}_{n=1..N}) / \text{mean}(\{\sqrt{\{F_n^2 - (F_n)^2\}}\}_n) \end{cases}$$

an example



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- S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 1997.
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- A. Papandreou-Suppappola (*ed.*), *Applications in Time-Frequency Signal Processing*, CRC Press, 2003.
- B. Boashash (*ed.*), *Time-Frequency Signal Analysis and Processing*, Elsevier, 2003.
- Ch. Doncarli & N. Martin (*eds.*), *Décision dans le Plan Temps-Fréquence*, Traité IC2, Hermes, 2004.
- F. Auger & F. Hlawatsch (*eds.*), *Temps-Fréquence — Concepts et Outils*, Traité IC2, Hermes, 2005.

(p)reprints, Matlab codes & contact

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