# "Chirps" everywhere

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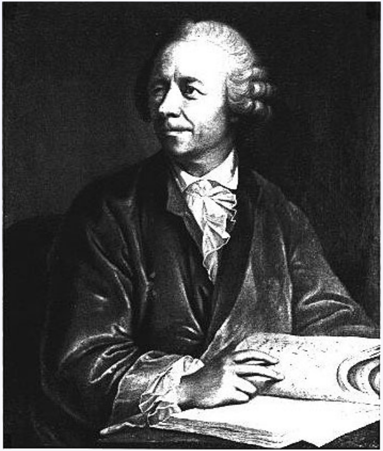
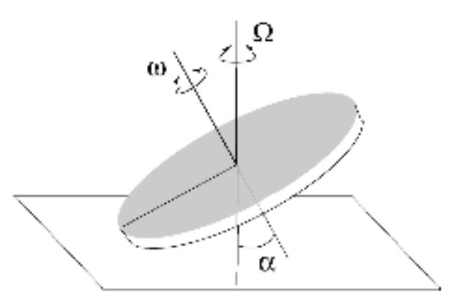
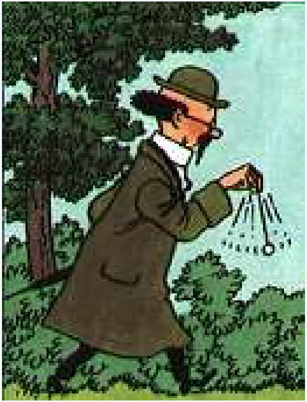


Leonhard Euler (1707-1783)

# Euler's disk



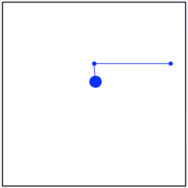
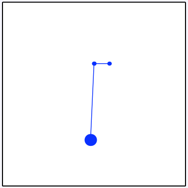
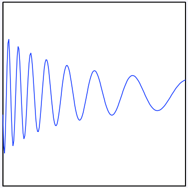


# oscillations

# oscillations

## Definition

$$\ddot{\theta}(t) + (g/L)\,\theta(t) = 0$$

- *Constant length $L = L_0$* — Small oscillations are sinusoidal, with *constant* period $T_0 = 2\pi\sqrt{L_0/g}$
- *"Slowly-varying" length $L = L(t)$* — Small oscillations are almost-sinusoidal, with *varying* pseudo-period $T(t) \sim 2\pi\sqrt{L(t)/g}$


Z.7001

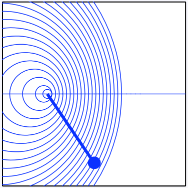
# waves

# waves

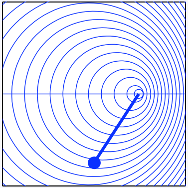
**Observation**

*Moving monochromatic source ⇒ differential perception of the emitted frequency*

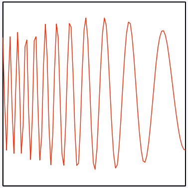
f + Δ f



f - Δ f



"chirp"



# chirps in time

**Definition**

*We will call "chirp" any complex-valued signal of the form $x(t) = a(t) \exp\{i\varphi(t)\}$, where $a(t) \geq 0$ is a low-pass amplitude whose evolution is slow as compared to the oscillations of the phase $\varphi(t)$*

- Slow evolution ?
- Usual heuristic conditions assume that
  1. $|\dot{a}(t)/a(t)| \ll |\dot{\varphi}(t)|$ : the amplitude is *almost-constant* at the scale of one pseudo-period $T(t) = 2\pi/|\dot{\varphi}(t)|$
  2. $|\ddot{\varphi}(t)|/\dot{\varphi}^2(t) \ll 1$ : the pseudo-period $T(t)$ is itself *slowly varying* from one oscillation to the next

# chirps in frequency

## Theorem (Stationary phase principle)

*Assuming that $\dot{\varphi}(t)$ has monotonic variation, with $t_s$ such that $\dot{\varphi}(t_s) = 2\pi f$, one can approach the chirp spectrum*

$$X(f) = \int a(t)\, e^{i(\varphi(t) - 2\pi f t)}\, dt$$

*by its stationary phase approximation* $\tilde{X}(f) \propto a^2(t_s)/|\ddot{\varphi}(t_s)|$

## Interpretation

*The "instantaneous frequency" curve $\dot{\varphi}(t)$ defines a one-to-one correspondence between one time and one frequency. The spectrum follows by weighting frequencies with durations*

# towards AM-FM

## Observation

*Given the harmonic model* $x(t) = a \cos(2\pi f_0 t + \varphi_0)$, *unambiguous definition of amplitude* $a$ *and frequency* $f_0$

## Aim

*Switch to an evolutive model* $x(t) = a(t) \cos \varphi(t)$, *with* $a(t)$ *time-varying and* $\varphi(t)$ *non linear*

## Problem

*Given one observation, no unicity anymore for the representation since, for any function* $0 < b(t) < 1$,

$$a(t) \cos \varphi(t) = [a(t)/b(t)]\, [b(t) \cos \varphi(t)] =: \tilde{a}(t) \cos \tilde{\varphi}(t)$$

# from monochromatic waves...

## Observation

*The real-valued harmonic model can indeed be written*

$$x(t) = a\cos(2\pi f_0 t + \varphi_0) = \mathrm{Re}\left\{a \exp i(2\pi f_0 t + \varphi_0)\right\},$$

*with*

$$a \exp i(2\pi f_0 t + \varphi_0) = x(t) + i\,(\mathbf{H}x)(t)$$

*and* **H** *the Hilbert transform (quadrature)*

## Interpretation

*A monochromatic wave (prototype of a deterministic "stationary" signal) is decribed, in the complex plane, by a rotating vector whose magnitude and rotation speed are time-invariant quantities*

# ... to AM-FM

## Idea

*Go to the complex plane and describe an AM-FM waveform by a rotating (Fresnel) vector whose magnitude and rotation speed are time-varying quantities, while mimicking the monochromatic construction*

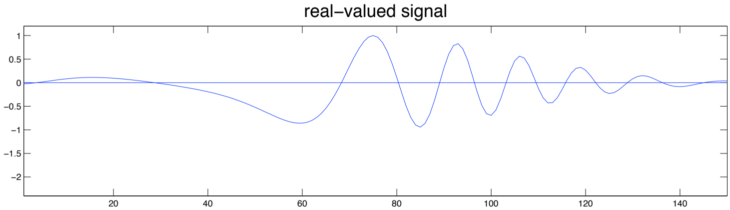
$$x(t) \to z_x(t) := x(t) + i\,(\mathbf{H}x)(t)$$

## Definition (Gabor, '46; Ville, '48)

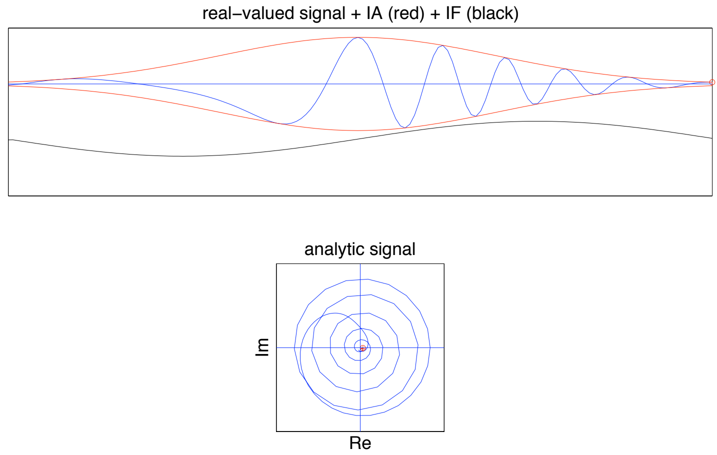
*The instantaneous amplitude and frequency follow from this complex-valued representation — referred to as analytic signal — as*

$$a_x(t) := |z_x(t)| \quad ; \quad f_x(t) := \frac{1}{2\pi}\frac{d}{dt}\arg z_x(t)$$

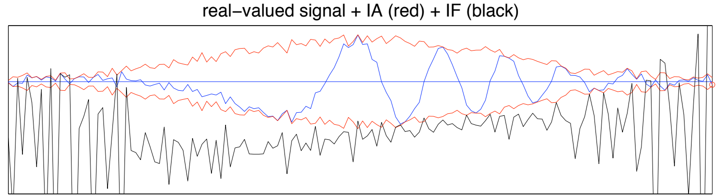
# example

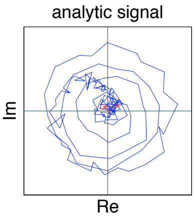


# example

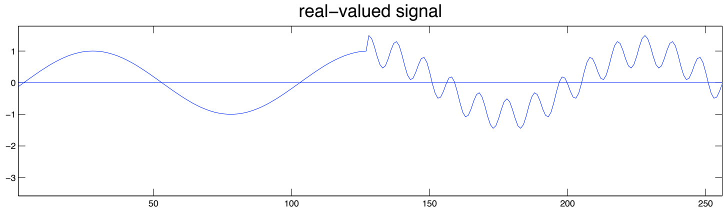


# limitation: noise



# assumption 1: monocomponent



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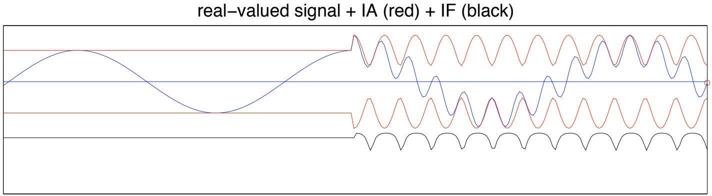
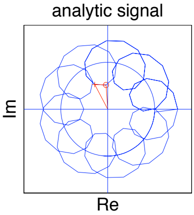
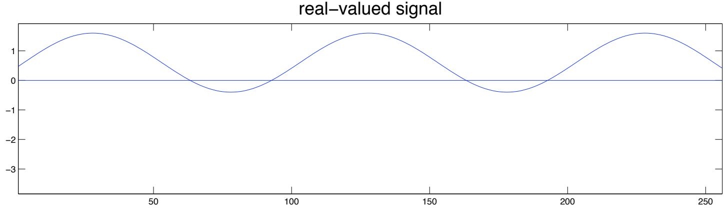
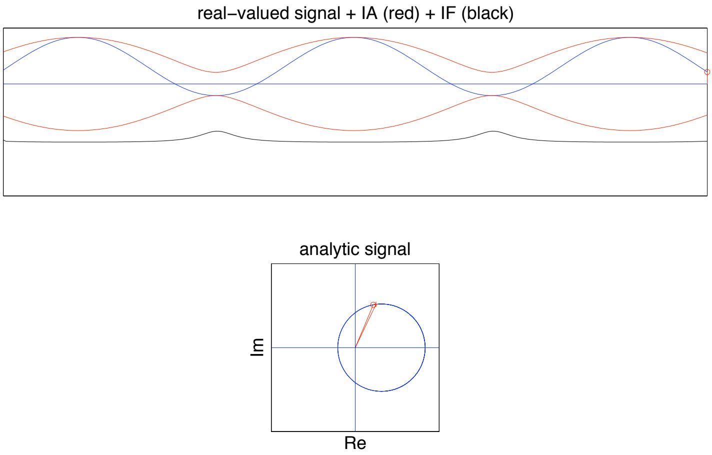


# assumption 2: zero-mean



# assumption 2: zero-mean



# alternatives

## Ideas

1. *Teager, '86 & Kaiser, '90: define IA-IF from a local energy operator*
2. *Huang, '98: pre-process the observation by an Empirical Mode Decomposition so as to get well-behaved modes for IA-IF extraction*
3. *Equis et al., '11: consider rotations relatively to a local, moving center and deduce IA-IF from the estimation of an osculating circle*

# wedding time and frequency

## Aim

*$t$ or $f$ (Fourier) $\rightarrow f(t)$ (Fresnel) $\rightarrow t$ and $f$ (time-frequency)*

## Problem (Heisenberg, '25; Gabor, '46)

*Localization trade-off, classically based on a second order (variance-type) measure: $\Delta t_x\, \Delta f_x \geq \|x\|/4\pi$ ($> 0$), with $\Delta t_x = (\int t^2\, |x(t)|^2\, dt)^{1/2}$ and $\Delta f_x = (\int f^2\, |X(f)|^2\, df)^{1/2}$*

## Interpretation

*No perfect pointwise localization*

## Remark

*Same limitation with other spreading measures, e.g., entropy (Hirschman, '57). Common denominator: minimum achieved with Gaussians*

# chirps as uncertainty minimizers

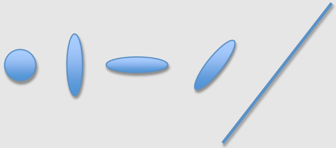
## Remark

*No pointwise localization does not mean no localization*

## Stronger uncertainty relation (Schrödinger, 1935)

$$\Delta t_x \, \Delta f_x \geq \frac{\|x\|}{4\pi} \sqrt{1 + 16\pi^2 \left( \int t \, (\partial_t \arg x(t)) \, |x(t)|^2 \, dt \right)^2}$$

*bound achieved for **"squeezed states"** of the form* $\{\exp(\alpha t^2 + \beta t + \gamma)\}$, *with **linear "chirps"** as a limit when* $\mathrm{Re}\{\beta\} = 0$ *and* $\mathrm{Re}\{\alpha\} \to 0_-$



# time-frequency alternatives

## From stationarity...

*Spectrum analysis “à la Wiener-Khintchine-Bochner” :*
$\Gamma_x(f) = \mathcal{F}\{\gamma_x\}(f)$, *with* $\gamma_x(\tau) := \langle x, \mathbf{T}_\tau x \rangle$ *correlation function*
*independent of time*

## ...to nonstationarities (Wigner, ’32; Ville, ’48)

$\gamma_x \rightarrow$ *time-frequency correlation* $\langle x, \mathbf{T}_{\tau,\xi} x \rangle$ + *2D Fourier transform* $\Rightarrow$ *Wigner-type transforms*

- *intrinsic definitions: no dependence on some measurement device (window, wavelet)*
- *perfect localization for linear chirps (with possible extensions to non linear cases)*

# "distribution/correlation" duality

## Definition

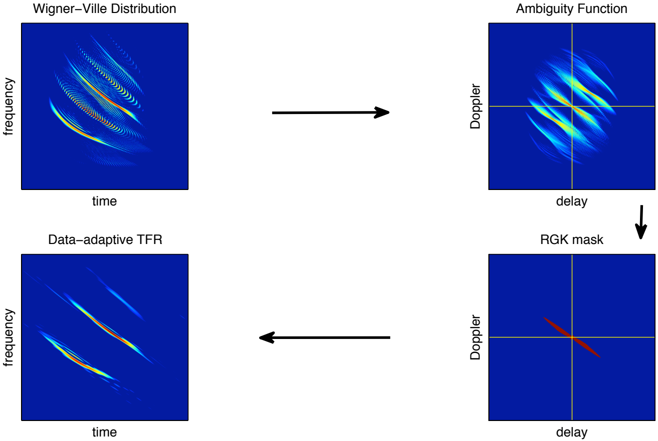
*The 2D Fourier transform $A_x(\xi, \tau)$ of the Wigner distribution $W_x(t, f)$ is referred to as the (narrowband) ambiguity function (AF)*

## Interpretation

*The TF-shift operator $(\mathbf{T}_{\xi,\tau} x)(t) := x(t-\tau)\, e^{-i2\pi\xi(t-\tau/2)}$ is such that $A_x(\xi, \tau) = \langle x, \mathbf{T}_{\xi,\tau} x \rangle \Rightarrow$ AF = TF correlation, with*

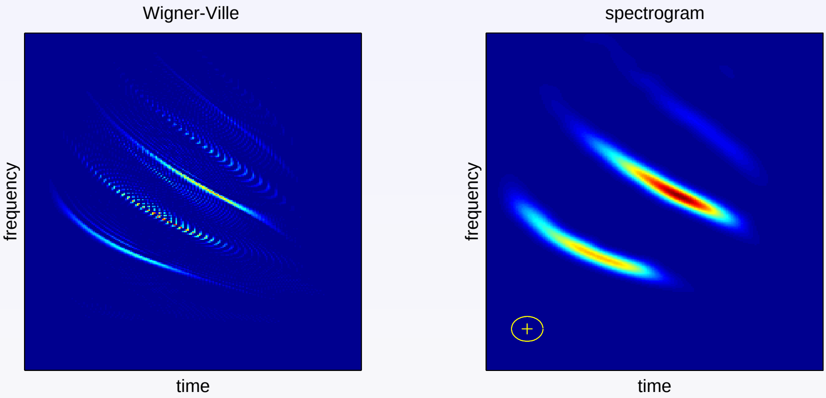
- *"auto-terms" neighbouring the origin of the plane*
- *"cross-terms" at a distance from the origin that equals the TF distance between components*

# the other trade-off and its “classical” way out

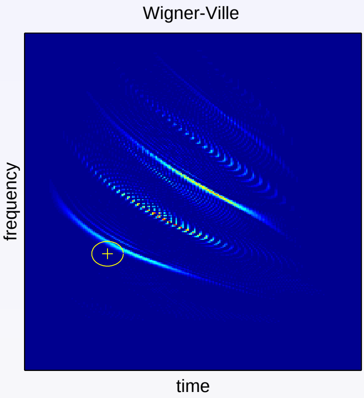


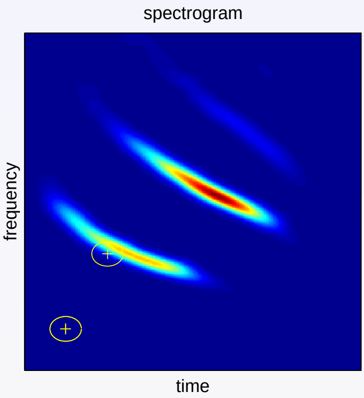
# from Wigner-Ville to spectrogram, and back

# spectrogram = smoothed Wigner

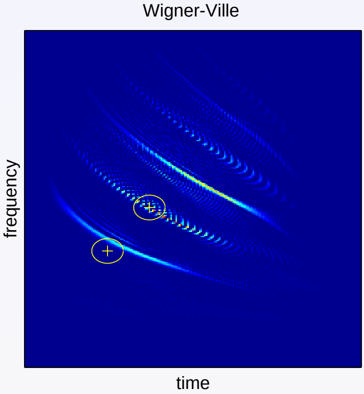


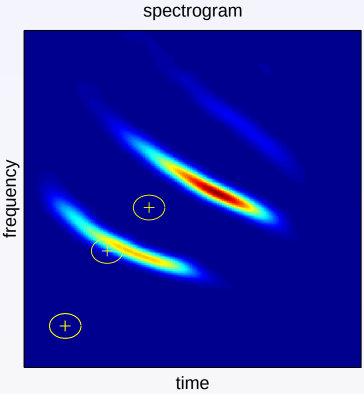
# spreading of auto-terms



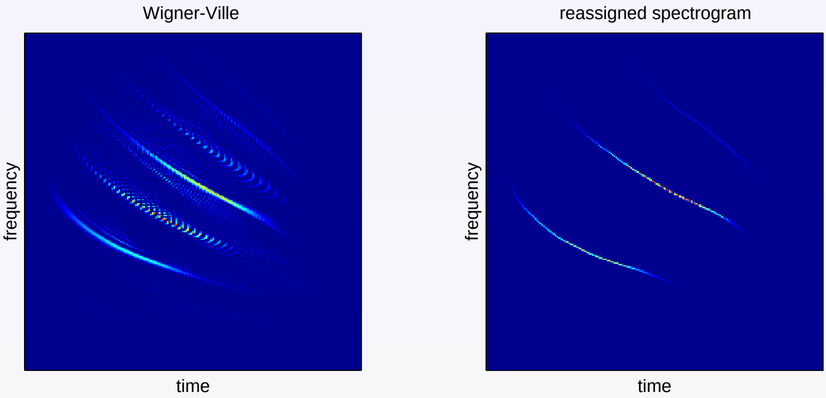



# cancelling of cross-terms

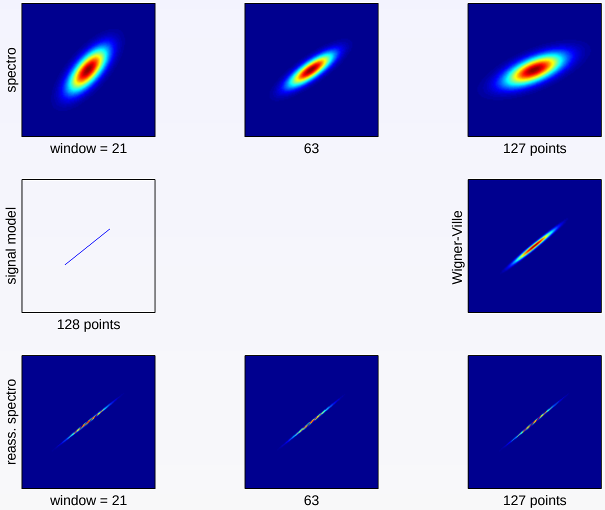


# reassignment (Kodera *et al.*, ’76, Auger & F., ’95)



# independence w.r.t. window size



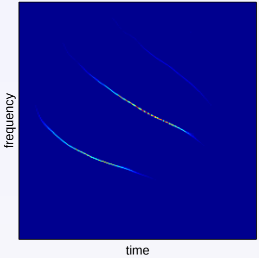
# a "compressed sensing" approach

**Discrete time**

*signal of dimension $N \Rightarrow$ TF distribution of dimension $\approx N^2$*

**Few components**

*$K \ll N \Rightarrow$ at most $KN \ll N^2$ non zero values in the TF plane*




**Sparsity**

*minimizing the $\ell_0$ "norm" not feasible, but almost optimal solution by minimizing the $\ell_1$ norm*

# a "compressed sensing" approach"

**Idea (F. & Borgnat, '08-10)**

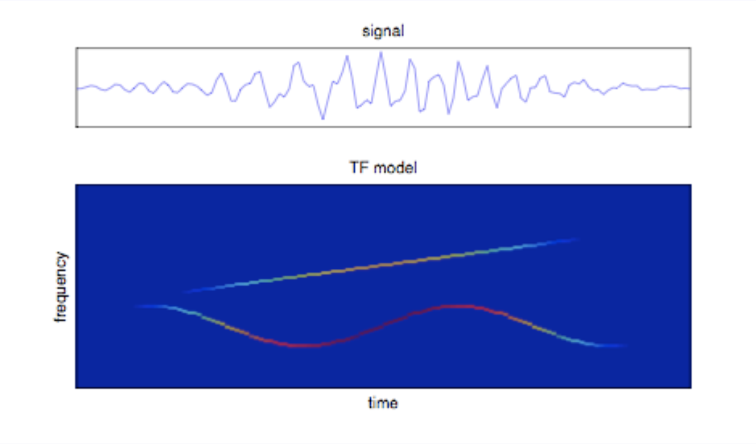
1. *choose a domain* $\Omega$ *neighnouring the origin of the AF plane*
2. *solve the program*

$$\min_{\rho} \|\rho\|_1 \; ; \; \mathcal{F}\{\rho\} - A_x = 0|_{(\xi,\tau)\in\Omega}$$

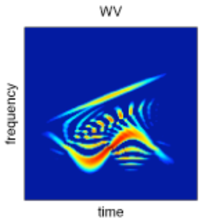
3. *the exact equality over* $\Omega$ *can be relaxed to*

$$\min_{\rho} \|\rho\|_1 \; ; \; \|\mathcal{F}\{\rho\} - A_x\|_2 \leq \epsilon|_{(\xi,\tau)\in\Omega}$$

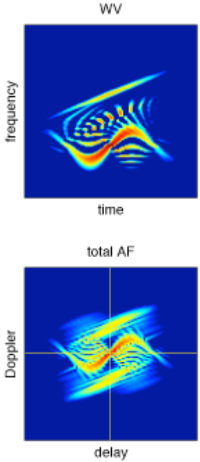
# a toy example



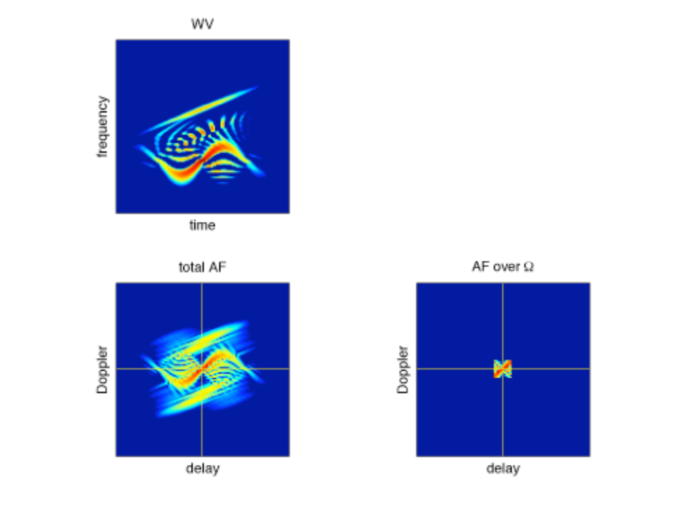
# Wigner



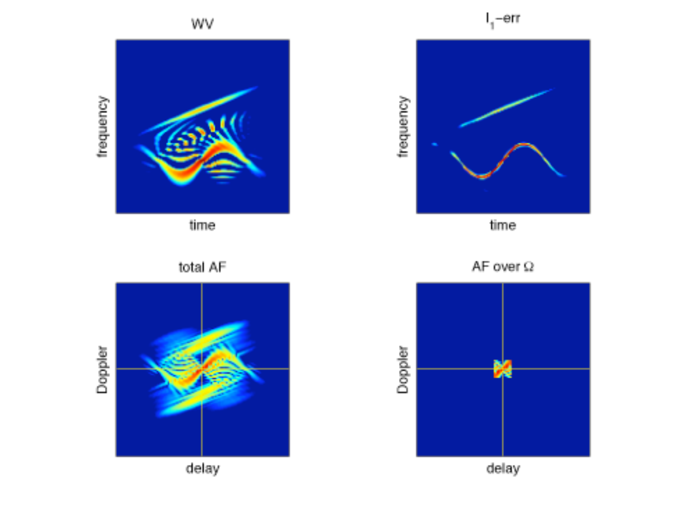
# ambiguity



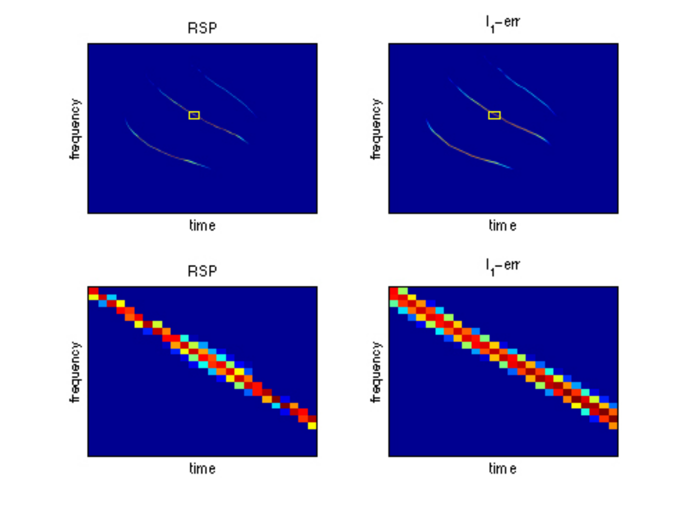
# selection


WV
frequency
time
total AF
Doppler
delay
AF over Ω
Doppler
delay

# sparse solution



# comparison sparsity vs. reassignment



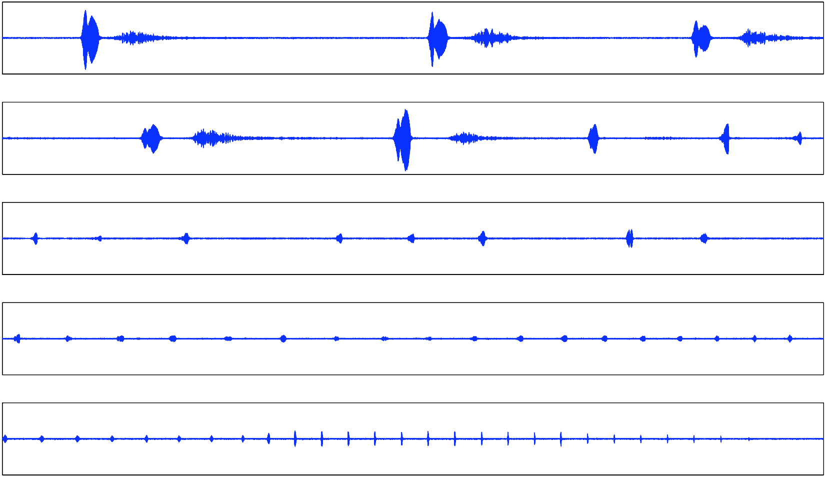
1. **Biology** — Bats echolocation calls
2. **Physics** — Gravitational waves
3. **Mathematics** — Riemann and Weierstrass functions

# bat sonar system

## Observation

- *Echolocation* — *Active system for navigation, natural airborne sonar*
- *Signals* — *Ultrasonic acoustic waves, transient (some ms) and wideband (some tens of kHz between 40 and 100kHz) chirp signals*
- *Performance* — *Close to optimality, with adaptation of the waveforms to multiple tasks (detection, estimation, recognition, interferences rejection,. . . )*

# a typical pursuit sequence (*Myotis mystacinus*)



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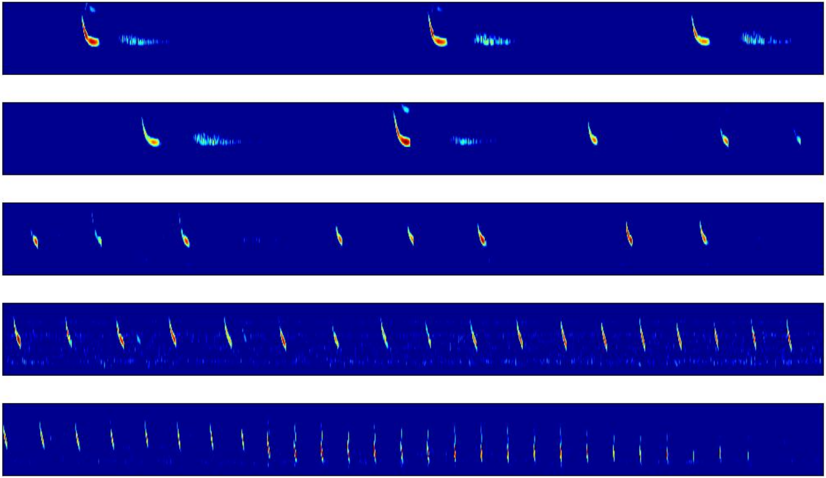

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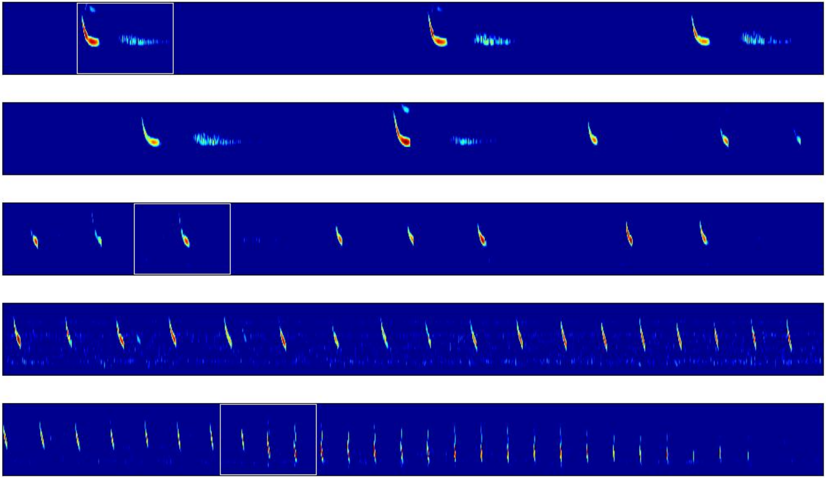
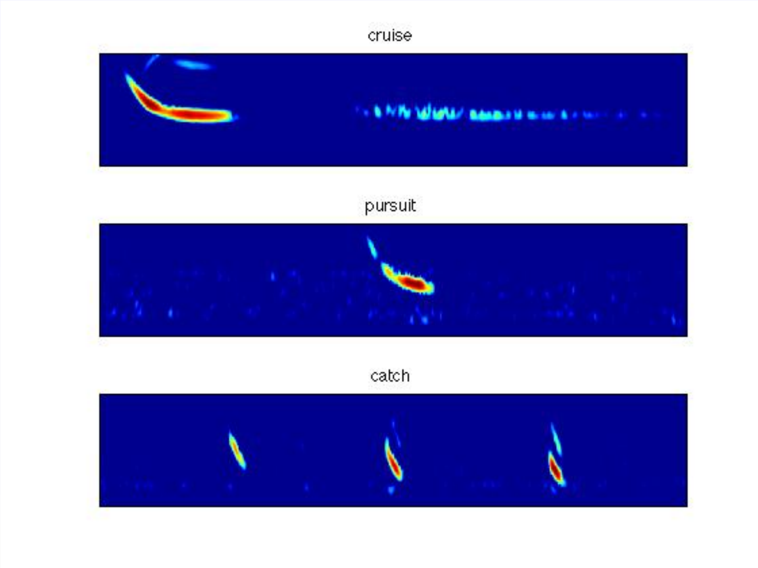
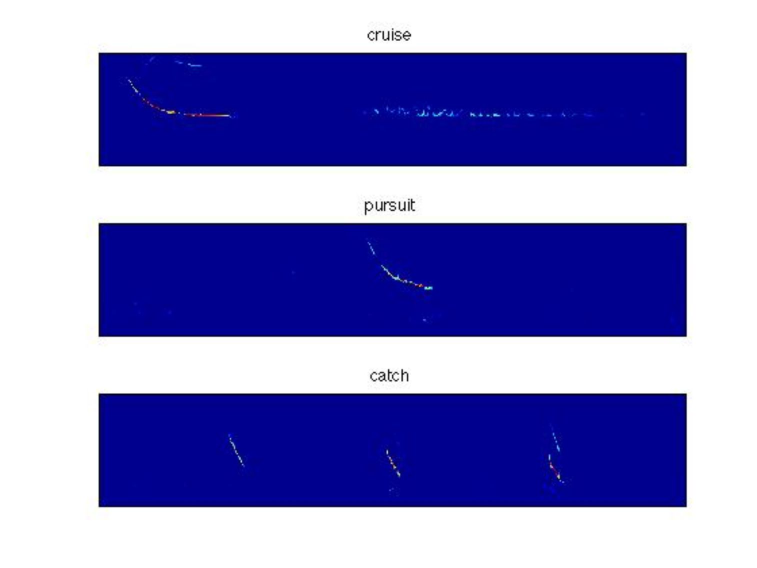


# close-up (spectrogram)



# close-up (reassigned spectrogram)



# bats and signal processing

## Aim

*Understand the signal design of bat echolocation calls:*

1. *evolution within a sequence?*
2. *adaptation to environment?*
3. *optimality if any?*
4. ...

## Result

*For an emitted signal of duration $T$ and bandwidth $B$, the accuracy in estimating the Doppler shift and the delay of the returning echo is roughly given by* $\delta f \sim T^{-1}$ *and* $\delta t \sim (B\sqrt{SNR})^{-1}$ *(Woodward's formula)*

# bats and signal processing

## Observation

1. ***Cruise*** *— Importance of estimating both distance (delay) and speed (Doppler)*
   $\Rightarrow$ *broadband chirp + almost Constant Frequency part*
2. ***Pursuit*** *— Importance of estimating distance whatever the Doppler*
   $\Rightarrow$ *adapted chirp + progressive suppression of the almost Constant Frequency part*
3. ***Catch*** *— Importance of precise localization with shorter pulses*
   $\Rightarrow$ *increasing the effective bandwidth B by lowering the fundamental and increasing distortion (harmonics)*

# chirp detection/estimation

## Interpretation (time-frequency)

1. *Matched filtering — Emitted signal $s(t)$ as a template + echo $e(t)$ as a delayed version of $s(t)$ embedded in wGn* $\Rightarrow$ *optimal estimation of delay by* $\hat{\tau} = \arg\max_\tau |\langle e, \mathbf{T}_\tau s\rangle|^2$
2. *Unitarity — Inner product equivalence* $\Rightarrow$ *(Moyal's formula)* $|\langle x, y\rangle|^2 = \iint \rho_x(t,f)\,\rho_y(t,f)\,dt\,df$
3. *Localization — Energy along instantaneous frequency* $f_s(t)$ $\Rightarrow \hat{\tau} = \arg\max_\tau \int a_s^2(t-\tau)\,\rho_e(t, f_s(t-\tau))\,dt$ *path integration in the time-frequency plane*

# optimal integration (Wigner-Ville)

# approximation (reassigned spectrogram)

# robustness

# Doppler tolerance

## Problem

*Coupled errors in the joint estimation of delay and Doppler*

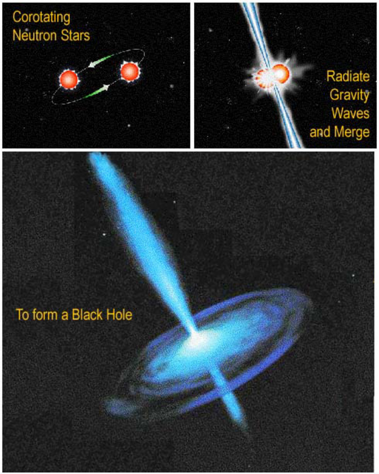
## Way out (Altes & Titlebaum, '70)

- *A waveform is said to be Doppler-tolerant if it permits an ubiased estimation of delay whatever the (unknown) Doppler*
- *For broadband signals, Doppler has to be considered as a dilation (shift = approximation for narrowband signals)*
- *Analytic solution* $=$ *hyperbolic chirps*

# Doppler tolerance

## Graphical solution

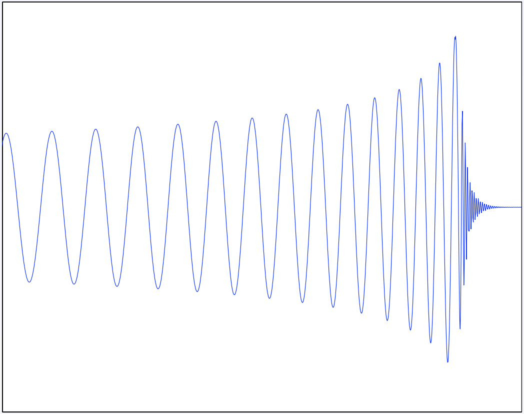
# coalescing binaries



# coalescing binaries

# expected waveform

gravitational wave



# the quest

**Challenge**

*First direct proof on Earth of the existence of gravitational waves (GWs), as predicted by general relativity*

- *Projects* — VIRGO (France-Italy) + LIGO (USA): giant Michelson interferometers ($\sim$ 3 km long arms)
- *Measurements* — GWs impinging the interferometer modify locally the space-time geometry and result in a differential variation of the arms length $\Rightarrow$ interference fringes
- *Difficulties* — Signals are very weak and can be efficiently observed only in a very short time window (some seconds) corrresponding to the frequency window above $\sim$ 10 Hz (seismic noise) and below $\sim$ 1 kHz (photon noise)

# GW detection/estimation

## Model

*The inspiral part of the GW radiated by a coalescing binary made of two objects of respective masses $m_1$ and $m_2$ can be modelled as a power-law chirp*

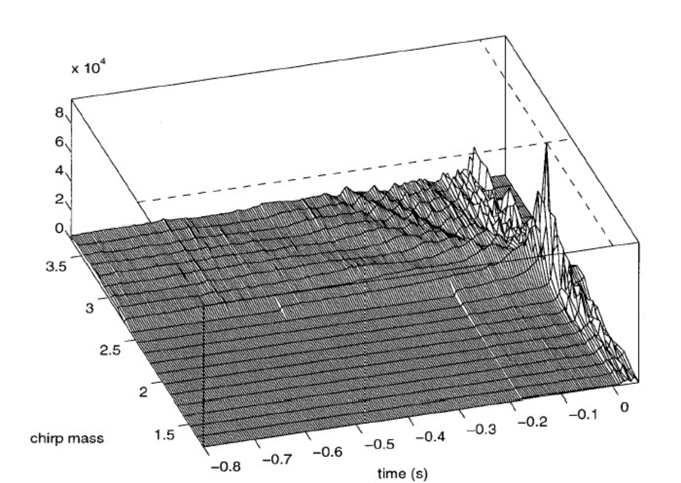
$$C_{\alpha,\beta}(t) = a\,(t_c - t)^{\alpha}\, \exp\{i(b\,(t_c - t)^{\beta} + c)\}\, U(t_c - t),$$

*with* $(\alpha, \beta) = (-1/4, 5/8)$

## Parameters

1. *the coalescence time* $t_c$
2. *the chirp mass defined as* $\mathcal{M} = (m_1 + m_2)^{2/5}(m_1^{-1} + m_2^{-1})^{-3/5}$*, and related to the "chirp rate" $b$ according to* $b \approx 38.6\,(\mathcal{M}/M_{\odot})^{-5/8}$*, where* $M_{\odot}$ *stands for the solar mass*

# reassigned spectrogram + path integration




(Chassande-Mottin & F., *ACHA* '98)

Bernhard Riemann (1826-1846)

# a very special function

## Definition

$$\sigma(t) := \sum_{n=1}^{\infty} n^{-2} \sin \pi n^2 t$$

## Result

*$\sigma(t)$ non differentiable if $t \neq t_0 = (2p+1)/(2q+1)$, $p, q \in \mathbb{N}$ (Hardy, '16) but differentiable in $t = t_0$ (Gerver, '70)*

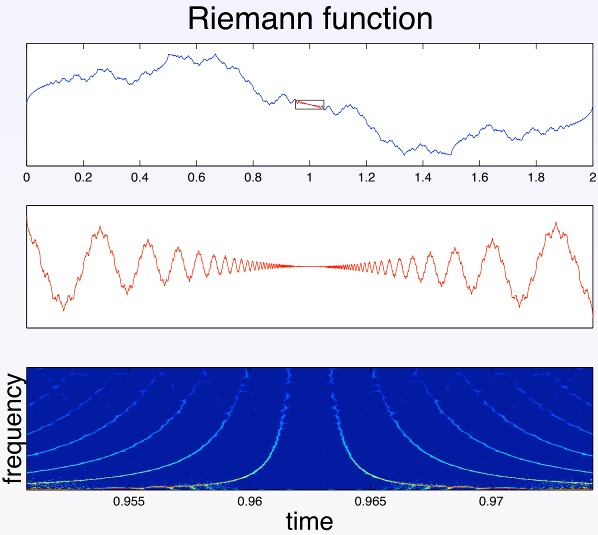
## Theorem (Meyer, '96)

*In the vicinity of $z = 1$, the holomorphic version of Riemann's function can be expressed as a combination of local chirps:*

$$\sigma(1+z) = \sigma(1) - \pi z/2 + \sum_{n=1}^{\infty} K_n(z)\, C_{3/2,-1}(z),$$

*leading to $\sigma(1+t) = \sigma(1) - \pi t/2 + O(|t|^{3/2})$ when $t \to 0$*

# power-law chirps



# from Fourier to Mellin

## Definition

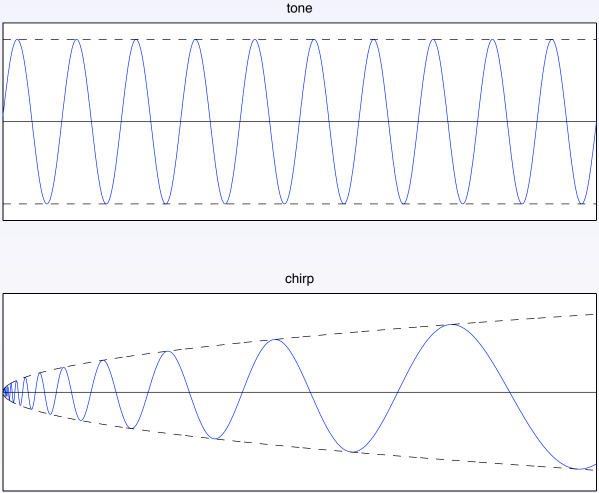
*The Mellin Transform (MT) of a signal* $x(t) \in L^2(\mathbb{R}^+, t^{-2\alpha+1}dt)$ *can be defined as the projection:*

$$(\mathcal{M}x)(s) := \int_0^{+\infty} x(t)\, t^{-i2\pi s-\alpha}\, dt =: \langle x, c\rangle$$

## Interpretation

1. *Analysis over hyperbolic chirps* $c(t) := t^{-\alpha} \exp\{i2\pi s \log t\}$
2. $\dot{\varphi}_c(t)/2\pi = s/t \Rightarrow$ *the Mellin parameter* $s$ *can be interpreted as a hyperbolic modulation rate*
3. *The MT can also be viewed as a warped FT, since* $\tilde{x}(t) := e^{(1-\alpha)t}\, x(e^t) \Rightarrow (\mathcal{M}x)(s) = (\mathcal{F}\tilde{x})(s)$

# Mellin as warped Fourier



Karl Weierstrass (1815-1897)

# another very special function

## Definition (Weierstrass, 1872)

*Geometrically spaced Fourier modes*

$$W(t) := \sum_{n=0}^{\infty} \lambda^{-nH} \cos \lambda^n t, \lambda > 1$$

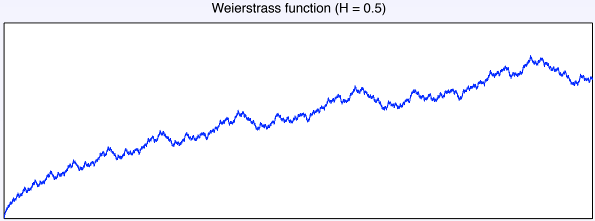
## Definition (Mandelbrot, 1977)

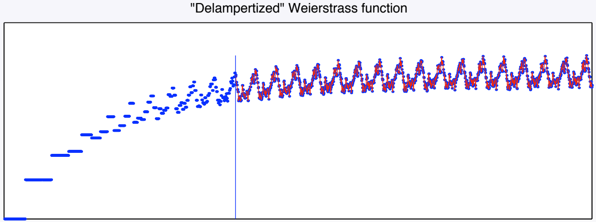
$$W_g(t) := \sum_{n=0}^{\infty} \lambda^{-nH} \left(g(0) - g(\lambda^n t)\right) e^{i\varphi_n}, \lambda > 1,$$

*with* $g(\cdot)$ $2\pi$*-periodic and* $\varphi_n \in \mathcal{U}(0, 2\pi)$

see also (Berry & Lewis, *Proc. Roy. Soc. London A, 1980)*

# “de-warping” (Lamperti, ’62)



# Weierstrass meets Mellin

## Result (Borgnat & F., '03)

*The Weierstrass-Mandelbrot admits the equivalent Mellin decomposition:*

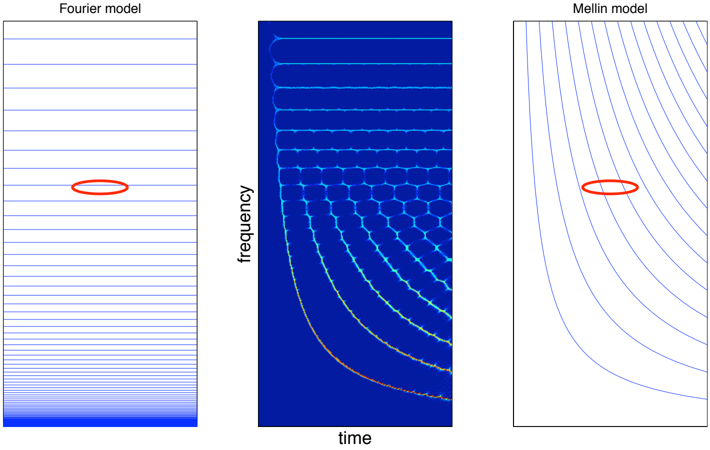
$$W_g(t) = \sum_{m=-\infty}^{\infty} \frac{(\mathcal{M}_H G)(m/\log\lambda)}{\log\lambda}\, m_{H,m/\log\lambda}(t),$$

*with* $G(t) := g(0) - g(t)$

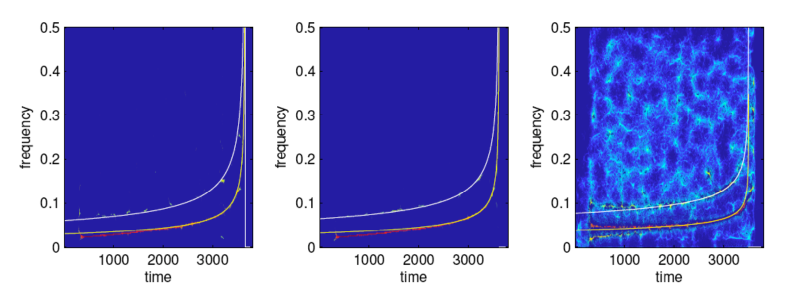
## Interpretation

*Natural co-existence of two readings (Fourier and Mellin) in the time-frequency plane*

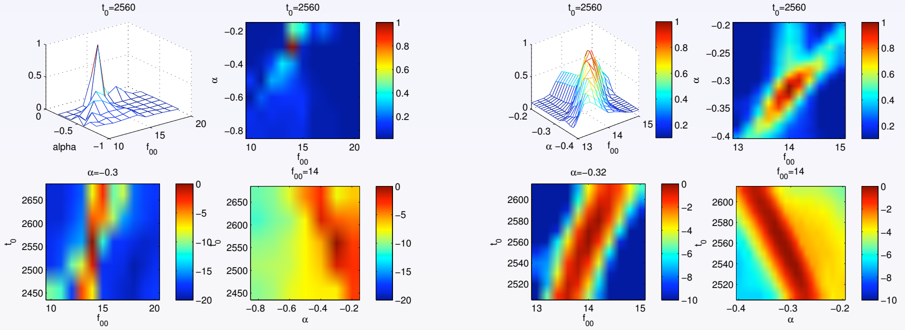
# Fourier vs. Mellin



# back to Euler



# Hough transform



# chirps “everywhere”

- bird songs
- ocean waves
- “whistling atmospherics"
- wideband impulses propagating in a dispersive medium
- vibroseismics
- EEG (epileptic seizure)
- uterine EMG
- coherent structures in turbulence
- precursors accumulation in earthquakes
- “speculative bubbles" prior a financial crash
- …

# (p)reprints, Matlab codes & contact

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