# TUTORIAL 1

## 1 Tensor Products

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$:

1. Compute $A \otimes B$ and $B \otimes A$.
2. Let $A, B, C, D$ matrices, $x, y$ vectors and $\lambda$ a complex number. For the following statements, say if it is true or false and prove it:
   (a) $A \otimes (\lambda B) = (\lambda A) \otimes B$
   (b) $A \otimes (BC) = (AB) \otimes C$
   (c) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$
   (d) If $A$ and $B$ are diagonalizable, then $A \otimes B$ is diagonalizable
   (e) $(A \otimes B)^\dagger = B^\dagger \otimes A^\dagger$
   (f) $(A + B) \otimes C = A \otimes C + B \otimes C$
   (g) $(A \otimes B)(x \otimes y) = (Ax) \otimes (By)$

## 2 Measurements and Probabilities

For all $|\varphi\rangle$ and $\mathcal{B}$, justify that $|\varphi\rangle$ is a state, $\mathcal{B}$ a basis, then measure $|\varphi\rangle$ in $\mathcal{B}$ and give the probability of each outcome:

1. $|\varphi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $\mathcal{B} = (|0\rangle, |1\rangle)$.
2. $|\varphi\rangle = \frac{|0\rangle-i|1\rangle}{\sqrt{2}}$, $\mathcal{B} = (|0\rangle, |1\rangle)$.
3. $|\varphi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $\mathcal{B} = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$.
4. $|\varphi\rangle = \frac{|0\rangle-i|1\rangle}{\sqrt{2}}$, $\mathcal{B} = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$.
5. $|\varphi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$, $\mathcal{B} = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$.
6. $|\varphi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$, $\mathcal{B} = \left(i|00\rangle, |01\rangle, -|10\rangle, e^{i\frac{\pi}{4}}|11\rangle\right)$.
7. $|\varphi\rangle = \frac{|00\rangle+(1+i)|01\rangle+|11\rangle}{2}$, $\mathcal{B} = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$.
8. $|\varphi\rangle = \frac{|00\rangle+(1+i)|01\rangle+|11\rangle}{2}$, $\mathcal{B} = \left(|00\rangle, |01\rangle, \frac{|10\rangle+|11\rangle}{\sqrt{2}}, \frac{|10\rangle-|11\rangle}{\sqrt{2}}\right)$.

## 3 Gates

1. Construct a SWAP-gate, that is to say a quantum circuit $U$ such that for any $a, b \in \{0, 1\}$ is such that $U \cdot |a\rangle|b\rangle \mapsto |b\rangle|a\rangle$, using CNOT gates.
   *Hint: how do you swap two digits using only XOR gate in the classical case?*

2. Suppose you are given access to a **bit-query**-oracle for a function $f : \{0, 1\}^n \to \{0, 1\}$, that is to say a gate $U_f$ such that $\forall a \in \{0, 1\}^n, \forall b \in \{0, 1\}, U_f|a\rangle|b\rangle = |a\rangle|b \oplus f(a)\rangle$.
   Construct a circuit giving acces to a **phase-query**-oracle, that is to say construct a circuit $U'_f$ such that for any $(a, b) \in \{0, 1\}^n \times \{0, 1\},\ U'_f \cdot |a\rangle|b\rangle = (-1)^{b \cdot f(a)}|a\rangle|b\rangle$.

## 4 Partial Measurements

Let $|\varphi\rangle, |\psi\rangle$ two normalized quantum states, let $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\varphi\rangle + |1\rangle|\psi\rangle)$, suppose we apply $H$ to the first qubit, then measure that qubit in the computational basis. Give the probability of measurement $1$ as a function of $|\varphi\rangle$ and $|\psi\rangle$.