Computer Science and Signal Processing C. Shannon: from LP to the MP3 standard

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"Applications of computer science to research and technological development"
École doctorale de Mathématiques et Informatique Fondamentale de Lyon. June 2007

Outline

- Historical milestones
- Digitalizing continuous data
- Digitalizing continuous operators

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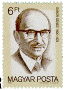
Historical milestones in (digital) SP (non exhaustive list)

1822. Fourier transform *Joseph*FOURIER

1925-1927. Uncertainty principle *Hermann*Weyl *Werner*HEISENBERG



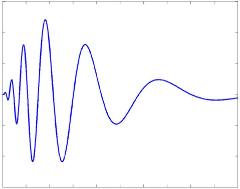
1946. Time-Frequency principle *Dennis*GABOR

1949. Sampling theory *Claude*SHANNON

1970 Continuous wavelets *Jean*MORLET *Alexander*GROSSMANN

1980. Orthogonal wavelet bases *Ingrid*DAUBECHIES

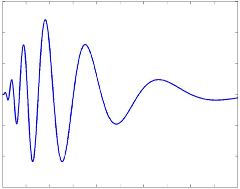
1980. Multiresolution analysis *Stéphane*MALLAT *Yves*MEYER



contains the “same” information as

1101001011010...

- sampling theory
- coding theory
- harmonic analysis
- information theory
- ...



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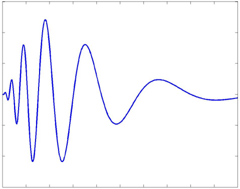

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Advantages – compression (up to thousands of LP’s in 2 Gbytes), disturbance immunity, rewritability, accessibility, portability

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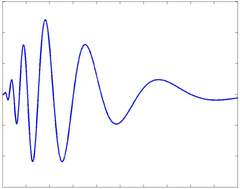

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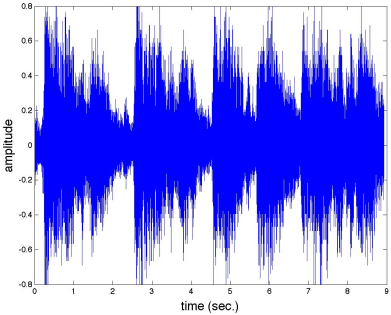
Advantages – compression (up to thousands of LP's in 2 Gbytes), disturbance immunity, rewritability, accessibility, portability

Drawbacks – information loss, quality loss (higher harmonics drop, dynamic squeeze, distorsion,...)

Fourier transform: a bridge between time and frequency



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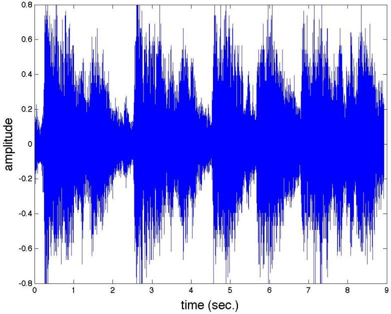

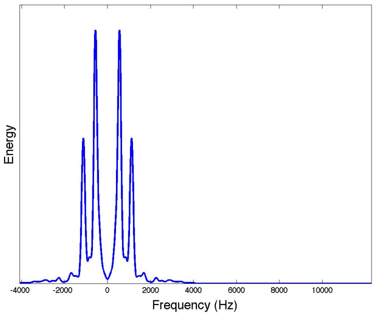

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Power Spectrum Density : $S(f) := |X(f)|^2$

analogic

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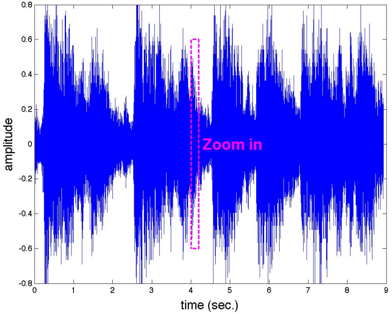

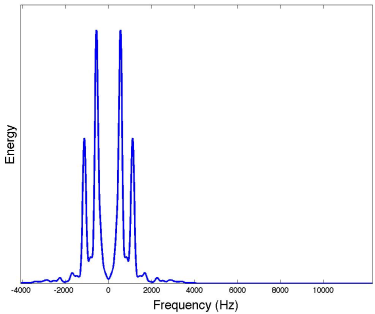

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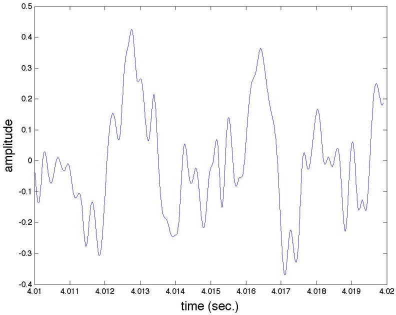

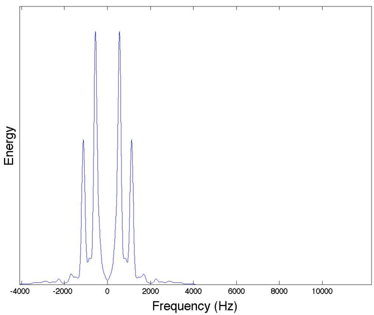

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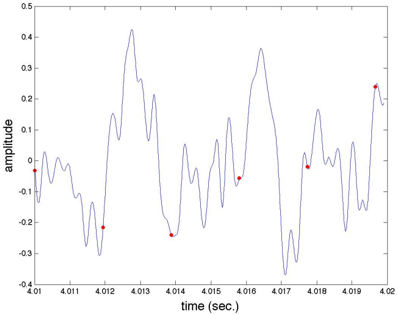

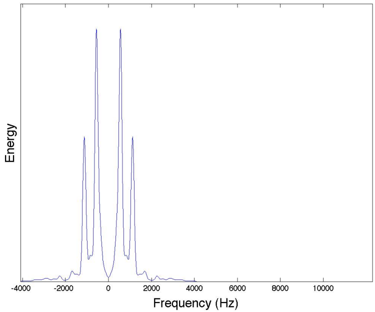

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analogic  512 Hz 

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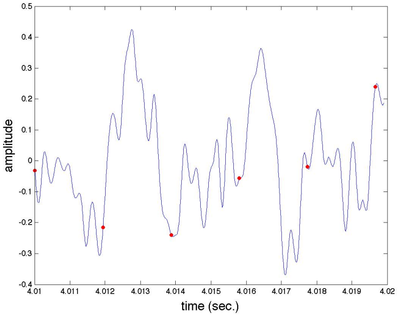

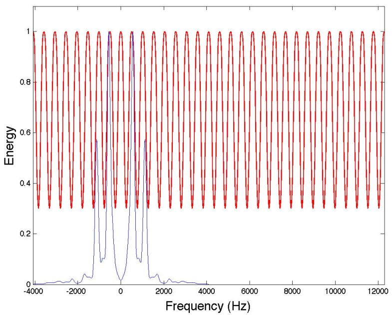

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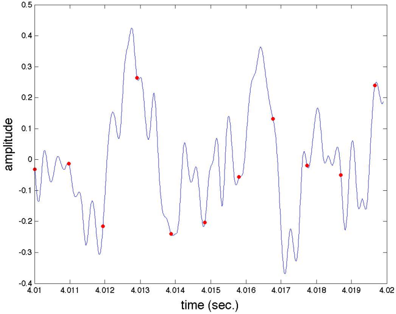

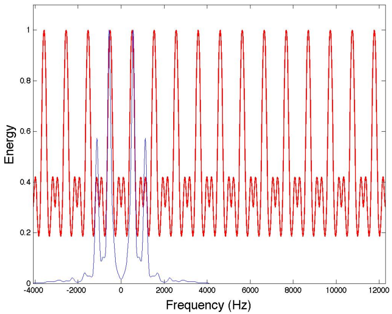

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analogic 512 Hz 1024 Hz

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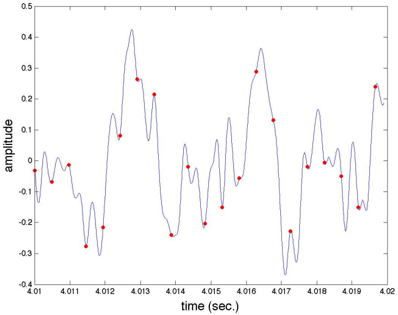

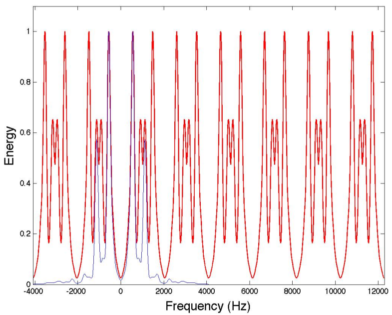

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analogic 512 Hz 1024 Hz 2048 Hz

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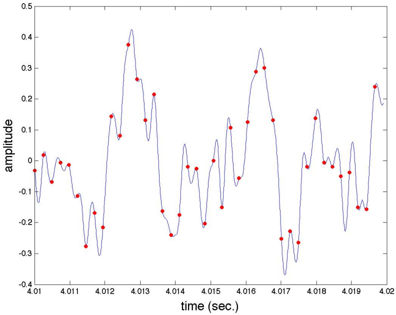

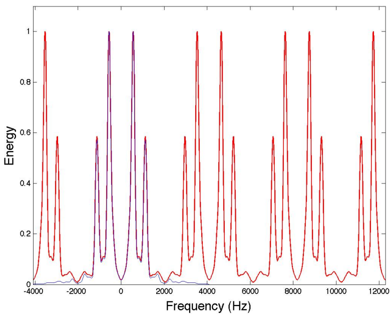

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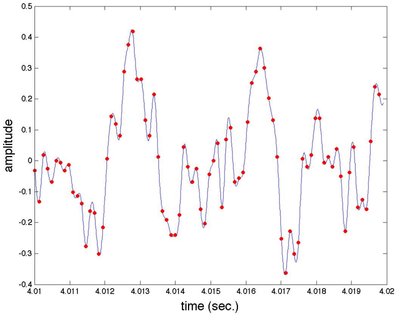

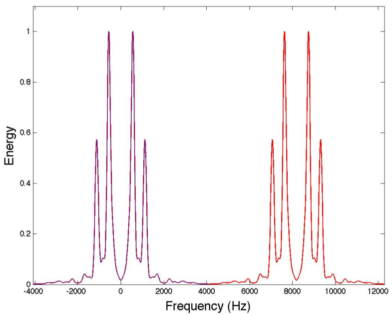

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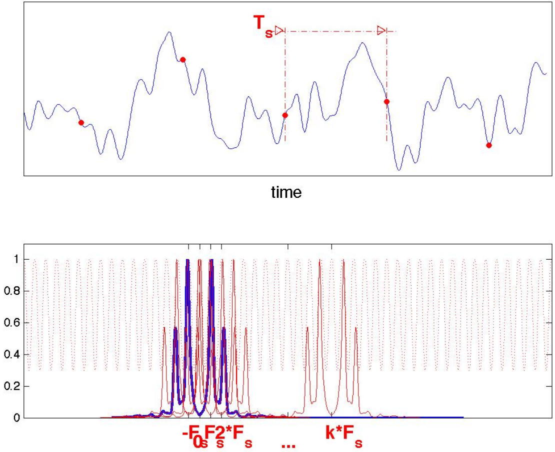
Discretization in time



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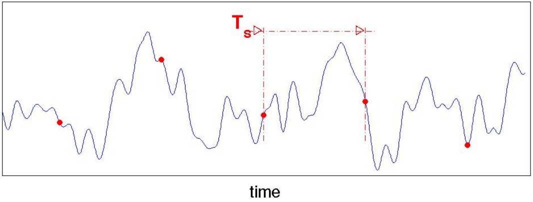

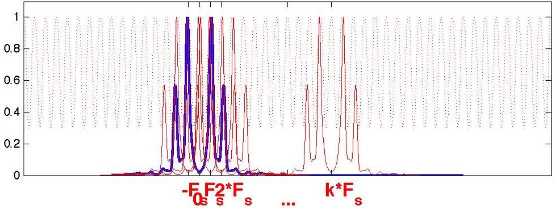

Discretizing:

$x[n] = x(nT_s),\ n \in \mathbb{Z}$

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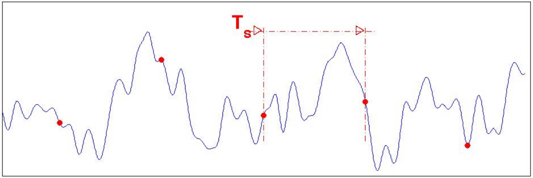

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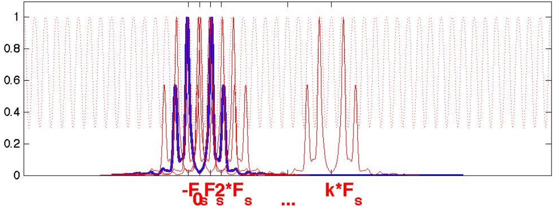

Periodizing:

$$\widetilde{X}(f) = \sum_{m=-\infty}^{m=\infty} X(f+mF_s)$$

The Poisson sum formula: $\widetilde{X}(f) = \sum_{n=-\infty}^{n=\infty} x(nT_s)\ \exp\{-i2\pi nT_s f\}$

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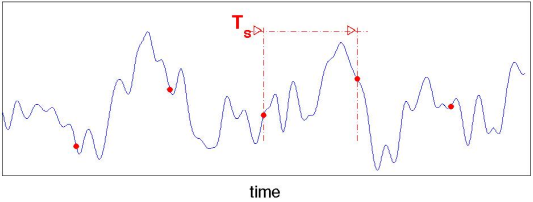

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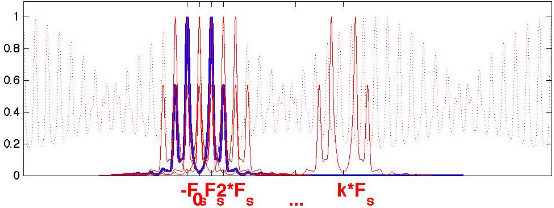

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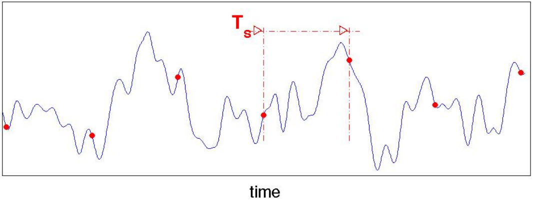

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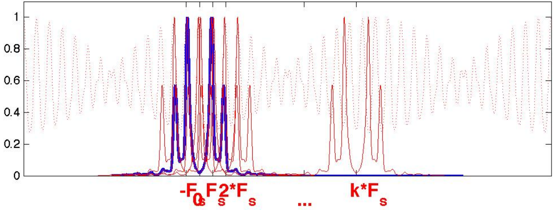

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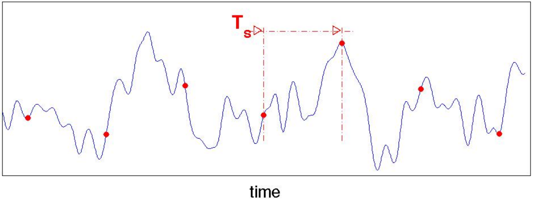

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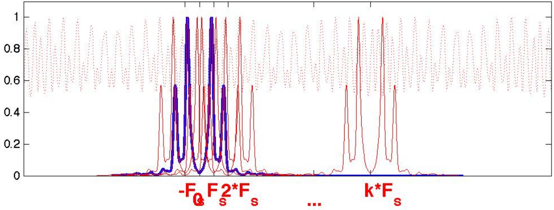

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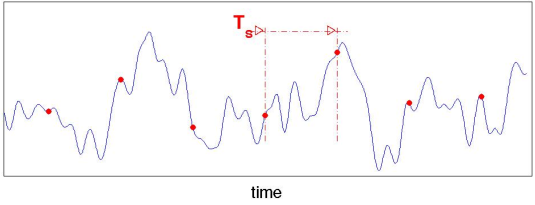

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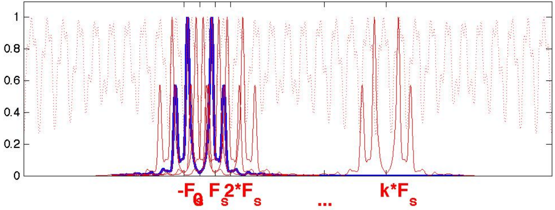

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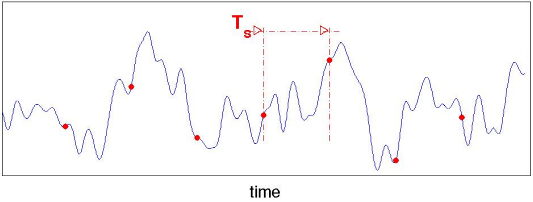

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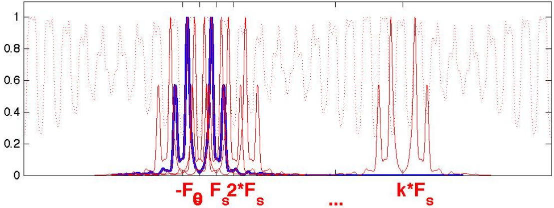

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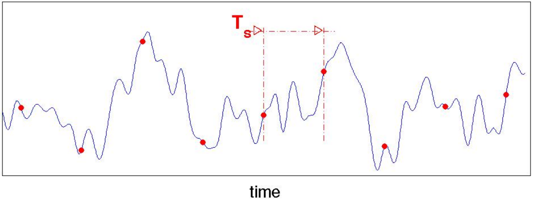

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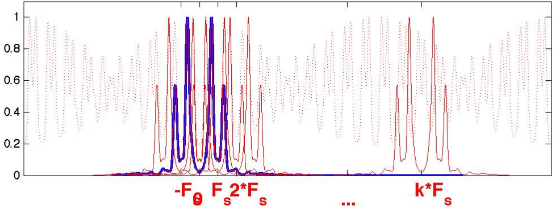

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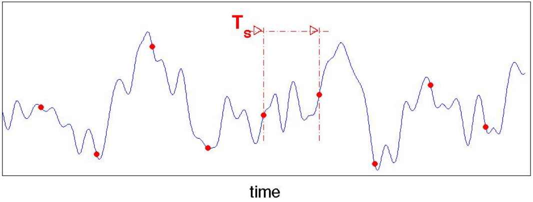

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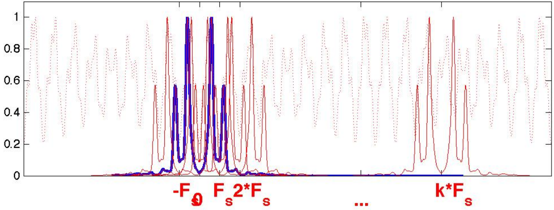

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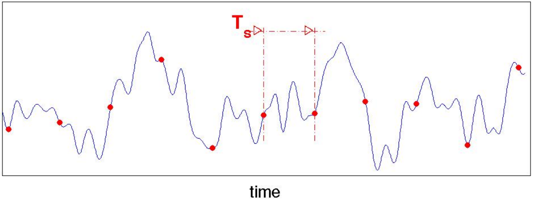

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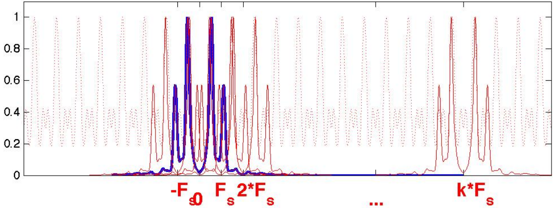

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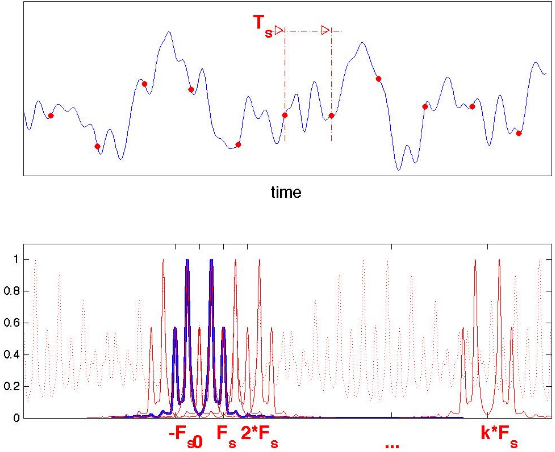

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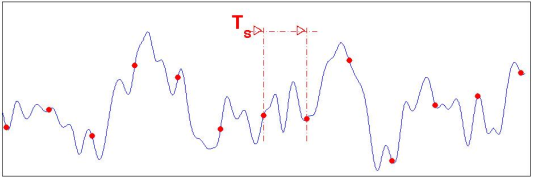

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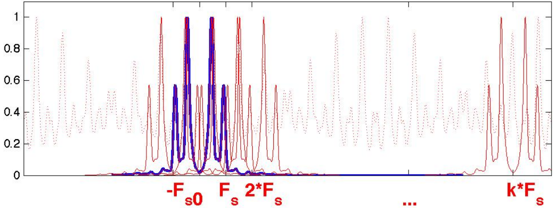

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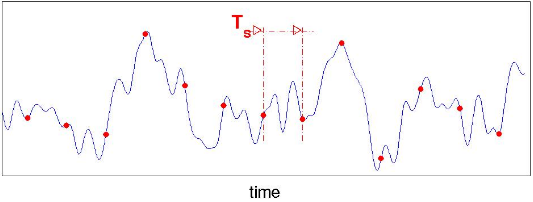

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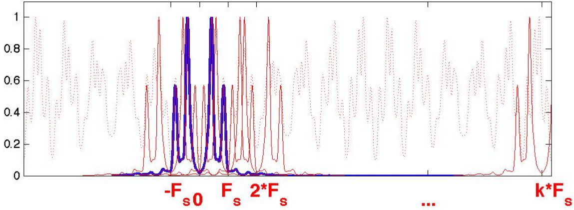

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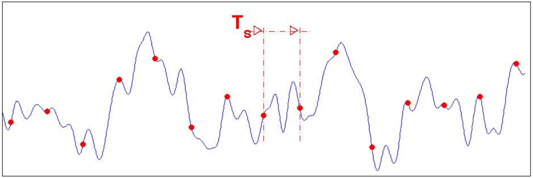

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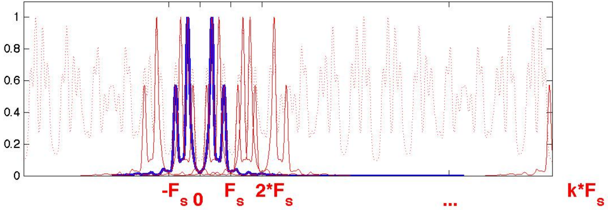

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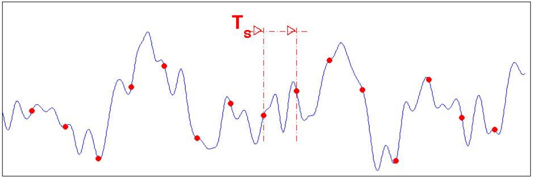

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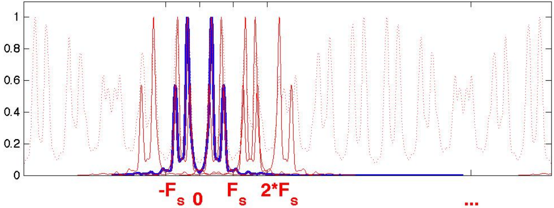

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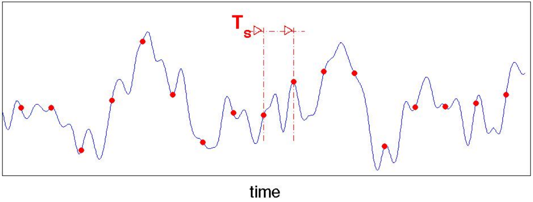

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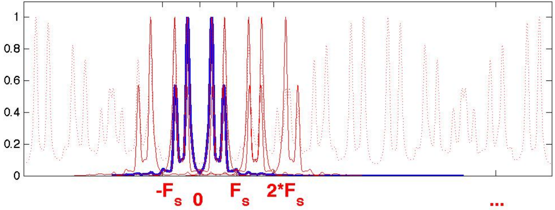

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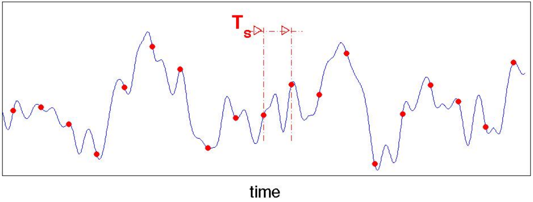

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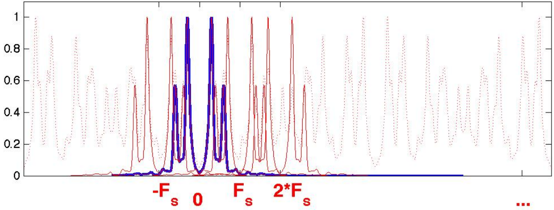

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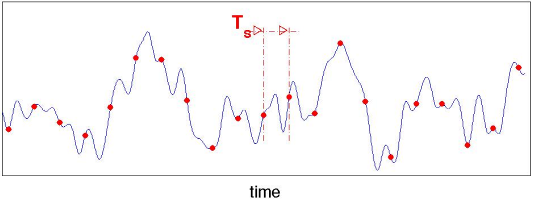

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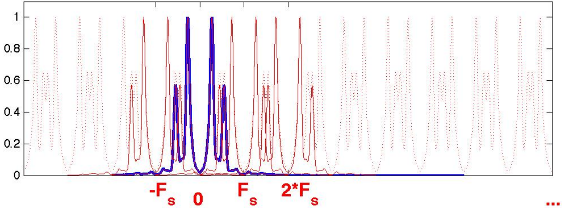

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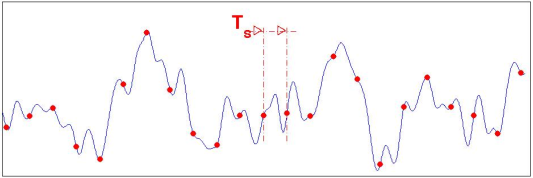

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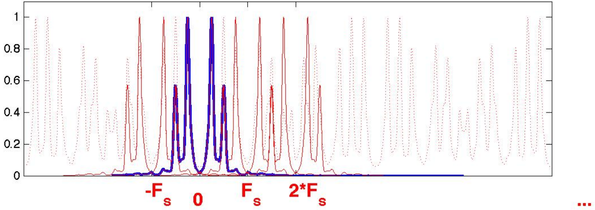

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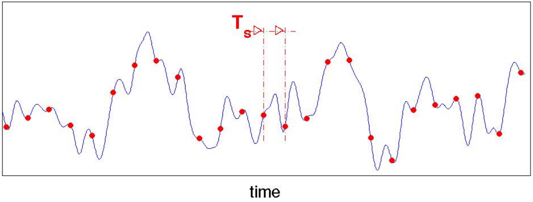

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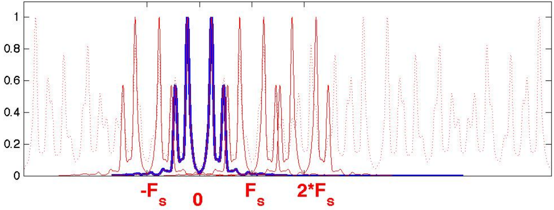

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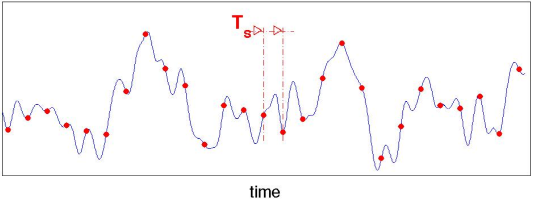

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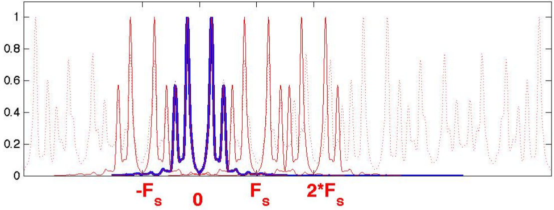

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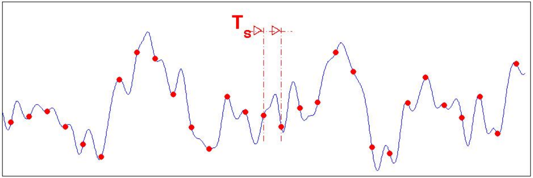

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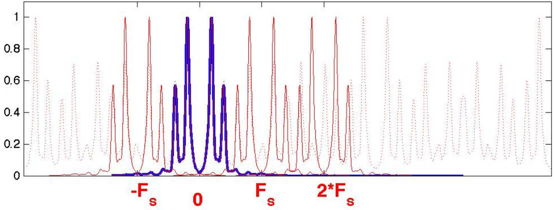

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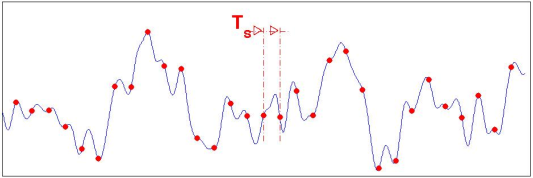

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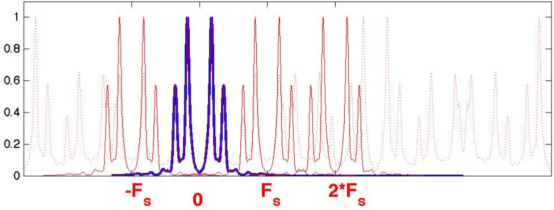

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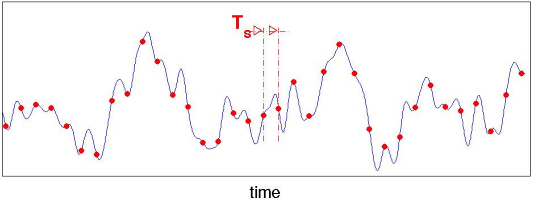

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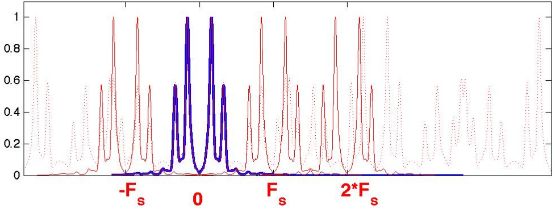

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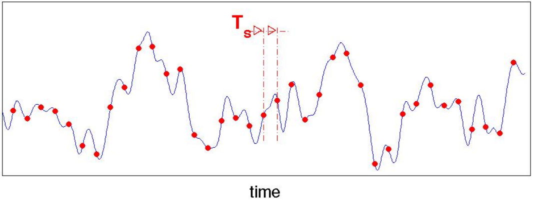

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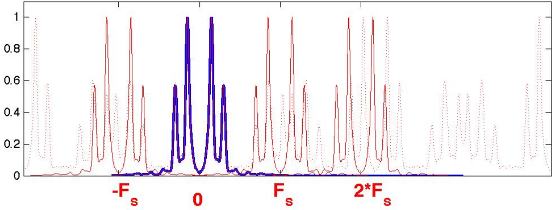

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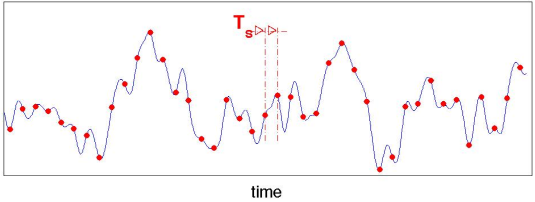

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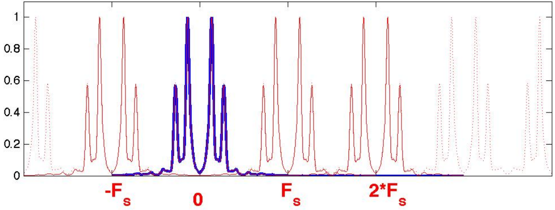

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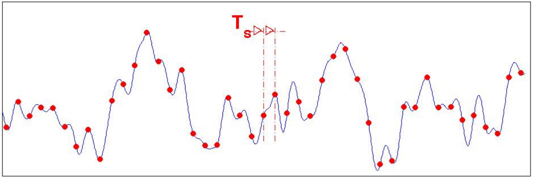

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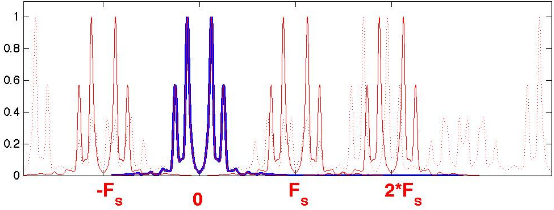

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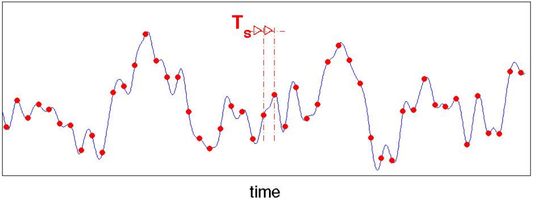

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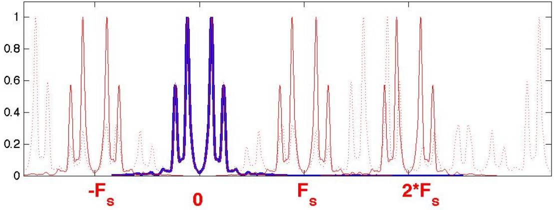

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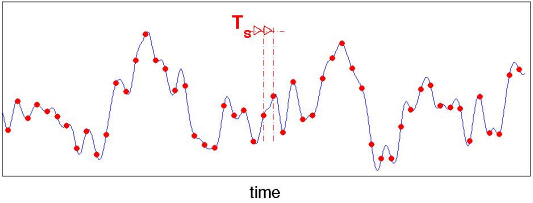

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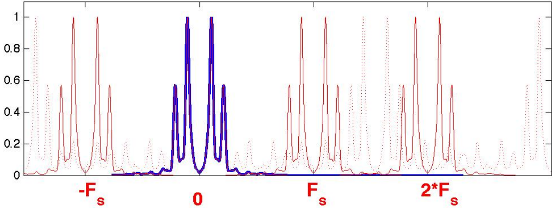

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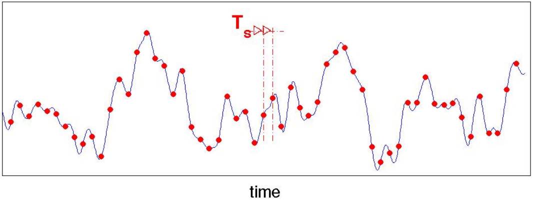

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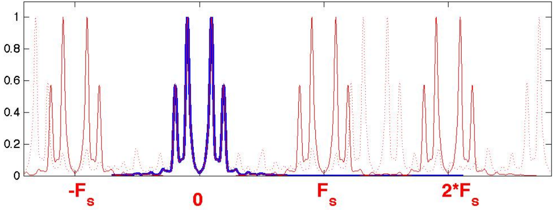

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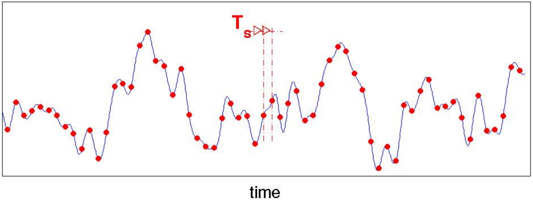

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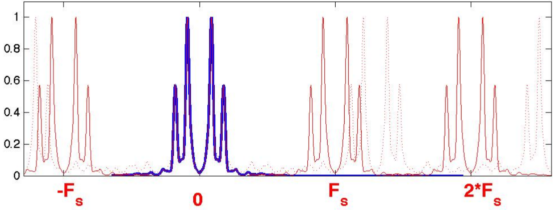

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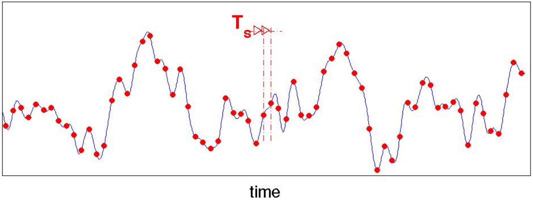

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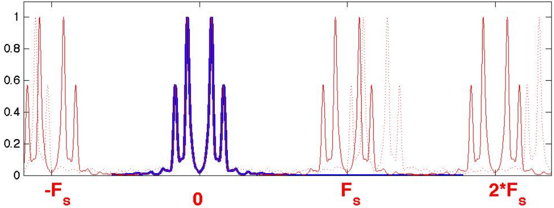

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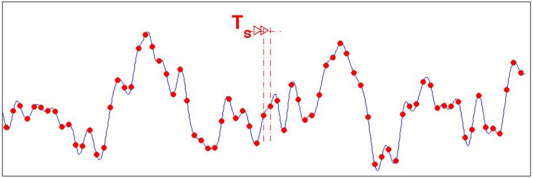

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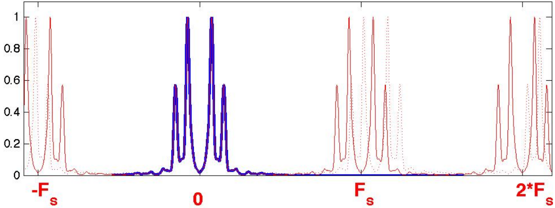

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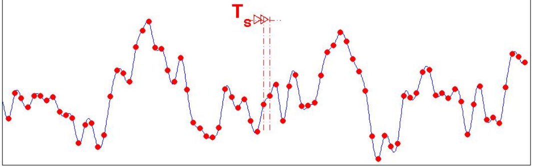

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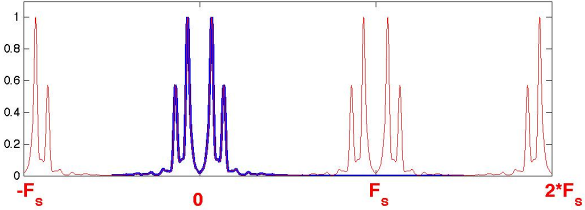

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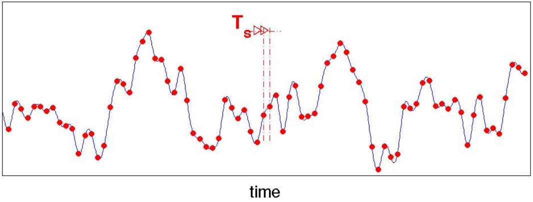

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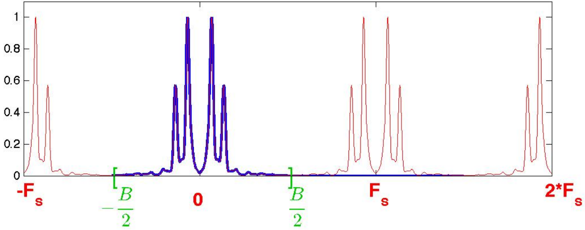

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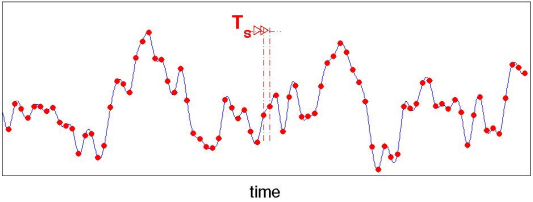

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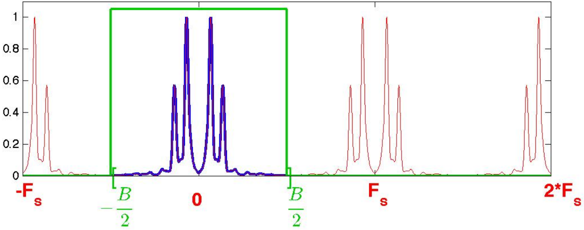

Filtering:

$$X(f) = X(f) \prod_{\left[-\frac{B}{2}, \frac{B}{2}\right]}(f)$$

iff $\mathbf{F_s} \geq \mathbf{B}$

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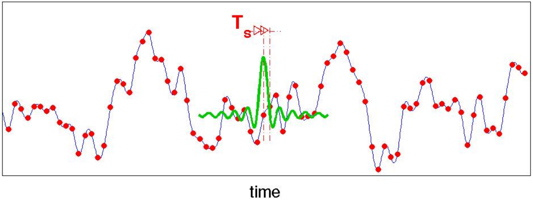
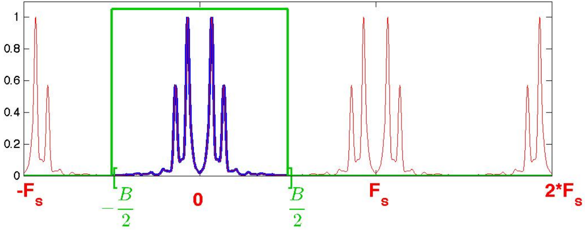


Interpolating:

$$x(t) = \sum_n x[n] \frac{\sin \pi B t}{\pi t}$$



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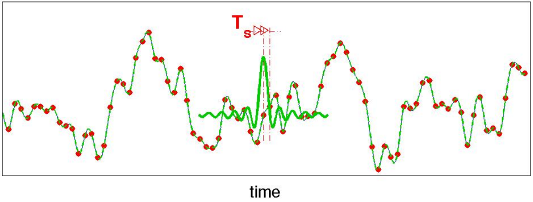

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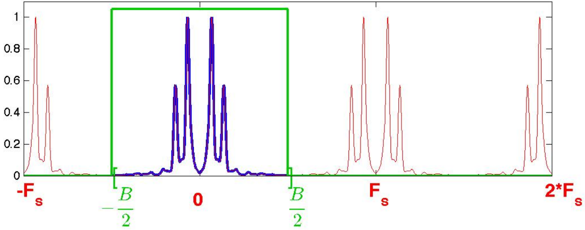

Filtering:

$$X(f) = X(f) \prod_{\left[-\frac{B}{2}, \frac{B}{2}\right]}(f)$$

iff $\mathbf{F_s} \geq \mathbf{B}$

The Poisson sum formula: $\widetilde{X}(f) = \sum_{n=-\infty}^{n=\infty} x(nT_s) \exp\{-i2\pi n T_s f\}$

Discretization in frequency

By isometry of Fourier transform, similarly in frequency:

$$\widetilde{x}(t) = \sum_{m=-\infty}^{m=\infty} X(m\Omega_s) \exp\{i2\pi m\Omega_s t\} \quad \text{and} \quad \widetilde{x}(t) = \sum_{m=-\infty}^{m=\infty} x\left(t + m\Omega_s^{-1}\right)$$

$x(t)$, $t \in [0, T[$ and $X(f)$, $f \in [-B/2, B/2[$ recoverable from $\widetilde{x}(t)$ and $\widetilde{X}(f)$ respectively

$$\text{iff} \begin{cases} T_s \cdot B \leq 1 \\ \Omega_s \cdot T \leq 1 \end{cases}$$ Sampling Shannon theorem

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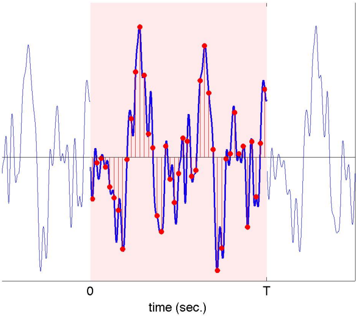
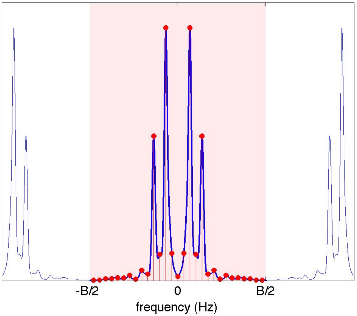


$\longrightarrow$
$\longleftarrow$



Discrete Fourier Transform

Combining both discretized representations yields the discrete Fourier transform:

$$X[m] = \sum_{n=0}^{N-1} x[n] \exp\{-i2\pi(T_s\Omega_s)mn\}, \;\; m = 0, \ldots, M-1, \quad \text{and} \quad T = NT_s, \; B = M\Omega_s$$

Moreover, setting $M = N = (T_s \cdot \Omega_s)^{-1}$, and $w_N := e^{i2\pi/N}$, the discrete Fourier series:

$$X[m] = \sum_n x[n]\, w_N^{-mn} \quad \leftrightarrow \quad x[n] = \sum_m X[m]\, w_n^{mn}$$

are $N-$periodic and form a one-to-one correspondence.

Straightforward computation of a DFT is in $\mathcal{O}(N^2)$.

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The Cooley Tukey algorithm (radix-2) relies on:

- the $N-$periodicity of the series
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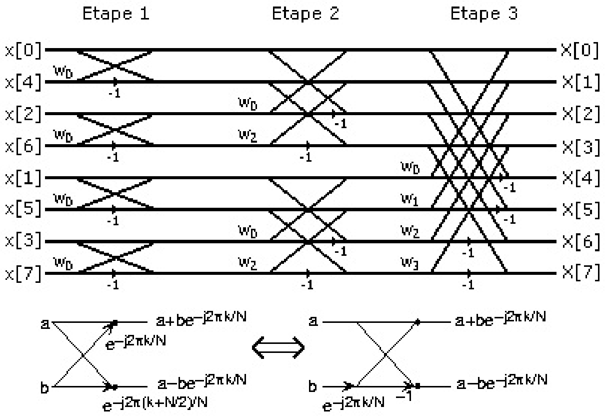

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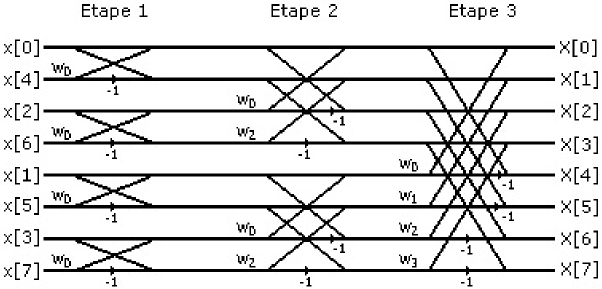
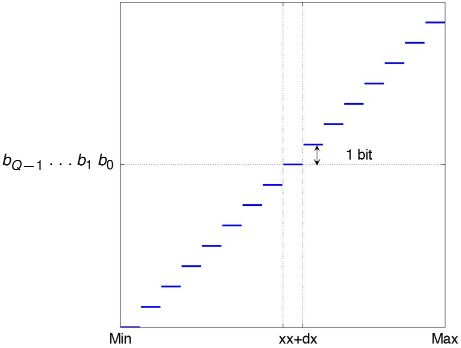


Computational cost in $\mathcal{O}(N \log N)$

Quantization



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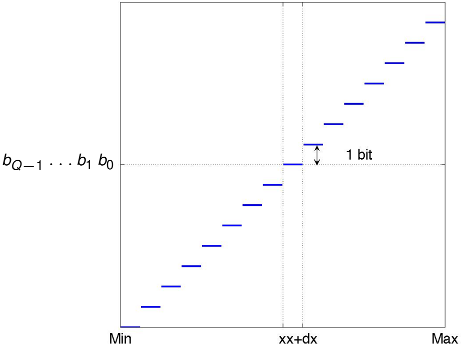

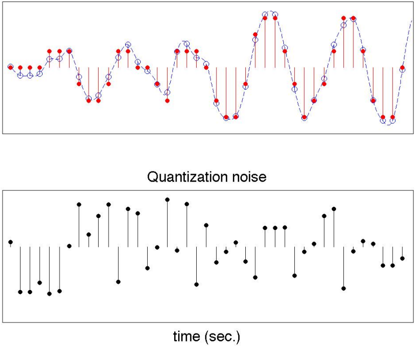

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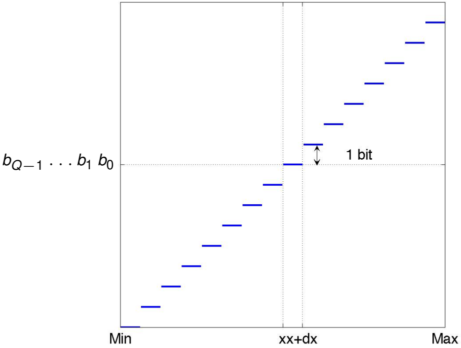
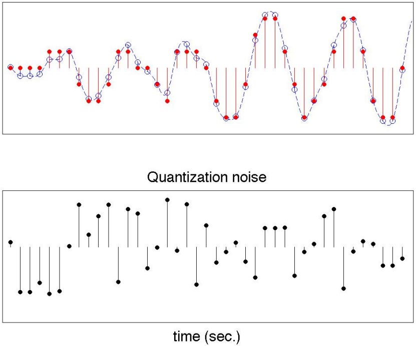
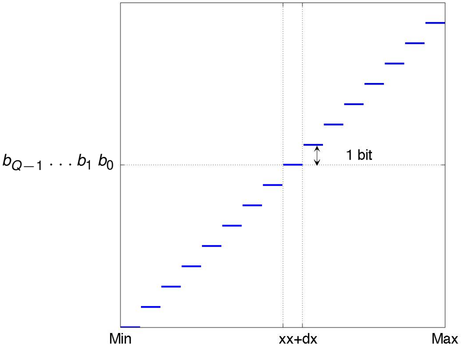


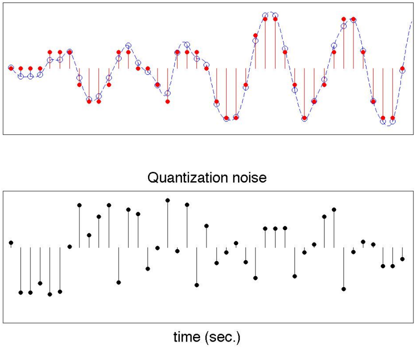


Quantization noise: uniformly distributed between -1/2 LSB and +1/2 LSB

Highly correlated with exact signal amplitude

Quantization





Q = 1 bit (2 levels quantization) - SNR $\approx$ 7.781 dB

Q = 2 bit (4 levels quantization) - SNR $\approx$ 13.801 dB

Q = 4 bit (16 levels quantization) - SNR $\approx$ 25.841 dB

Q = 8 bit (256 levels quantization) - SNR $\approx$ 49.92 dB

Compression / Coding

Remove contextual redundancy (lossless compression) or imperceptible information (lossy compression) from signals.

- psychoacoustic models (e.g. MP3 for audio)
- entropic coding (e.g. JPEG 2000 for images)
- statistical description (e.g. Lempel-Ziv-Welch for texts)
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Time Frequency decompositions

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An intuitive starting-point —

Discrete versions —

Orthogonal bases —

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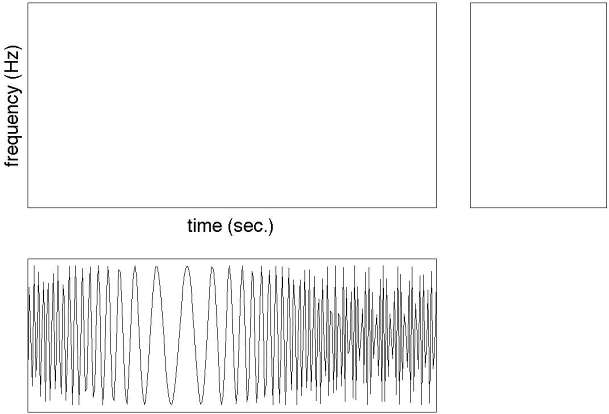

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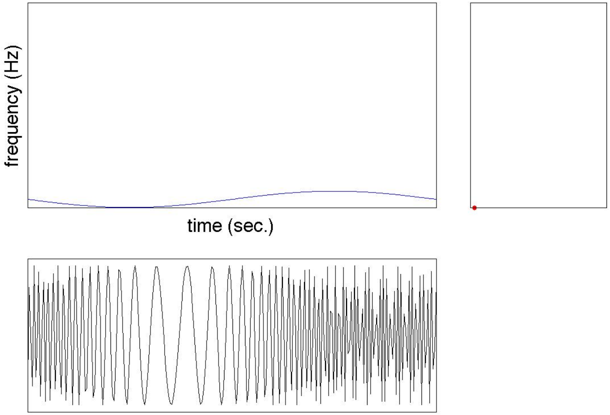

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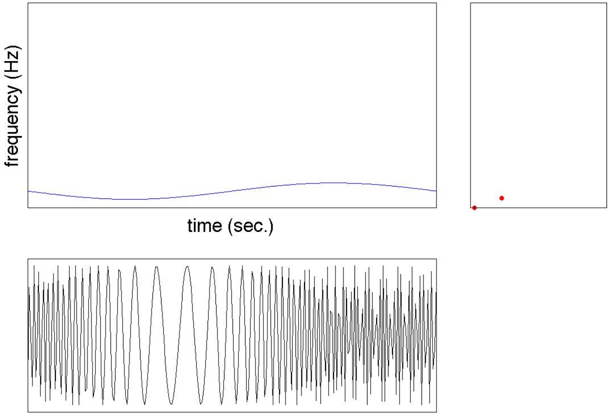

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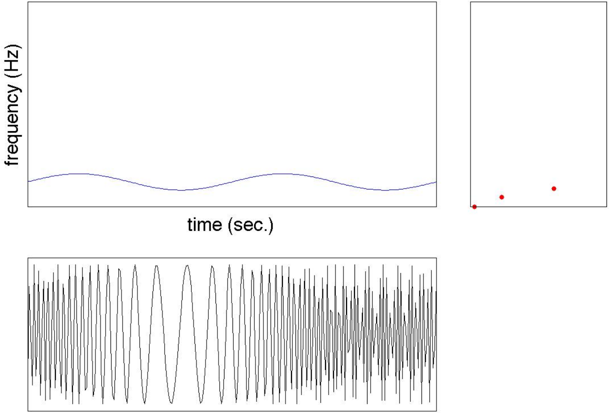

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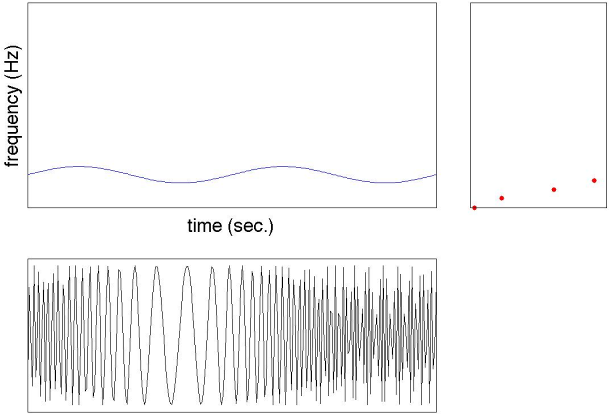

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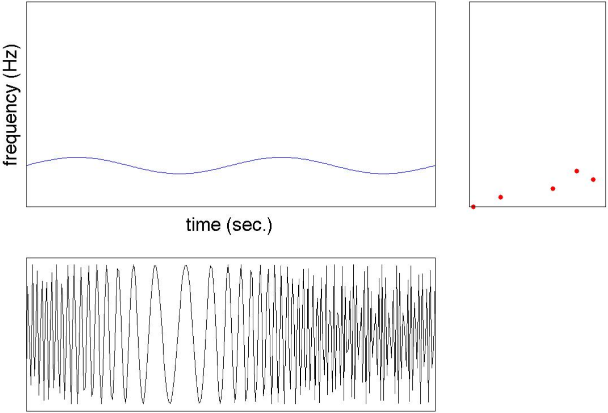

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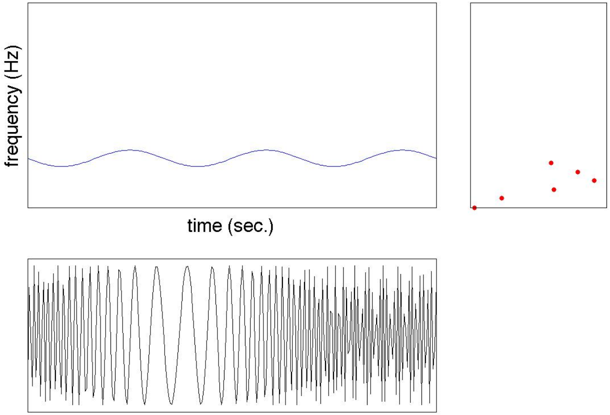

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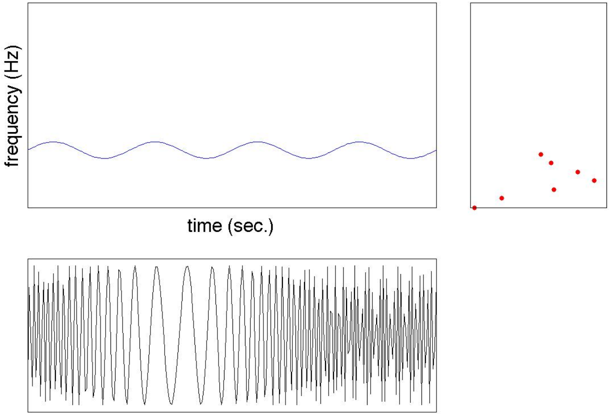

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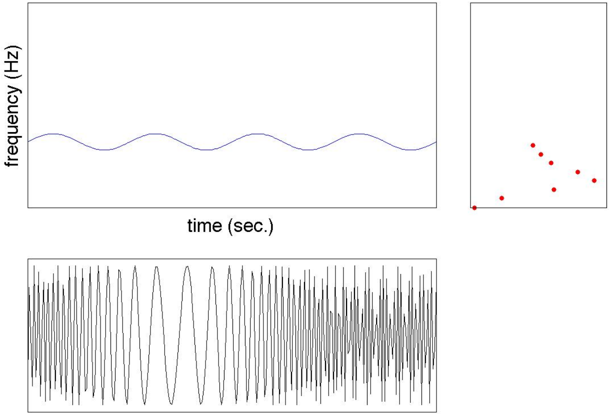

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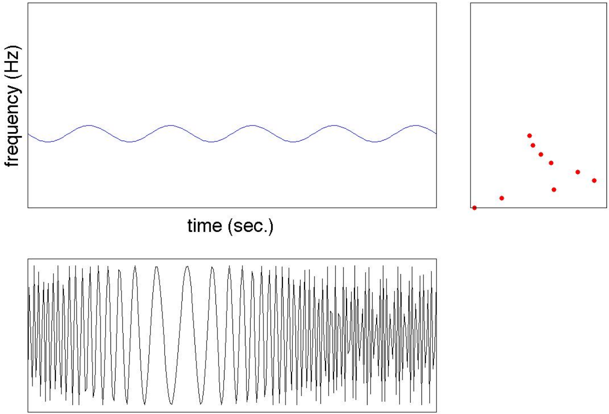

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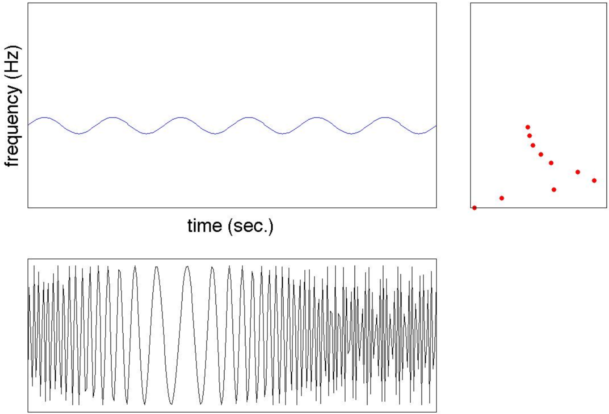

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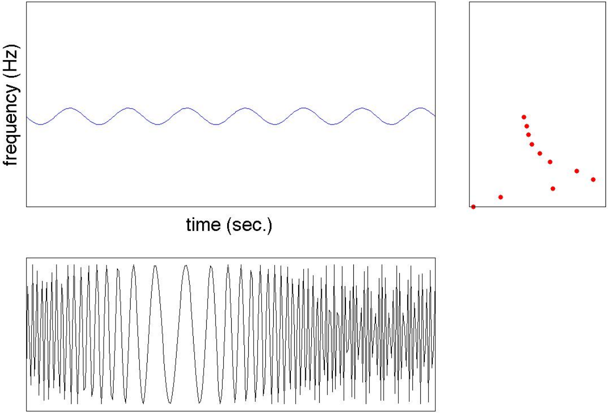

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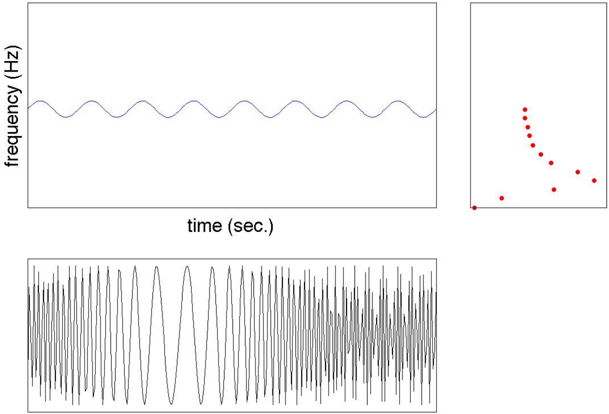

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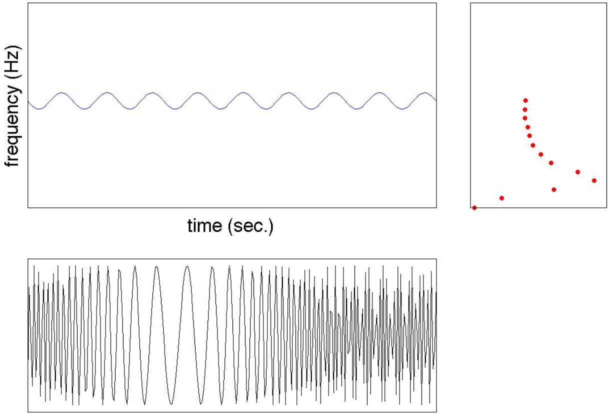

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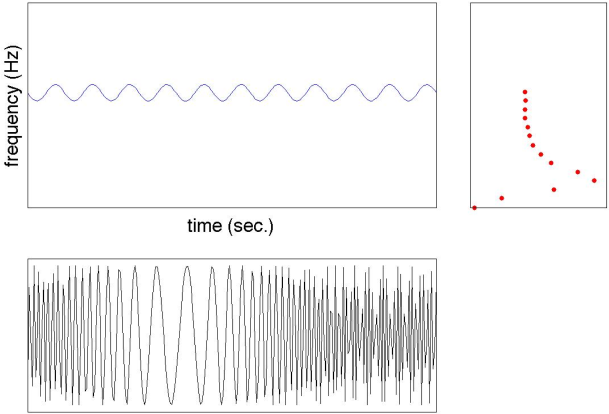

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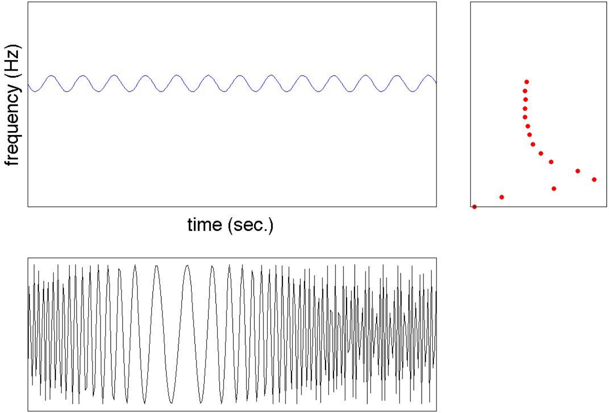

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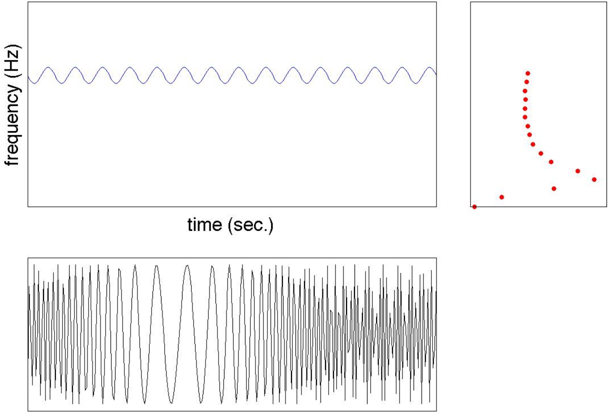

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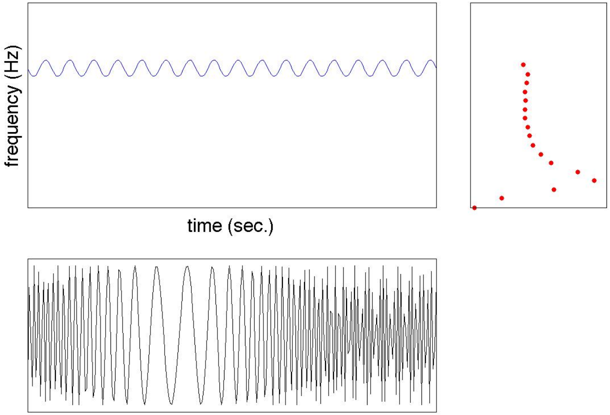

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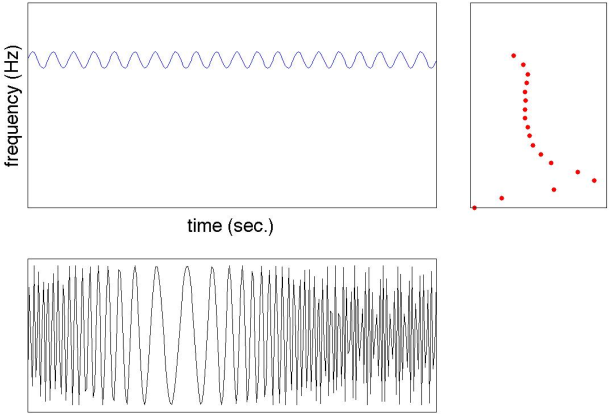

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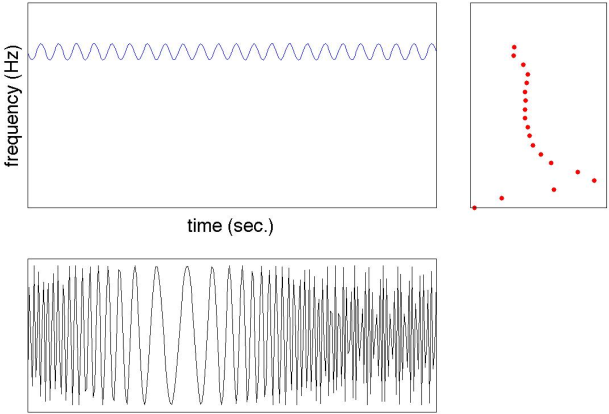

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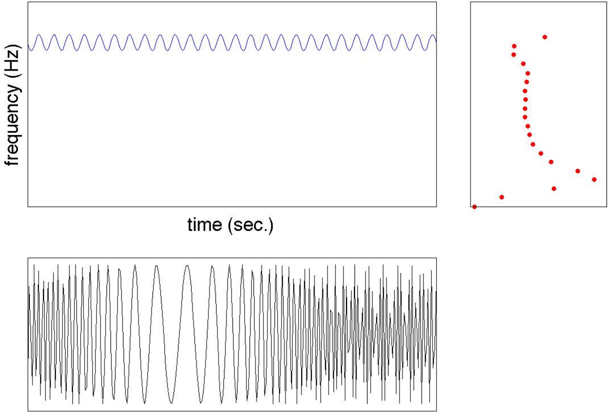

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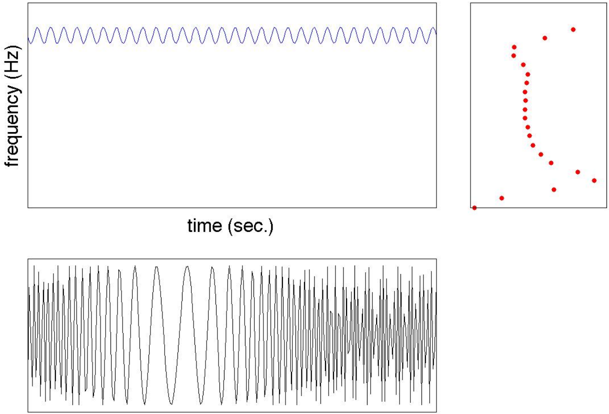

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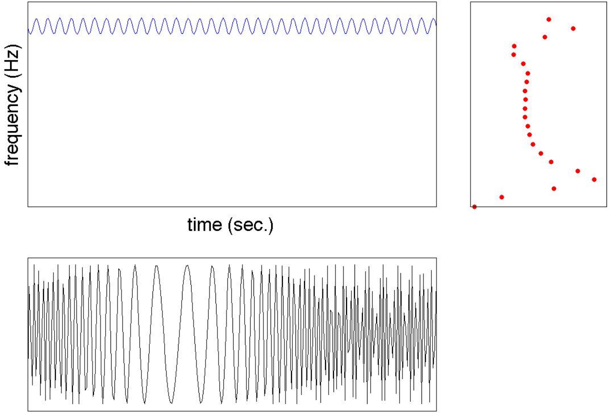

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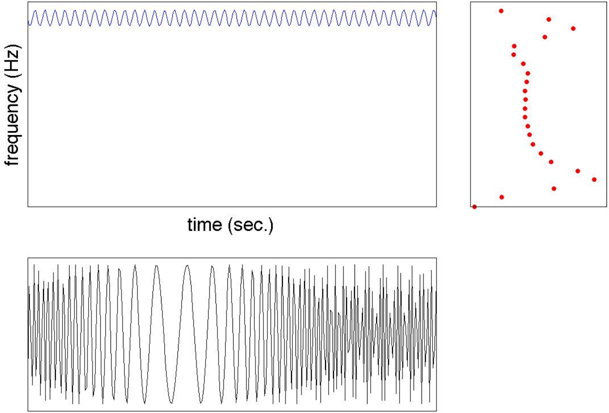

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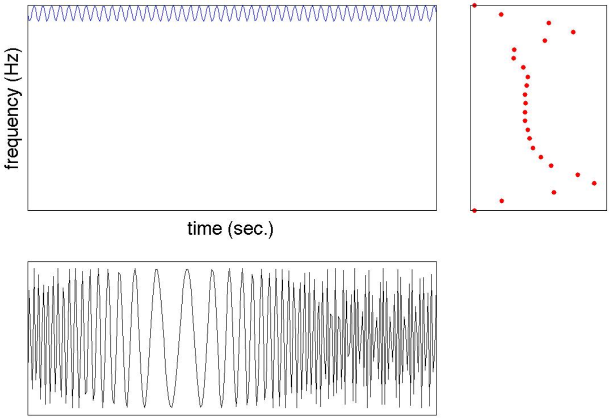

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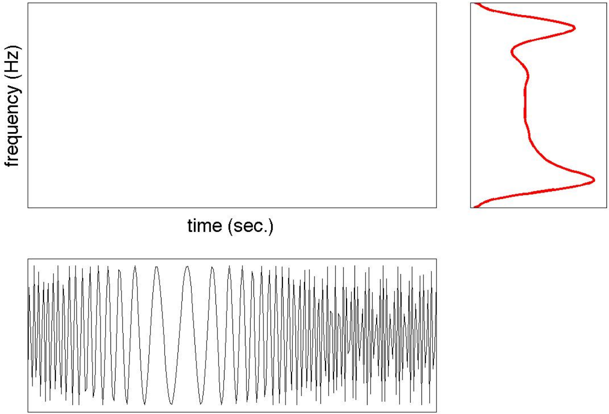

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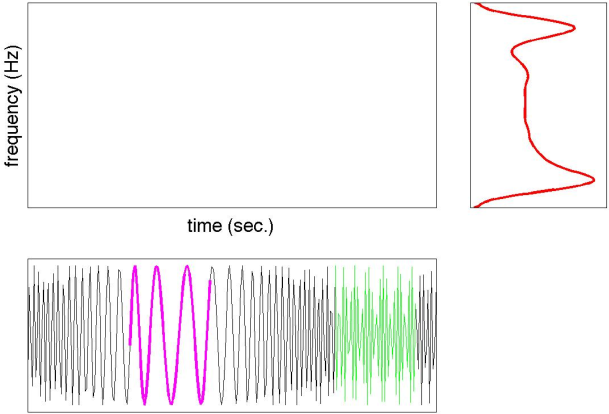

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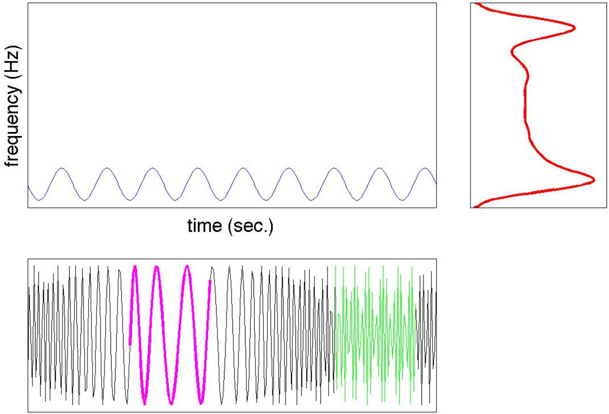

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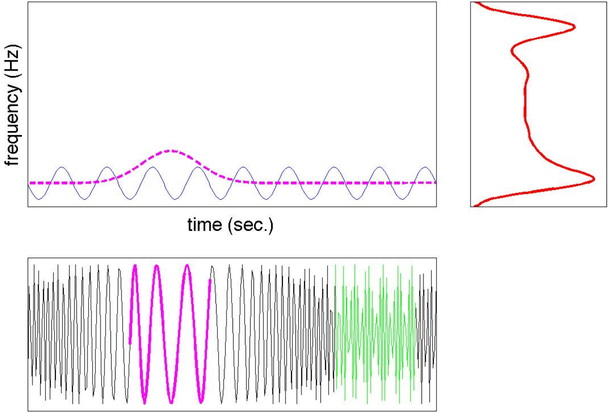

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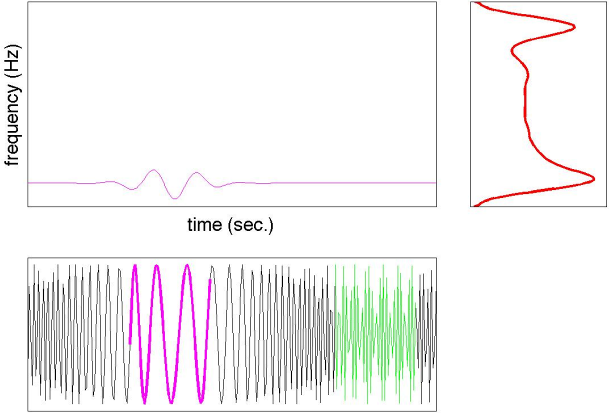

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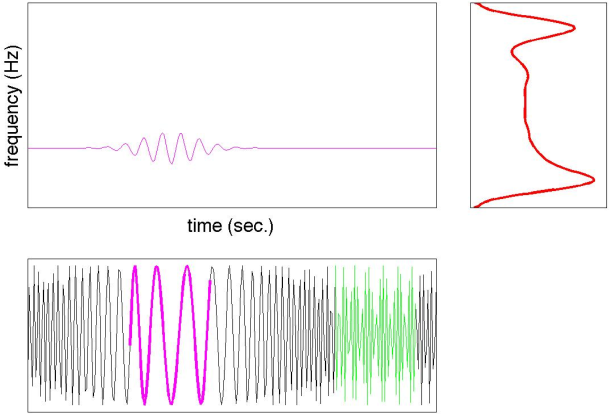

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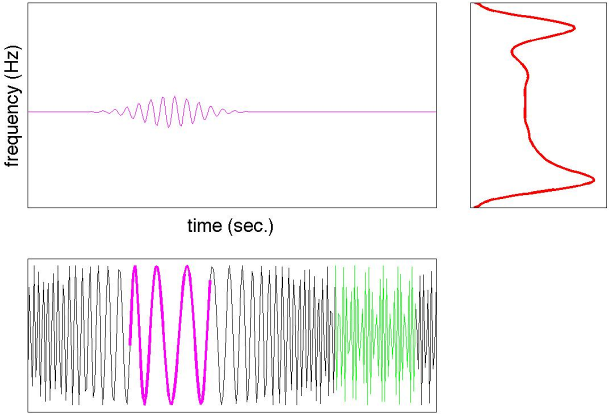

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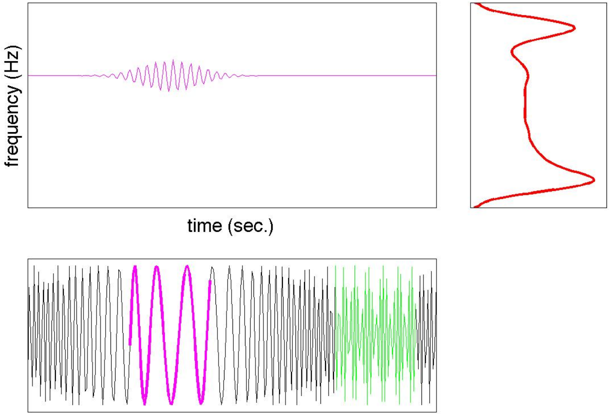

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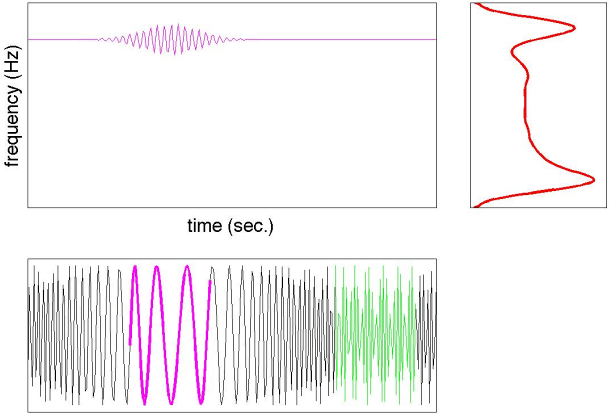

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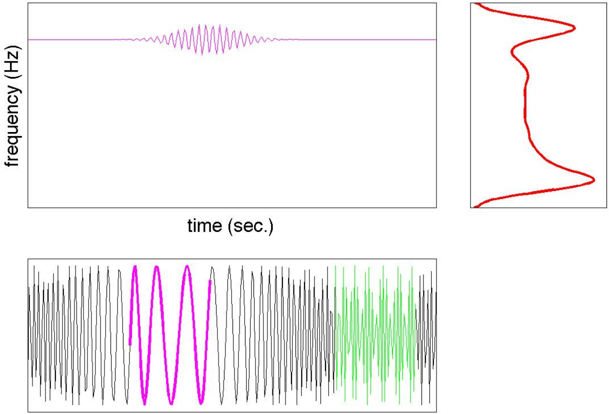

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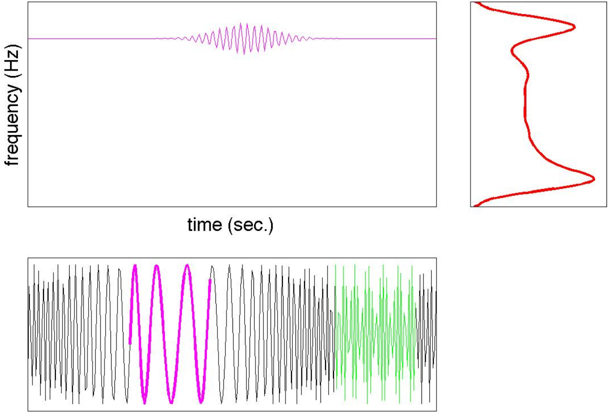

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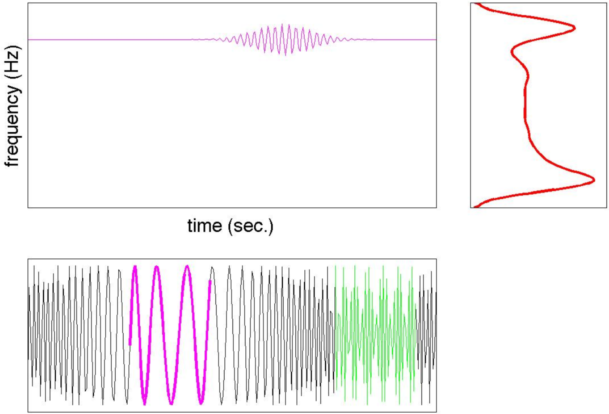

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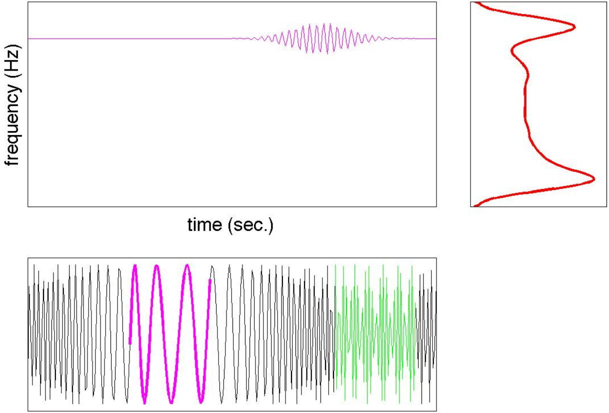

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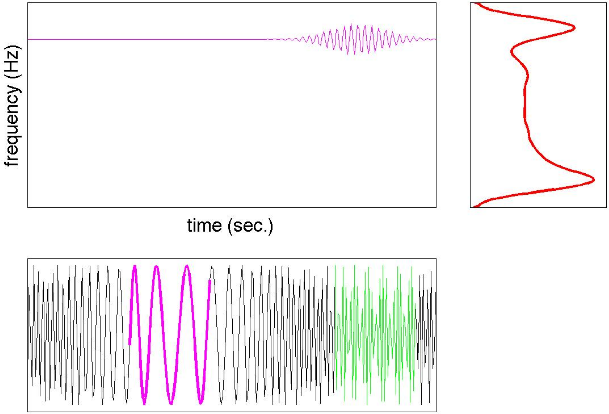

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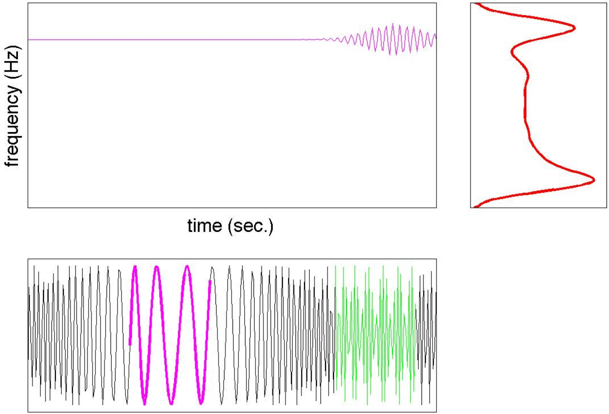

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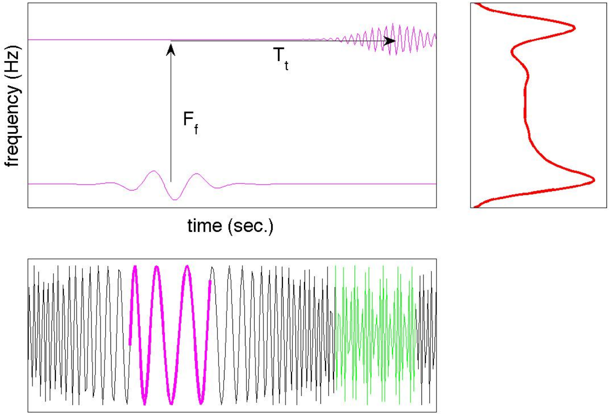

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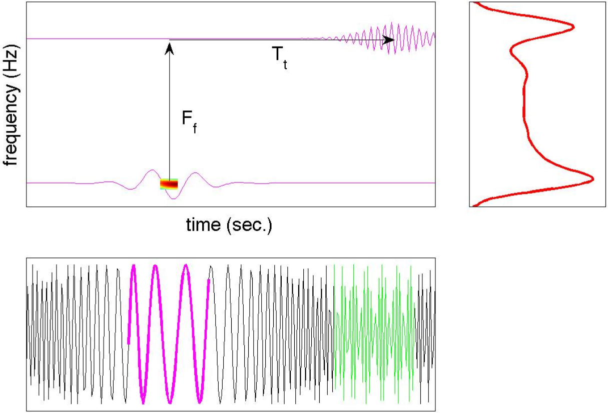

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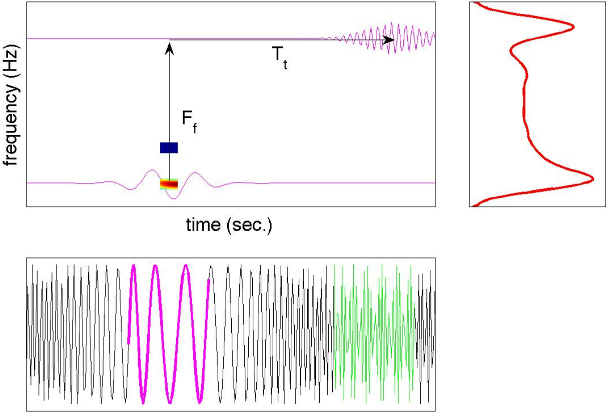

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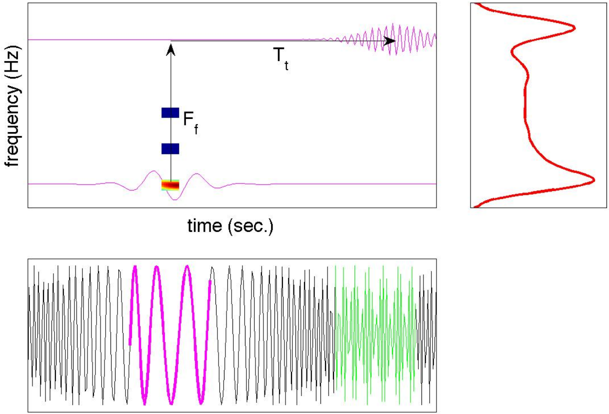

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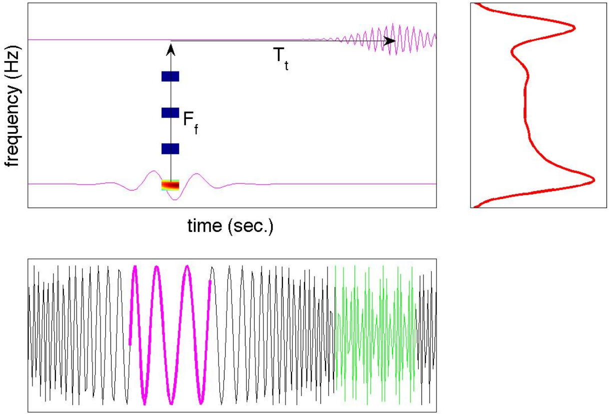

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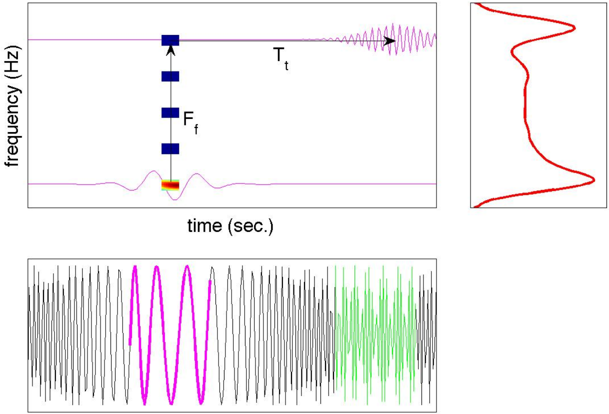

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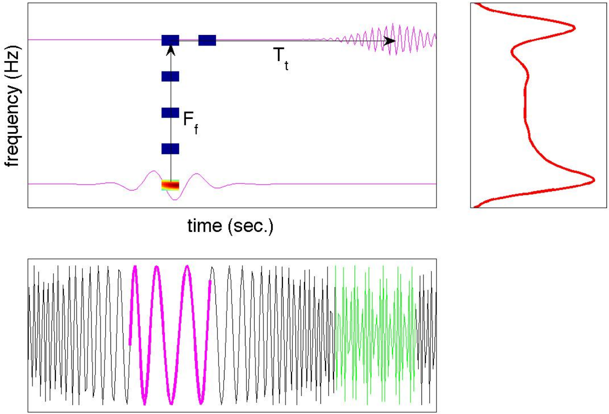

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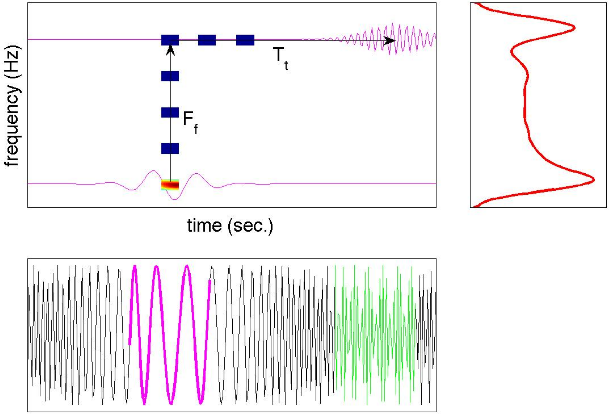

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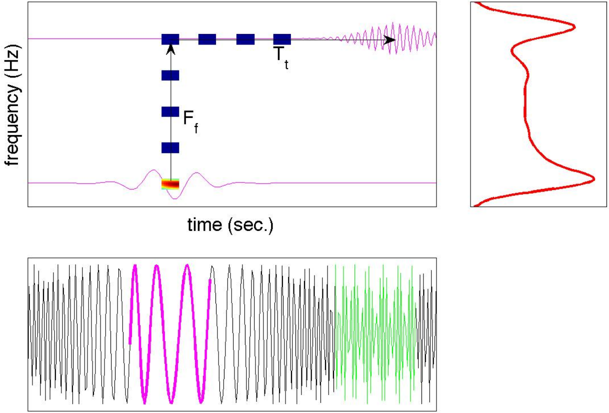

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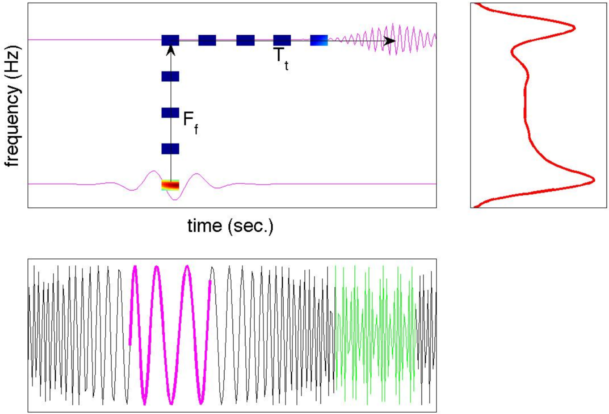

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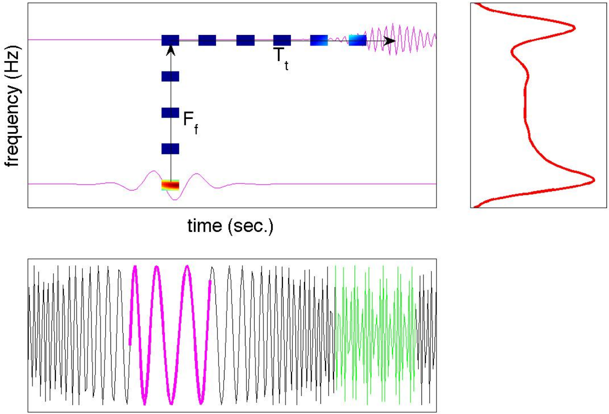

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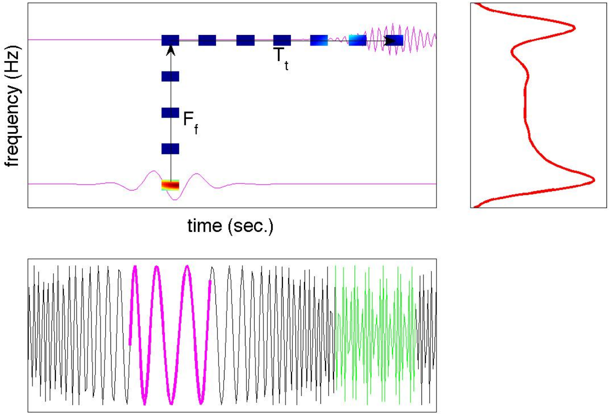

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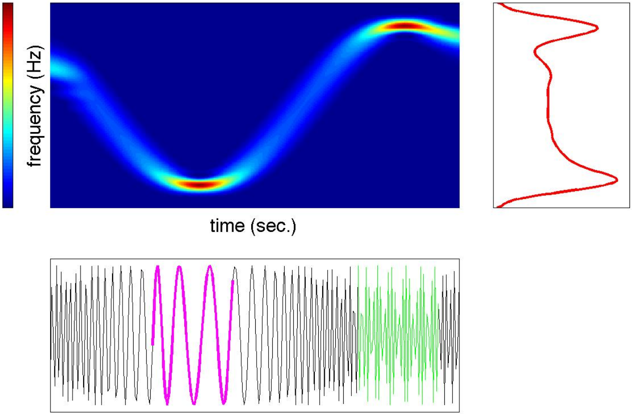

Discretizing the time-frequency plane

Under mild conditions, linear time-frequency decompositions:

$$L_x(t, f; g) = \int x(u)\, g_{t,f}(u)\, du = \int x(u)\, g(u-t)\, e^{-i2\pi f u}\, du,$$

are invertible,

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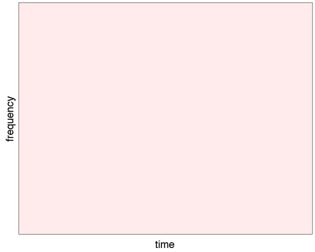

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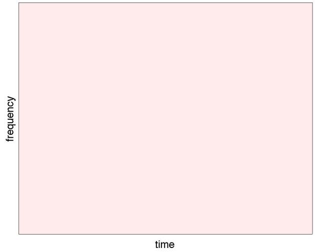

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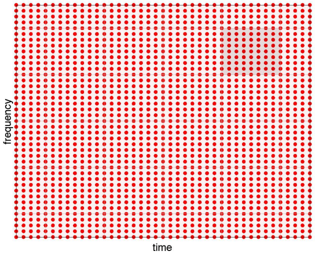

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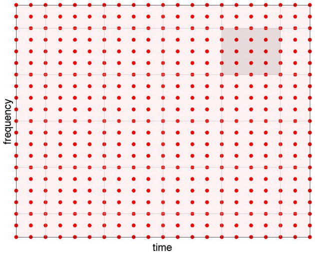

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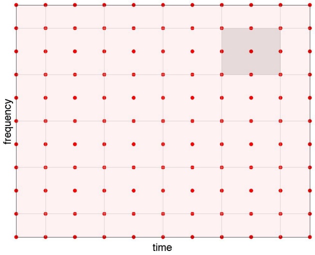

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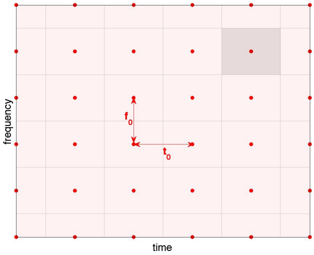

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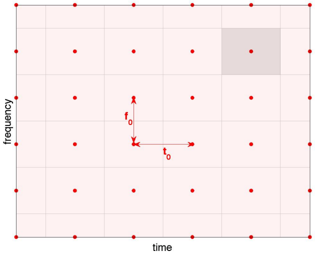
- revert $x(t)$ from a uniform tiling of the time-frequency plane:

$$x(t) = \sum_n \sum_m L_x[n, m]\, \widetilde{g}_{n,m}(t)$$

needs to introduce dual frames.



Frames

Definition — Frame: The sequence $\{g_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$ is a frame of $\mathcal{H}$ if there exist two constants $0 < A \leq B$, s.t. for any $f \in \mathcal{H}$:

$$A \parallel f \parallel^2 \leq \sum_{n,m} | \langle f, g_{n,m}\rangle |^2 \leq B \parallel f \parallel^2 .$$

Definition — Dual frame: Let $\{g_{n,m}\}_{(n,m)}$ be a frame. The dual frame defined by

$$\widetilde{g}_{n,m} = (L^* L)^{-1} g_{n,m} \quad \text{where} \quad L^* L x = \sum_{n,m} \langle x, g_{n,m}\rangle g_{n,m}$$

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$$\forall f \in \mathcal{H}, \quad x = \sum_{n,m} \langle x, g_{n,m}\rangle \widetilde{g}_{n,m} = \sum_{n,m} \langle x, \widetilde{g}_{n,m}\rangle g_{n,m}$$

Theorem (Balian-Law) — If $\{g_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$ is a *windowed Fourier* frame with $t_0 \cdot f_0 = 1$, then

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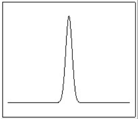
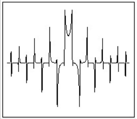
Windowed Fourier frames: Gabor transform

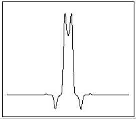


(e) Gabor function g

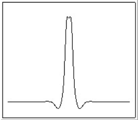


(f) Dual function for critical sampling

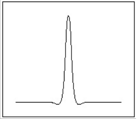
(g) Dual function for oversampling rate 1.06



(h) Dual function for oversampling rate 1.25

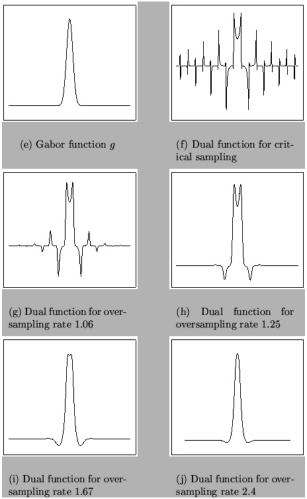


(i) Dual function for oversampling rate 1.67



(j) Dual function for oversampling rate 2.4

Windowed Fourier frames: Gabor transform



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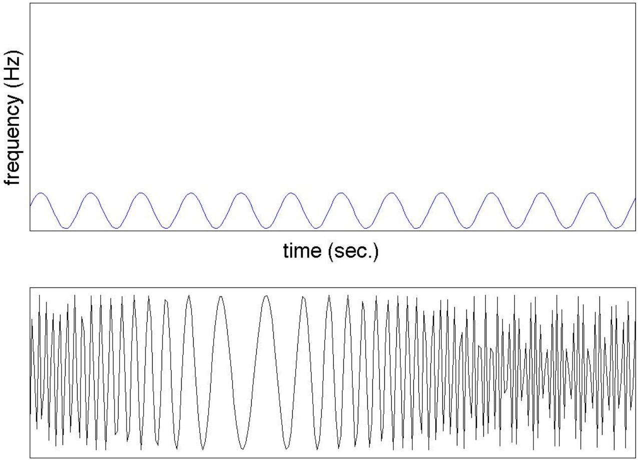
There exists no

orthogonal windowed Fourier basis,

images of a compactly supported function g,

by time and frequency shift operators.

Wavelet decomposition



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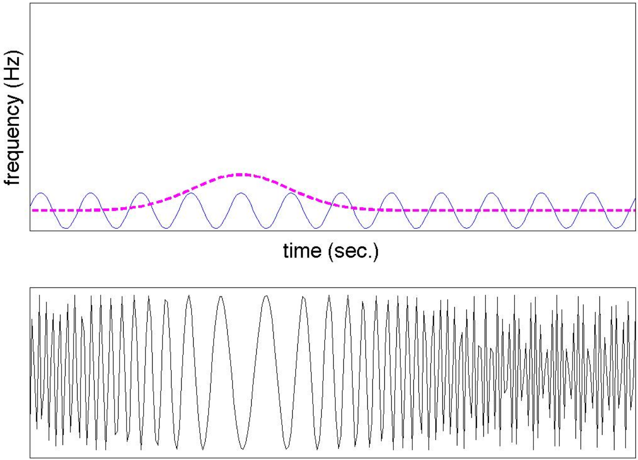

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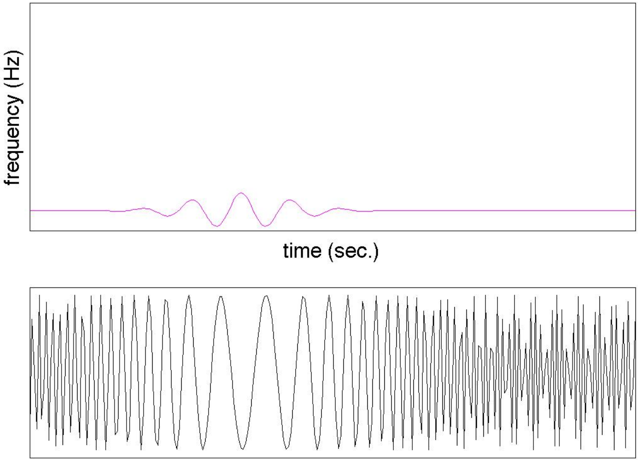

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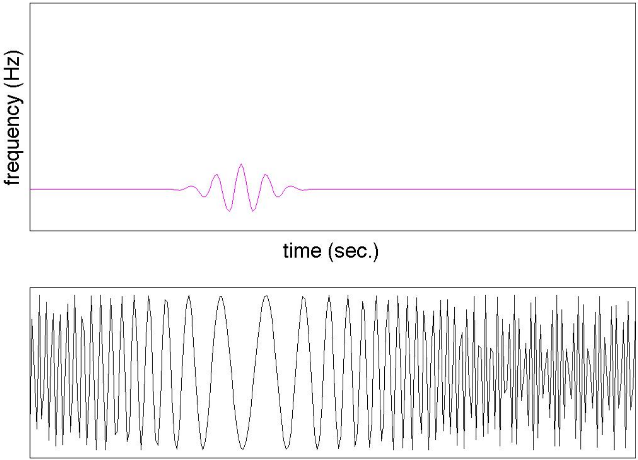

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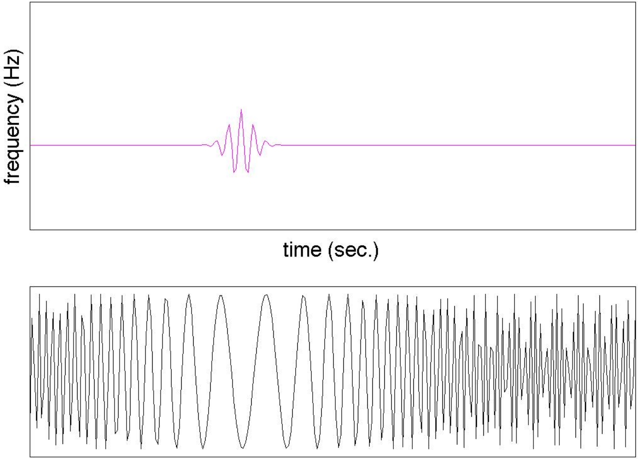

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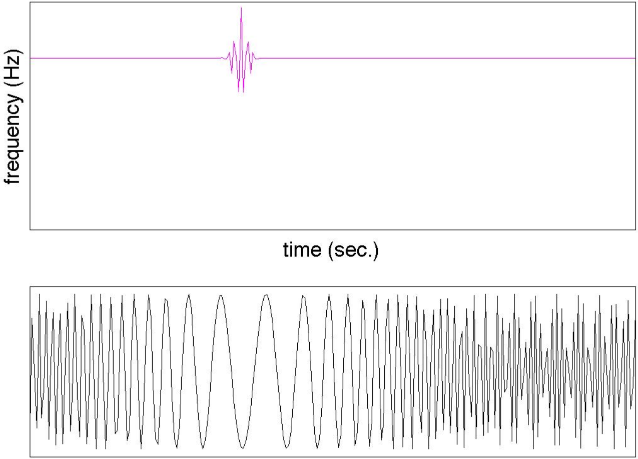

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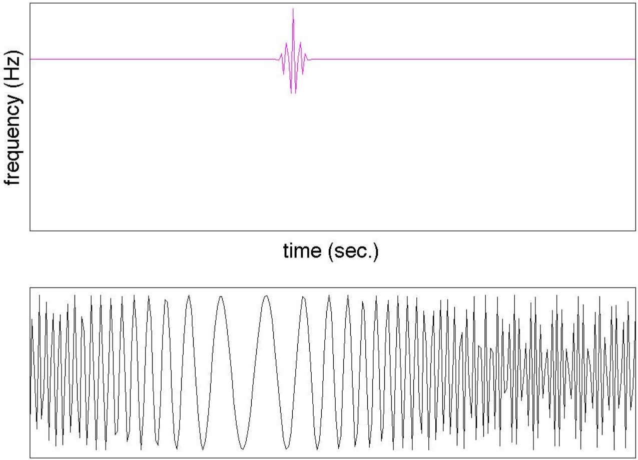

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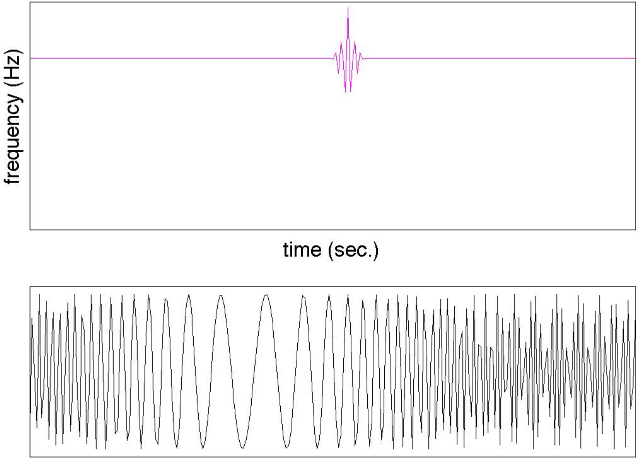

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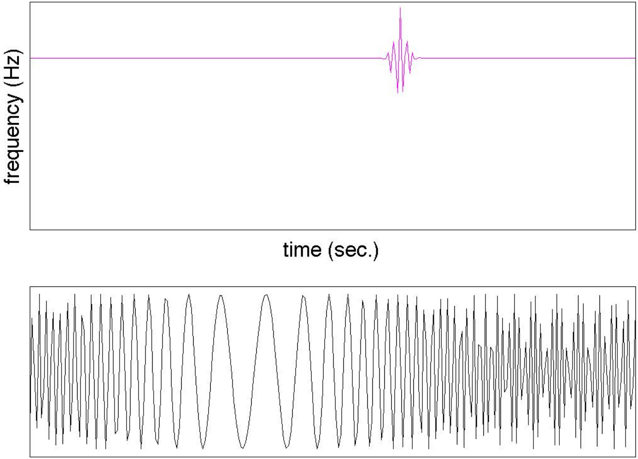

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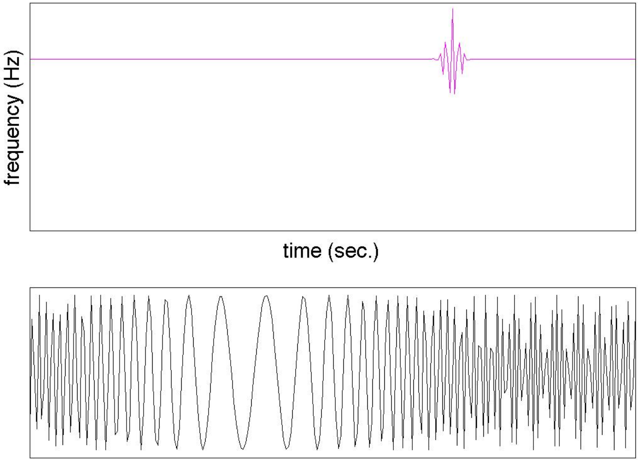

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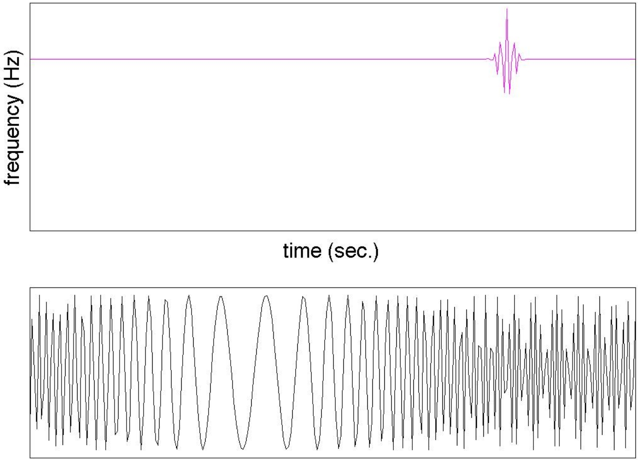

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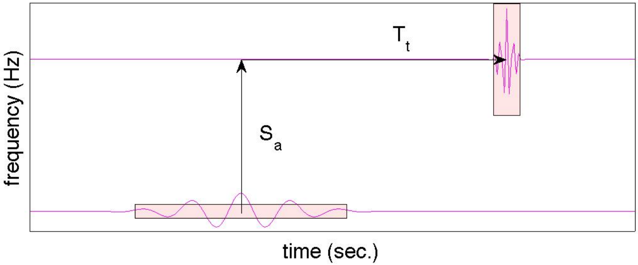

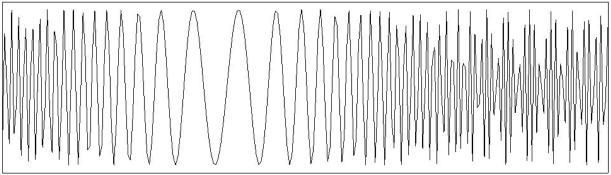

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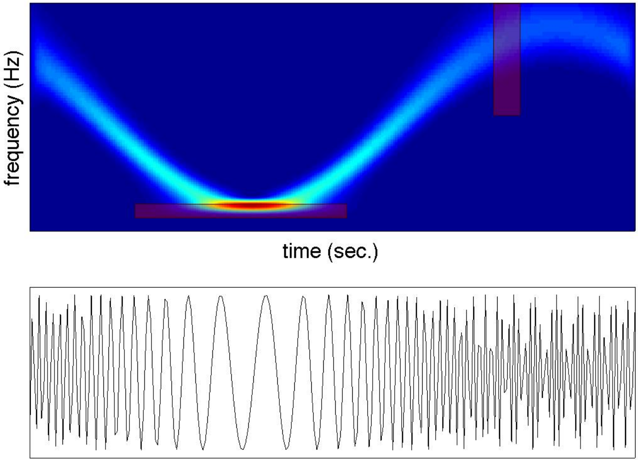
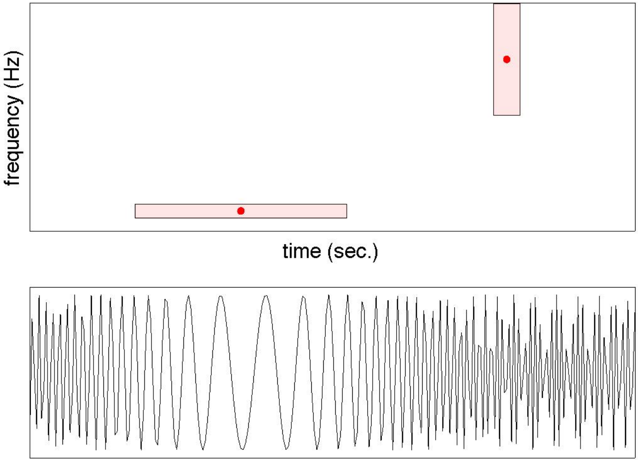


Wavelet decomposition



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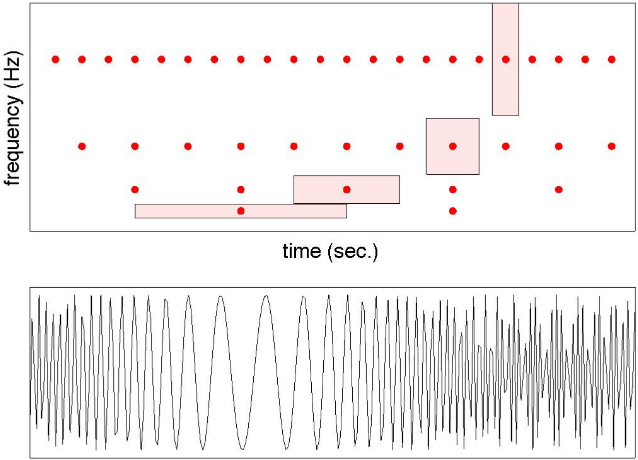

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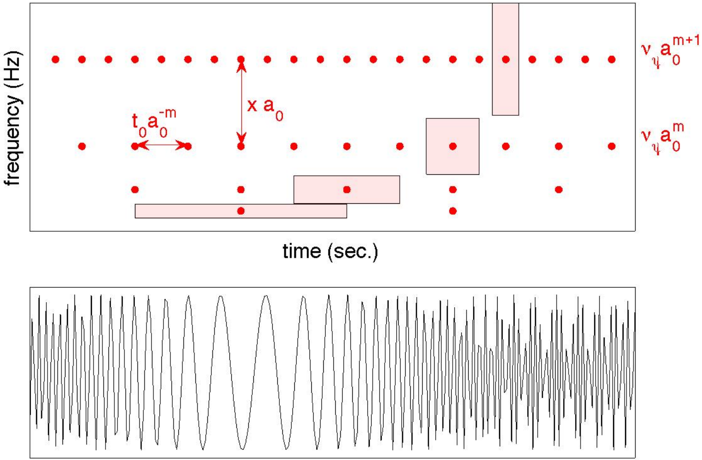

Wavelet transform

Definition — Continuous wavelet transform:

$$W_x(t,a) = \int x(u)\,\psi_{t,a}(u)\,du \quad \text{with} \quad \psi_{t,a}(u) := \frac{1}{\sqrt{a}}\,\psi\left(\frac{u-t}{a}\right)$$

Definition — Discrete wavelet transform:

$$\forall k,j \in \mathbb{Z},\; d^x_{j,k} := W_x\left(t \mapsto kt_0 a_0^{-j}, a \mapsto a_0^{-j}\right) \quad \text{and} \quad \psi_{k,j} := a_0^{j/2}\psi\left(a_0^j t - kt_0\right)$$

Theorem (I. Daubechies) — There exists compacty supported functions ψ that generate orthonormal wavelet bases $\{\psi_{j,k}(t);\, j,k \in \mathbb{Z}\}$ of $\mathcal{H}$.

Computation — Wavelet bases are associated to multiresolution analysis schemes (S. Mallat), with efficient pyramidal filter-bank implementation.

Properties — Computational cost is $\mathcal{O}(N)$ *vs* $\mathcal{O}(N \log N)$ for a FFT

Sparse decomposition

Large coefficients localize on singularities of the signal

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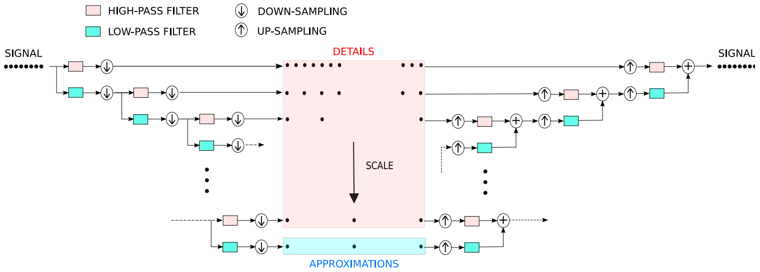
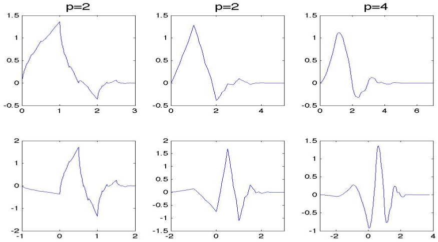
Quadrature filter-banks

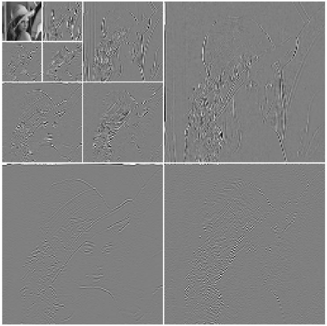


Daubechies scaling function ϕ (top) and wavelet ψ (bottom) with p vanishing moments
[from *A wavelet tour of signal processing*, S. Mallat]



2-D Wavelet Decomposition

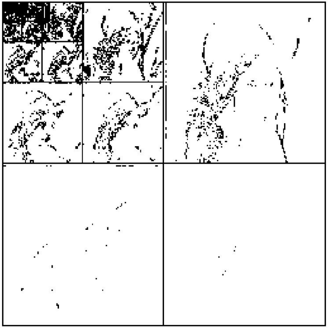
Original image (256×256 pixels)



Separable wavelet transform

[from *A wavelet tour of signal processing*, S. Mallat]

Non-linear image analysis



Maxima wavelet coefficients ($N^2/16$ coefs.)

Non linear approximation

[from *A wavelet tour of signal processing*, S. Mallat]

Wavelet based compression: JPEG 2000

JPEG compression (LCT)

JPEG 2000 compression (wavelets)

Message...

Computer science allows for a numerical implementation of continuous operators

But, combined with signal and image processing, it led to a discipline on its own: the Digital Signal Processing

DSP opened up the scope of a new mathematical field with inherent concepts and theorems that might not have been obtained otherwise

- Filter design
- Machine learning (classification, estimation, prediction,. . .)
- Information theory (coding, compression)
- Communication
- Fractal analysis
- . . .