Time-Frequency Representations

Wavelet Decompositions

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CPE Lyon
5ETI - Majeure Image

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Contents

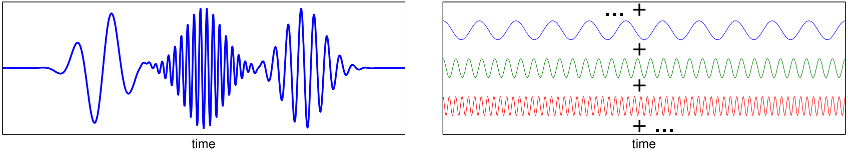
Fourier representation : *a paradox* for non-stationary signals

$$x(t) \in L^2(\mathbb{R}) \qquad \left(\int_{\mathbb{R}} |x(t)|^2 dt < \infty\right)$$

$$X(f) = \int_{\mathbb{R}} x(t)\, e^{-i2\pi ft}\, dt \quad = \quad \langle x, e_f \rangle$$

$$x(t) = \int_{\mathbb{R}} X(f)\, e^{i2\pi ft}\, df \quad = \quad \langle X, e_t^* \rangle$$

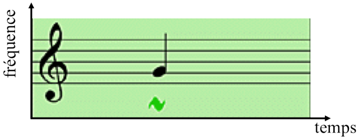
Signal $\equiv$ (continuous) superposition of harmonic functions of infinite support...



– **dynamic cancellation** (destructive interferences) reproduces static cancellation
– time reversal keeps **spectral density** unchanged (phase encodes time)

Principle of time-frequency representations

Formalise the concept of **musical score**



Linear decomposition on a family of time and frequency localised analysing functions

Frequency (Fourier) : $\langle x, \delta_f \rangle$ Time (Shannon) : $\langle x, \delta_t \rangle$ Time-Frequency : $\langle x, g_{t,f} \rangle$

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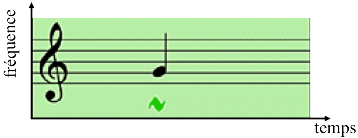

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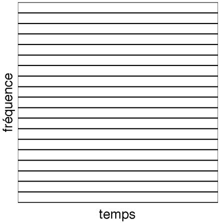

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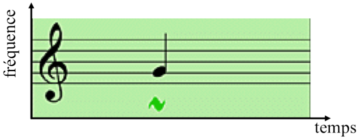

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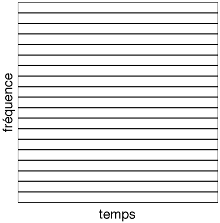

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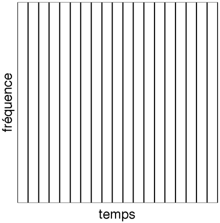

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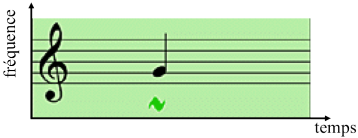

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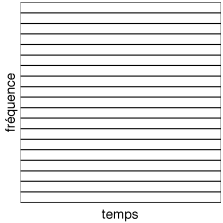

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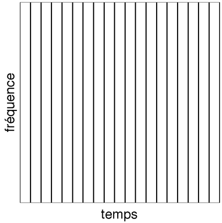

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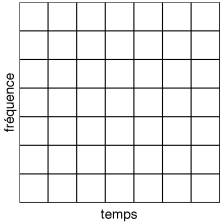

Time-Frequency duality

Uncertainty principle (Weyl-Heisenberg)

$$\text{Define} \begin{cases} \sigma_t^2 = \frac{1}{||g||^2} \int t^2 |g(t)|^2 \, dt & \text{(equivalent time support of } g\text{)} \\ \sigma_f^2 = \frac{1}{||g||^2} \int f^2 |G(f)|^2 \, df & \text{(equivalent bandwidth of } g\text{)} \end{cases} \text{, then :}$$

$$\sigma_t^2 \, \sigma_f^2 \geq \frac{1}{4\pi} \quad \left(\text{with equality if } |g(t)| = C\, e^{-\alpha t^2}\right)$$

Compact supports (Slepian-Pollack-Landau)

Theorem – *If $g(t) \neq 0$ has compact support in time, then $G(f)$ cannot be zero on a whole interval. Reciprocally, if $G(f) \neq 0$ has compact support, then $g(t)$ cannot be zero on a whole interval.*

g cannot be simultaneously time limited and frequency limited...

(*prolate spheroidal wave functions* achieve the **best energy concentration** in both domains)

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One motivation for time-frequency representation is to characterise the local frequency content of a signal and to display its time evolution

Example of a pure sine wave : $x(t) = sin(2\pi\nu_0 t)$

the frequency component is invariant with time : $\nu_x(t) = \nu_0$

Instantaneous frequency

Define the analytical signal : $z_x(t) := \mathcal{F}^{-1}\left\{2\,X(f)\,\mathbb{1}_{[0,\infty)}\right\} = |z_x(t)|\,e^{i\Phi_x(t)}$

and the instantaneous frequency : $\nu_x(t) \;:=\; \frac{1}{2\pi}\frac{\mathrm{d}\Phi_x(t)}{\mathrm{d}t}$

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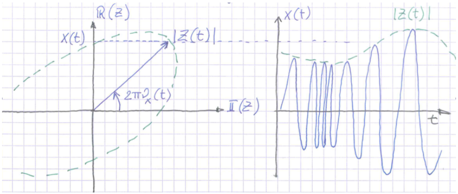
(Fresnel representation of a rotating vector)



Local measures

Instantaneous frequency is an intricate notion that *is not always meaningful*.

In particular, it can be questionable in the case of multi-components signals. . .

It is of particular interest for AM-FM type signals :

$$x(t) = a(t) \cos \Phi(t)$$

Theorem (Bedrosian)

- If $a(t)$ is a low-pass function on $[-B, B]$
- if $\Phi(t)$ is a high-pass function supported on $]-\infty, B'] \bigcup [B', \infty[$, with $B' > B$
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$$z_x(t) = a(t)\, e^{i\Phi(t)} \quad \text{and} \quad \nu_x(t) = \frac{1}{2\pi} \frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}$$

Average frequency

$$\langle \nu \rangle = \frac{2}{E_x} \int_0^{\infty} \nu |X(\nu)|^2 \,\mathrm{d}\nu = \frac{1}{E_z} \int_{-\infty}^{\infty} \nu_x(t) |z_x(t)|^2 \,\mathrm{d}t$$

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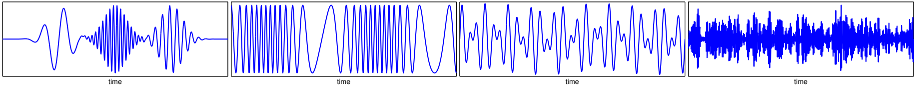

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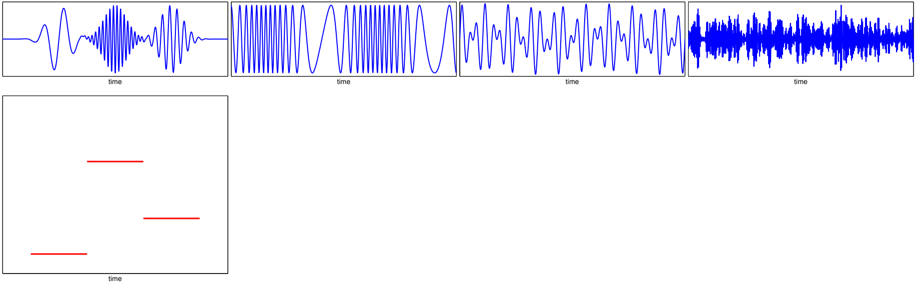

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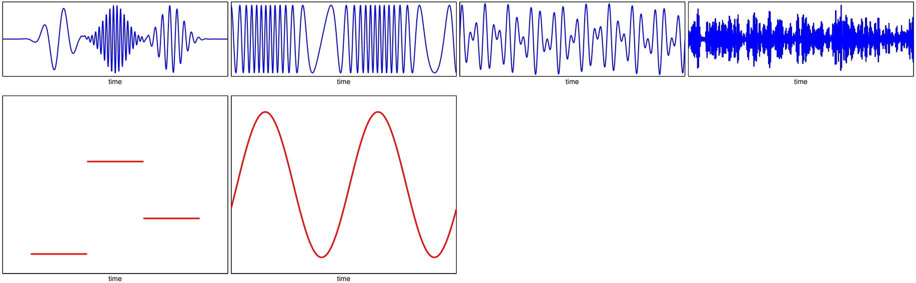

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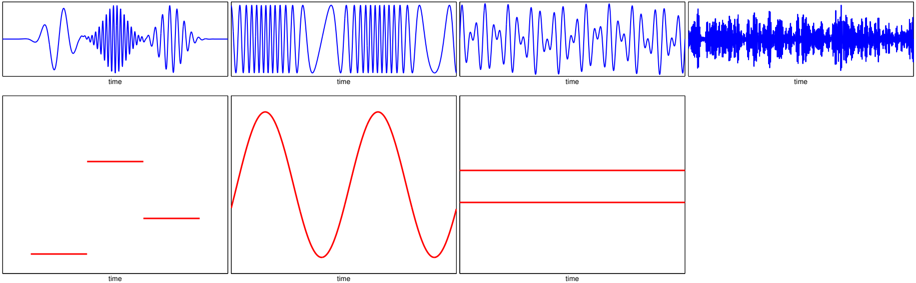

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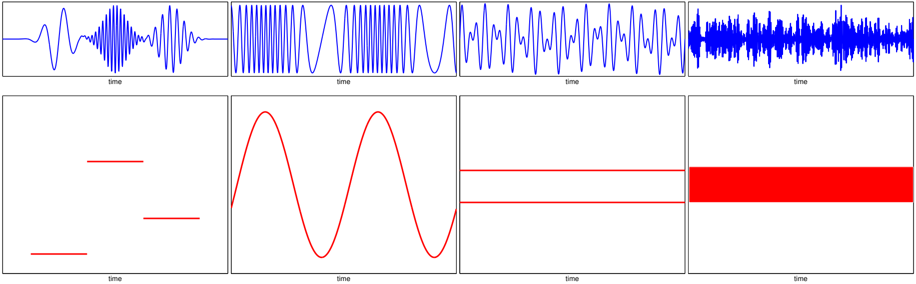

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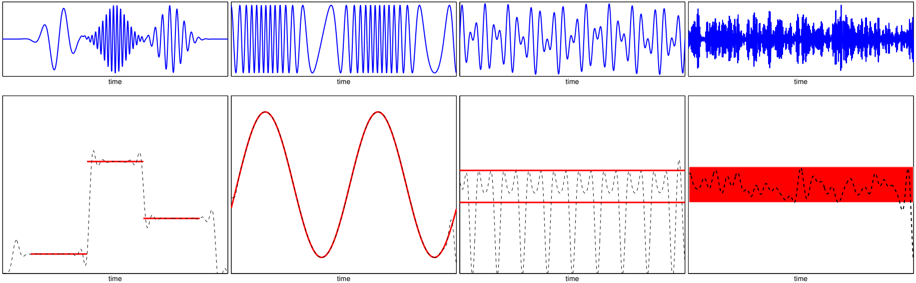
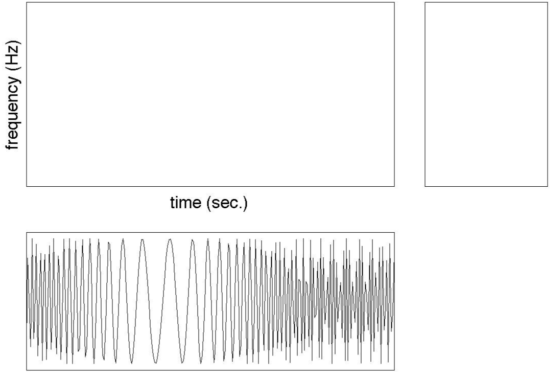


Short-time Fourier Transform (STFT)



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g : analysing function (template) localised in time and in frequency, **simultaneously**

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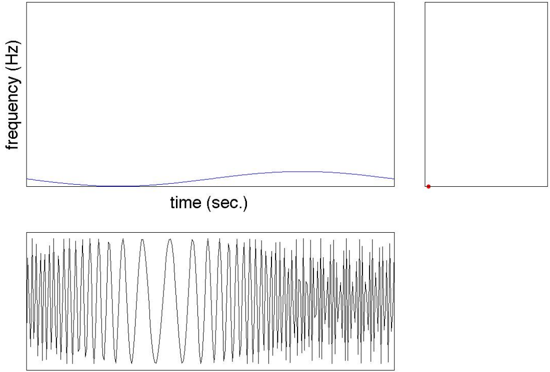

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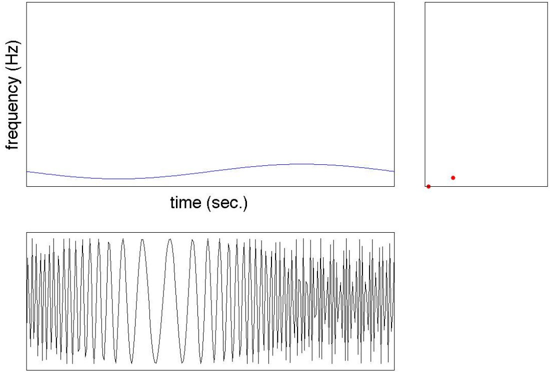

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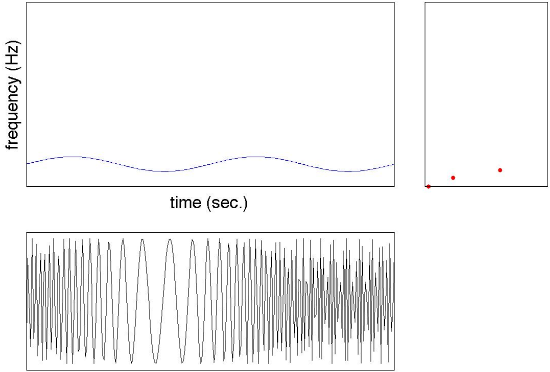

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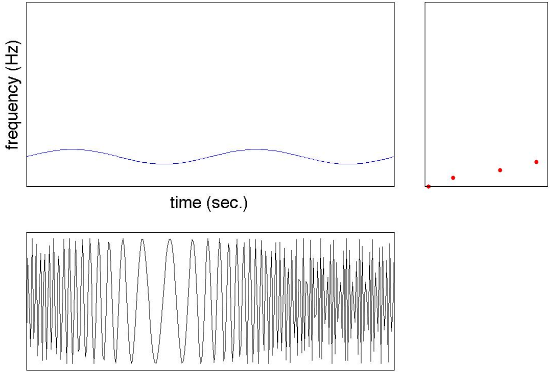

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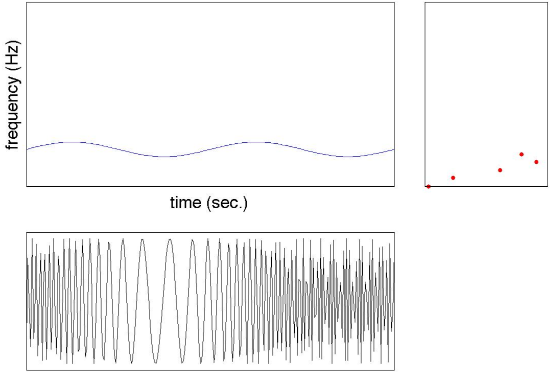

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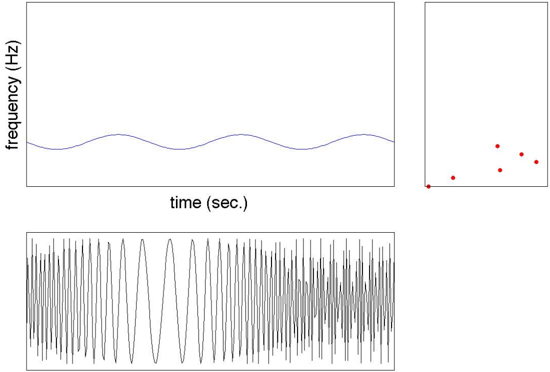

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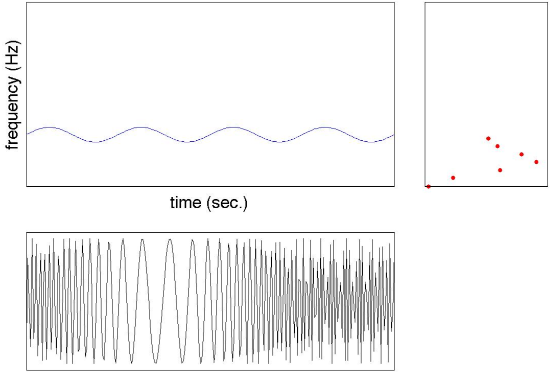

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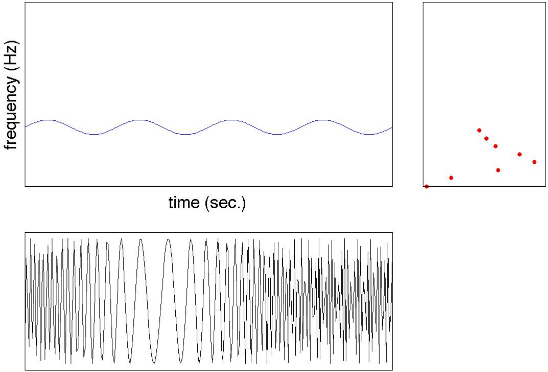

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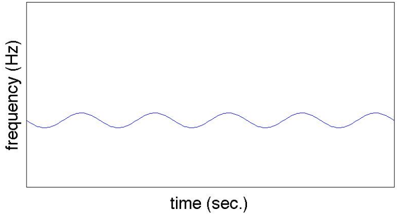

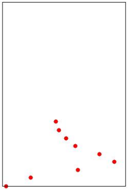

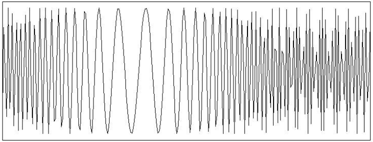

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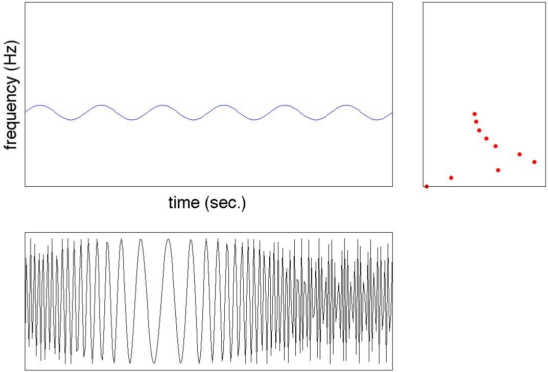

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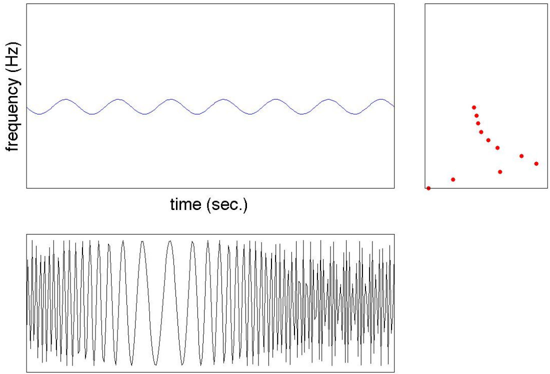

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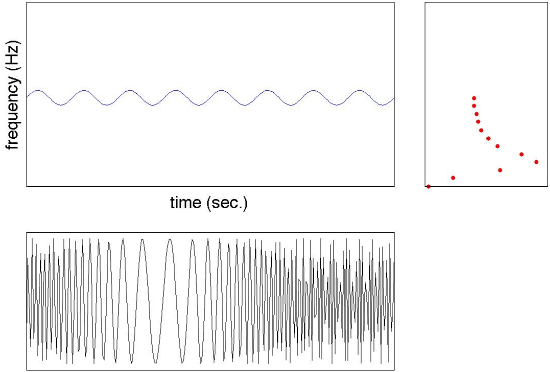

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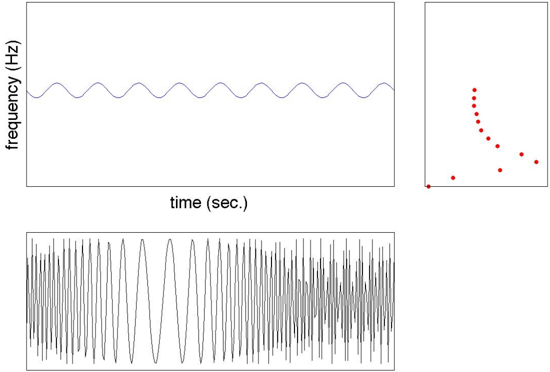

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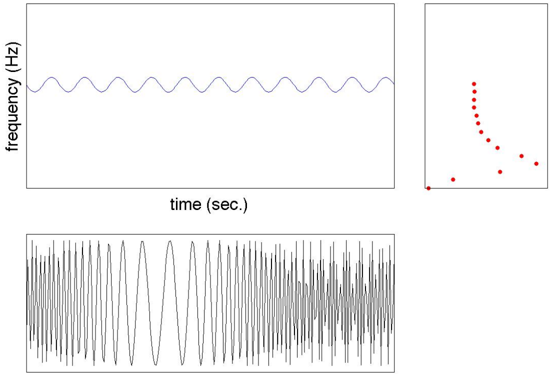

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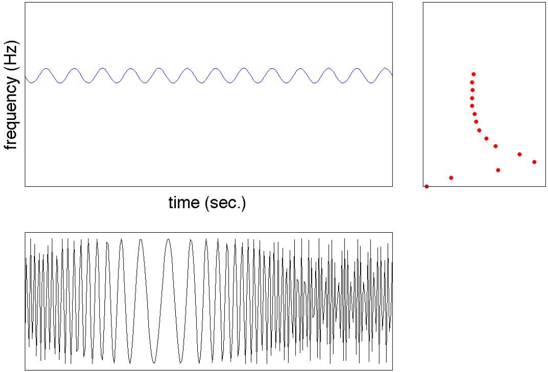

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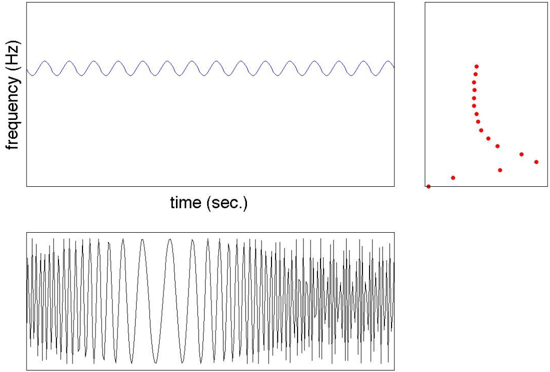

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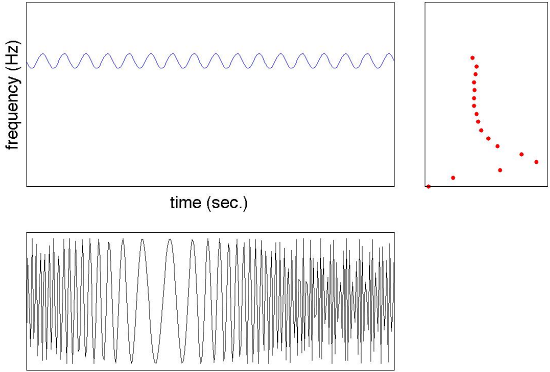

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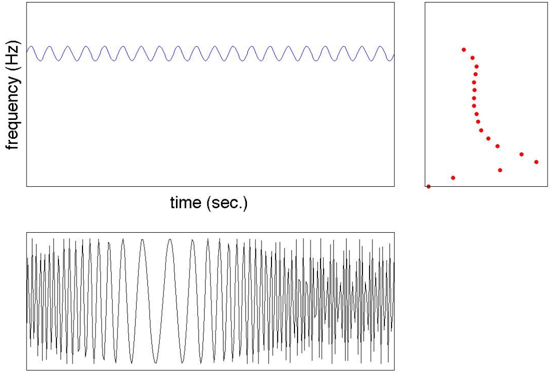

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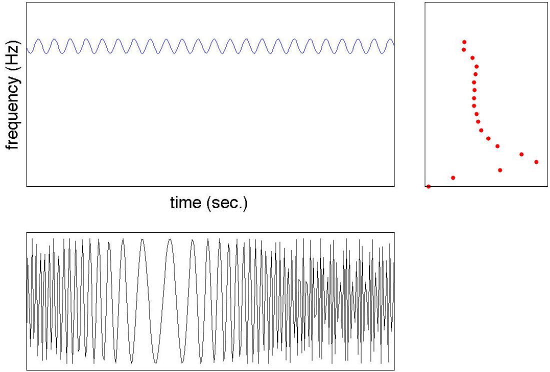

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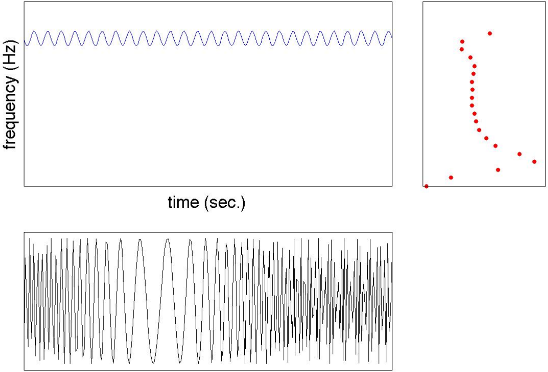

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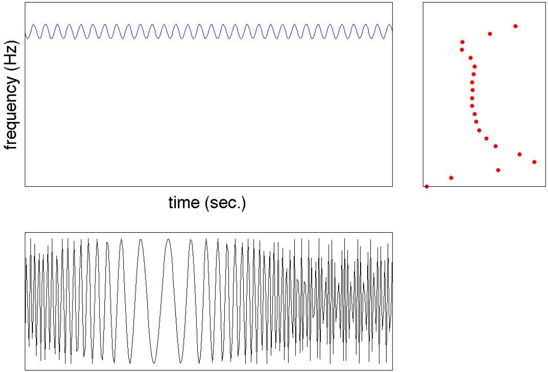

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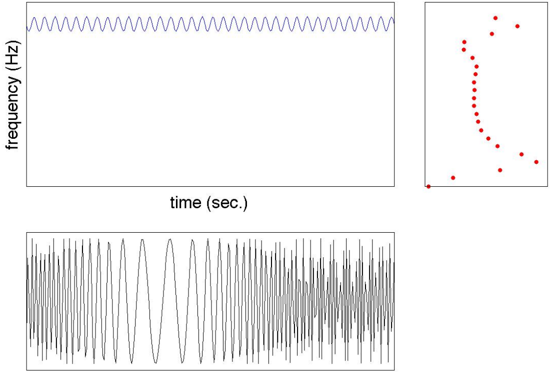

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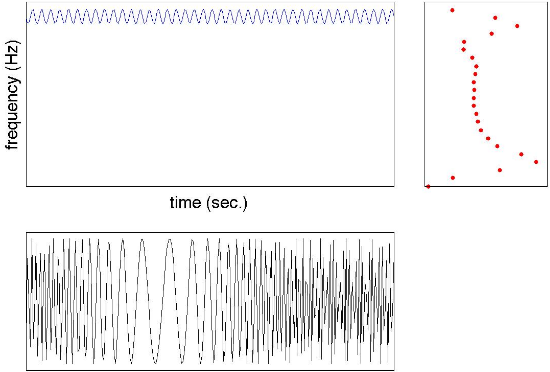

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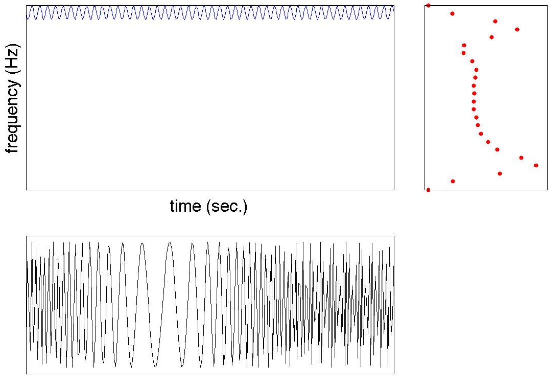

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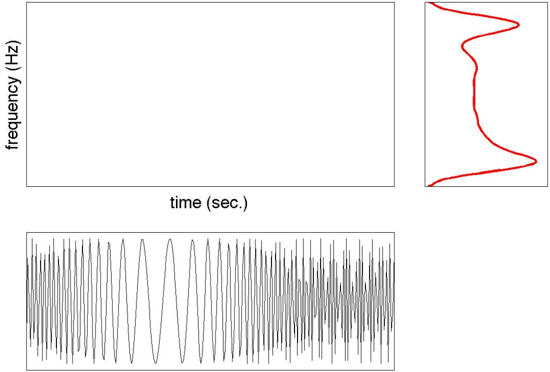

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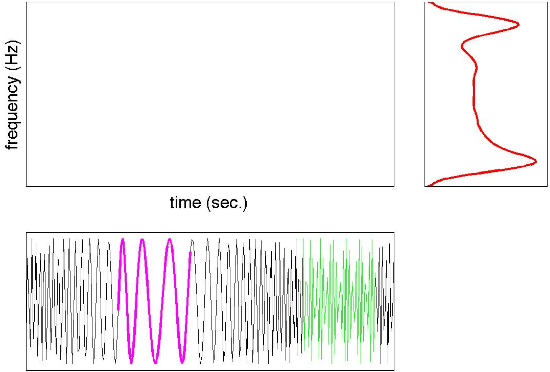

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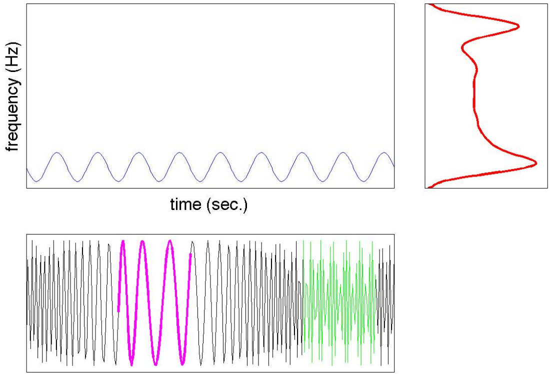

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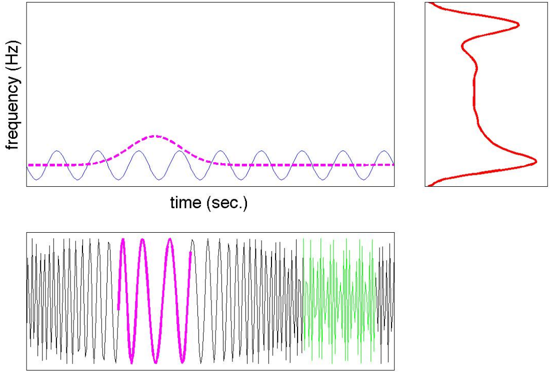

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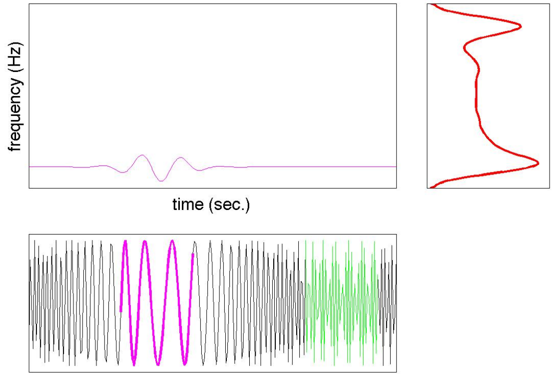

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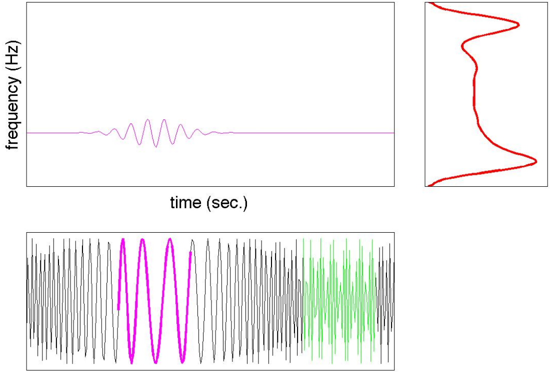

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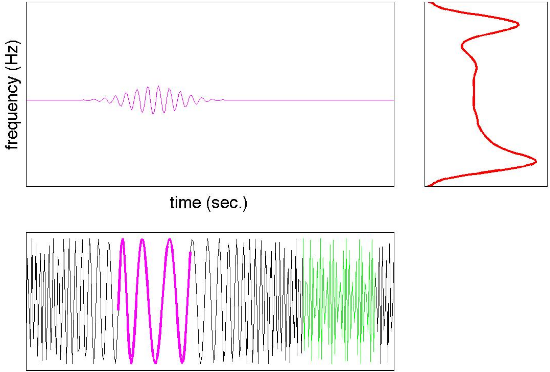

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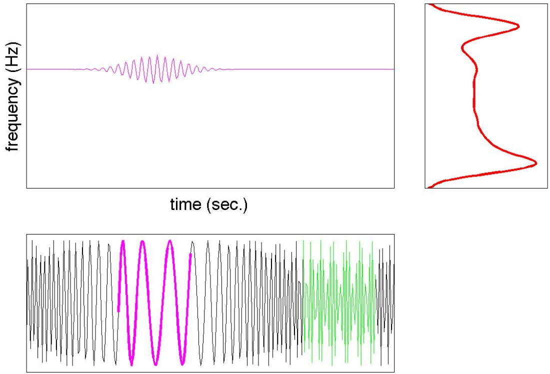

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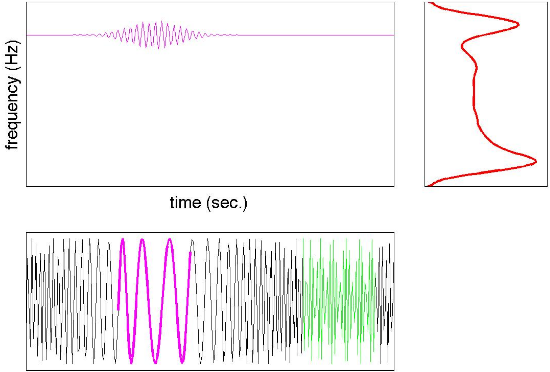

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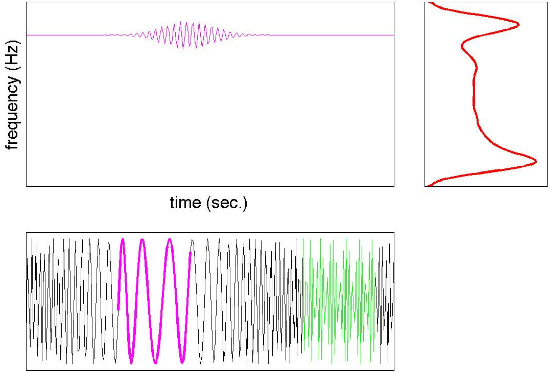

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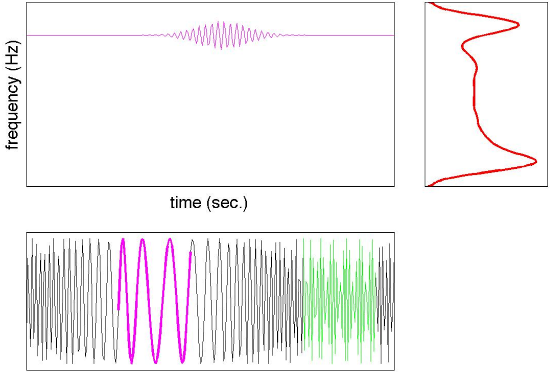

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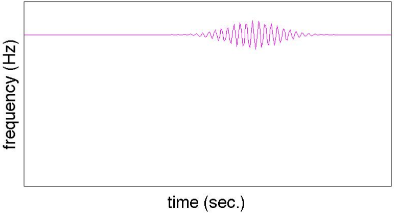

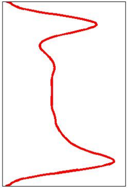

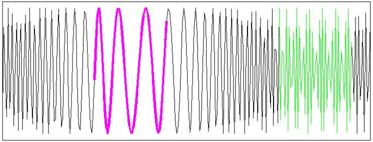

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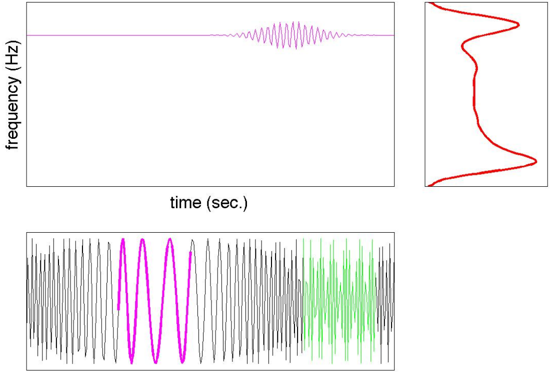

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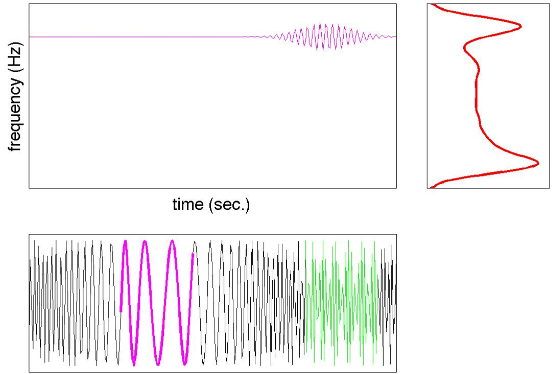

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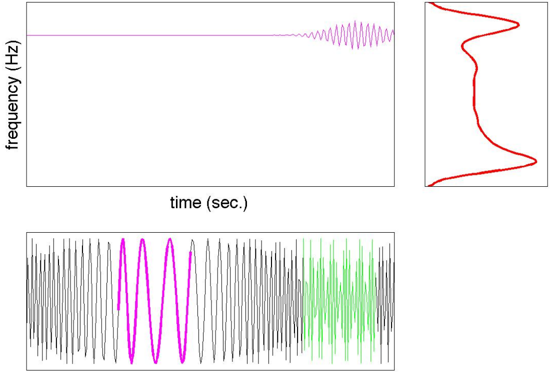

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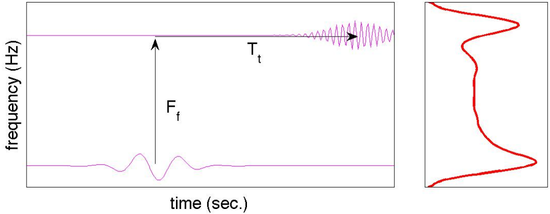

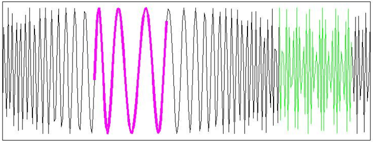

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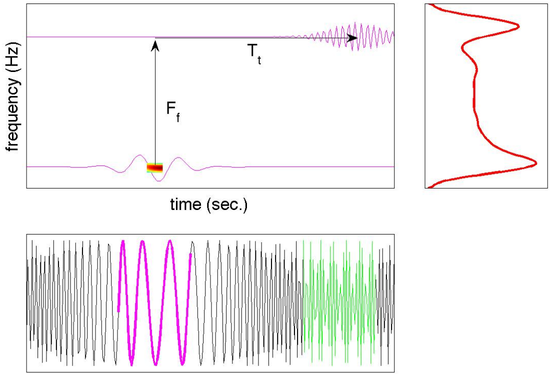

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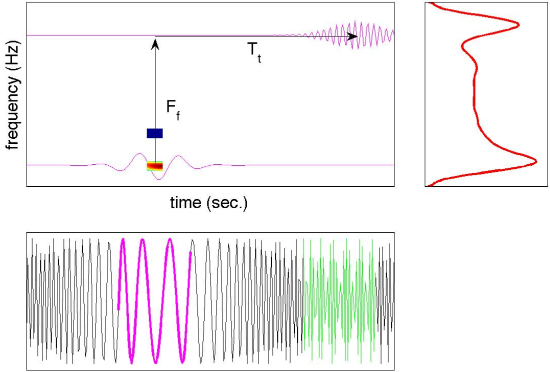

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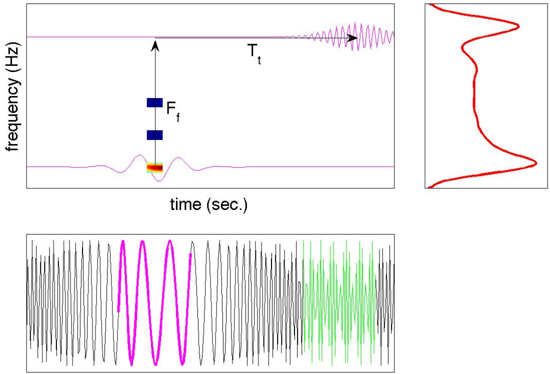

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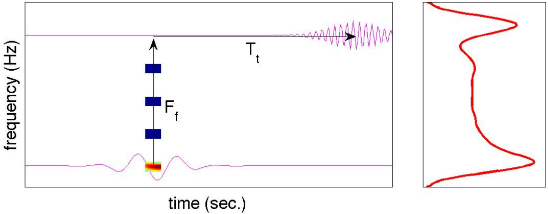

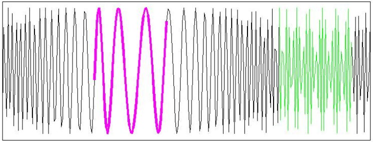

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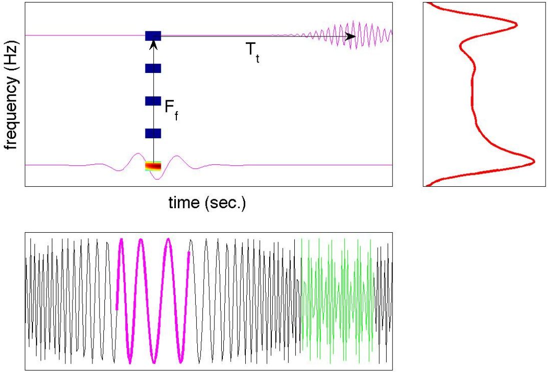

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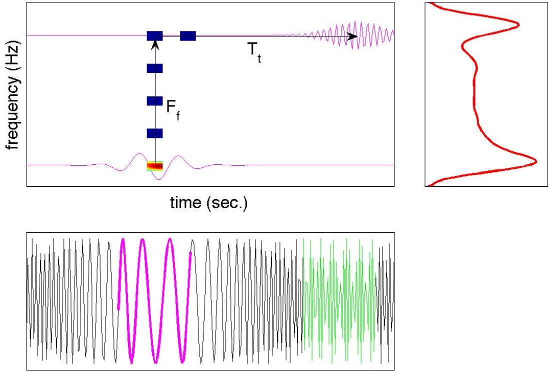

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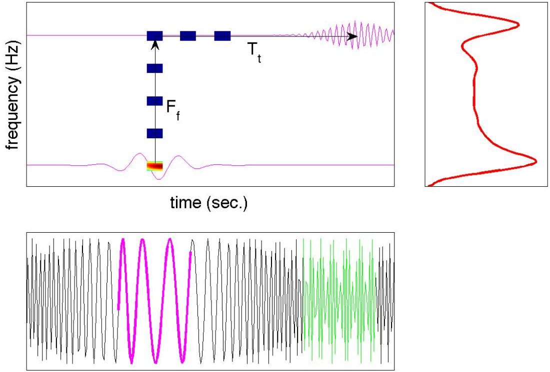

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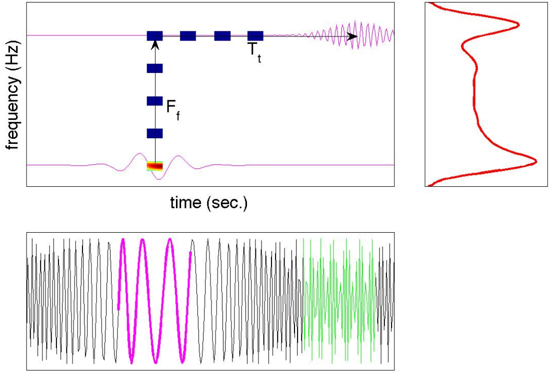

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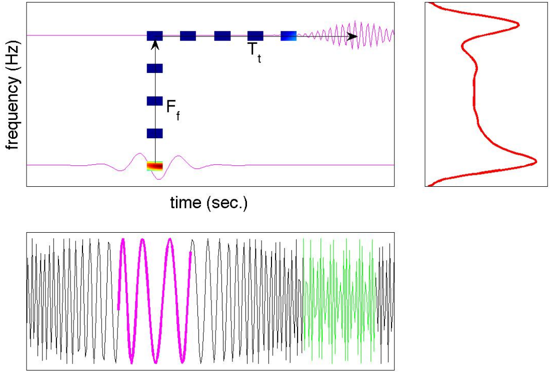

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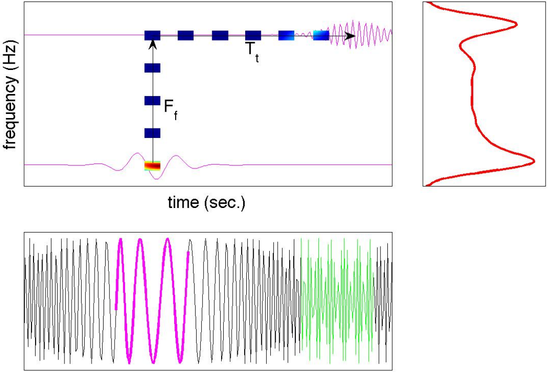

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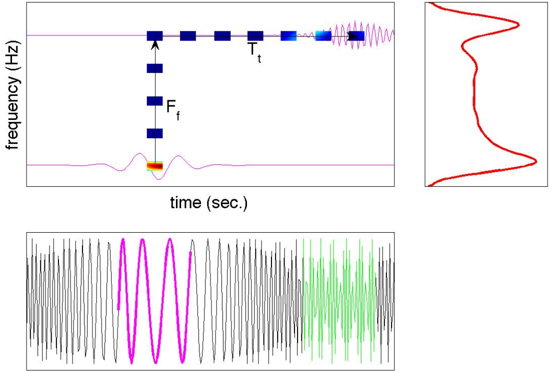

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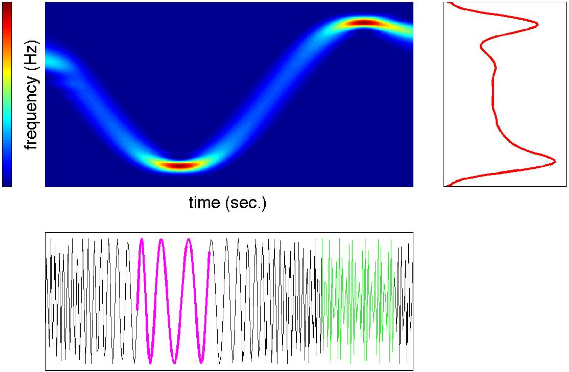

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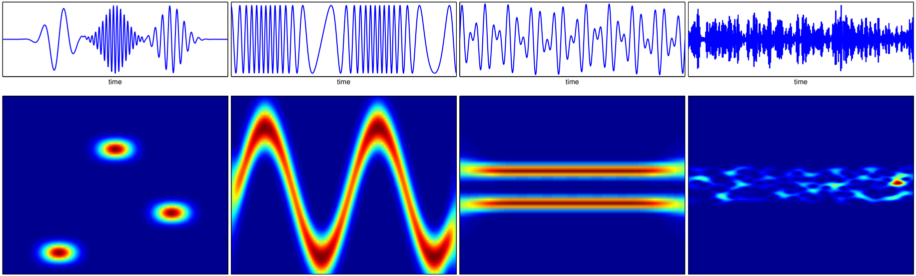
STFT on *non-stationary* signals



Gabor transform : Gaussian window

$$g(u) = C\, e^{-\alpha u^2}$$

achieves the best joint time & frequency resolution (uncertainty principle)

Linear transform

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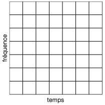

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STFT – properties

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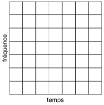

Invertible

$$x(u) = \int_t \int_f L_x(t, f; g)\, g_{t,f}(u)\, \mathrm{d}t\, \mathrm{d}f \qquad \left(\text{iff closure cond.} \ \int |g(u)|^2\, \mathrm{d}u = 1\right)$$

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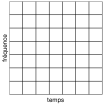

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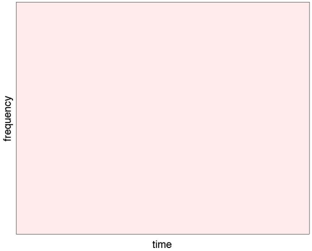

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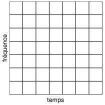
$$L_x(t, f; g) = \langle x, g_{t,f} \rangle = \Big\langle x,$$ 

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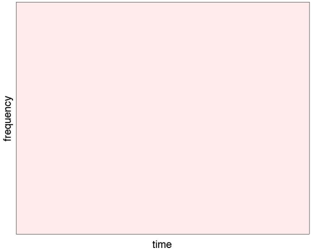

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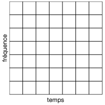
$$L_x(t,f;g) \;=\; \langle x, g_{t,f}\rangle \;=\; \Big\langle x,$$  $$\Big\rangle$$

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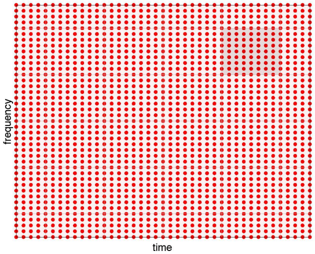

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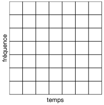

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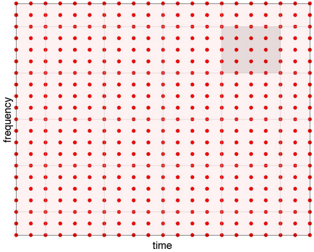

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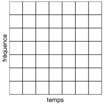

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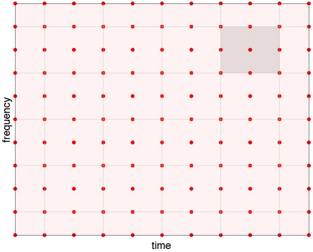

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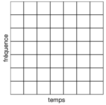

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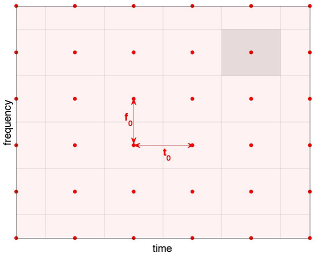

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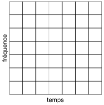

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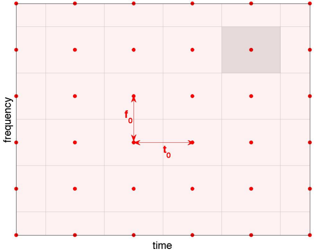
- revert $x(t)$ from a uniform tiling of the time-frequency plane :

$$x(u) = \sum_n \sum_m L_x[n, m]\, \widetilde{g}_{n,m}(u)$$

needs to introduce dual frames.



Frames

Frame (definition) The sequence $\{g_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$ is a frame of $\mathcal{H}$ if there exist two constants $0 < A \leq B$, s.t. for any $f \in \mathcal{H}$:

$$A \parallel f \parallel^2 \leq \sum_{n,m} \mid \langle f, g_{n,m}\rangle \mid^2 \leq B \parallel f \parallel^2 .$$

Dual frame (defintion) Let $\{g_{n,m}\}_{(n,m)}$ be a frame. The dual frame defined by

$$\widetilde{g}_{n,m} = (L^* L)^{-1} g_{n,m} \quad \text{where} \quad L^* L\, x = \sum_{n,m} \langle x, g_{n,m}\rangle g_{n,m}$$

satifies

$$\forall f \in \mathcal{H}, \quad x = \sum_{n,m} \langle x, g_{n,m}\rangle \widetilde{g}_{n,m} = \sum_{n,m} \langle x, \widetilde{g}_{n,m}\rangle g_{n,m}$$

Balian-Law (theorem) If $\{g_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$ is a *windowed Fourier* frame with $t_0 \cdot f_0 = 1$, then

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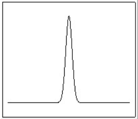
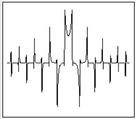
Windowed Fourier frames : Gabor transform

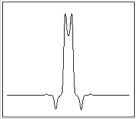


(e) Gabor function g

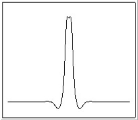


(f) Dual function for critical sampling

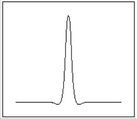
(g) Dual function for oversampling rate 1.06



(h) Dual function for oversampling rate 1.25

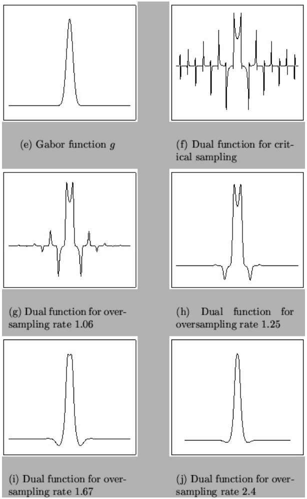


(i) Dual function for oversampling rate 1.67



(j) Dual function for oversampling rate 2.4

Windowed Fourier frames : Gabor transform



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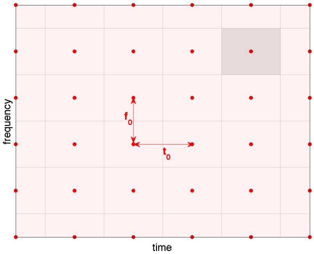
We cannot construct an

orthogonal windowed Fourier basis

with a

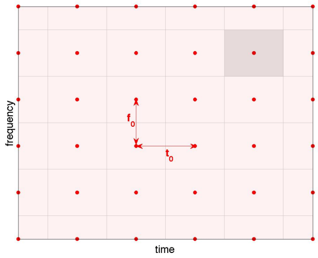
differentiable window g of compact support

STWT

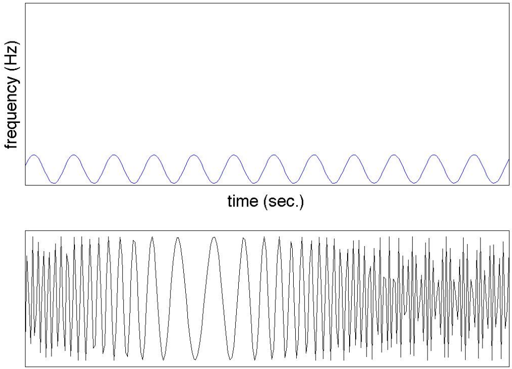


The Wavelet revolution

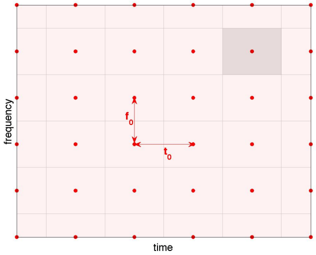
STWT



Wavelet transform



STWT



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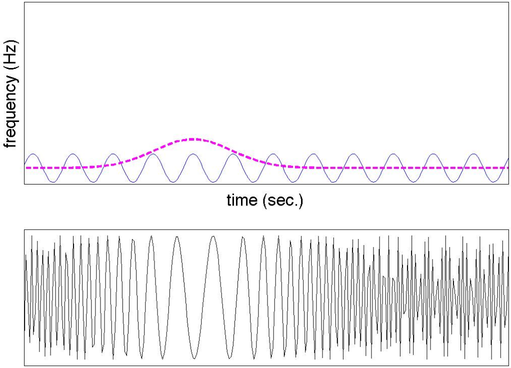

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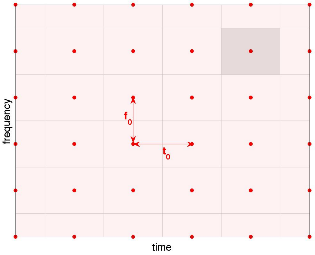

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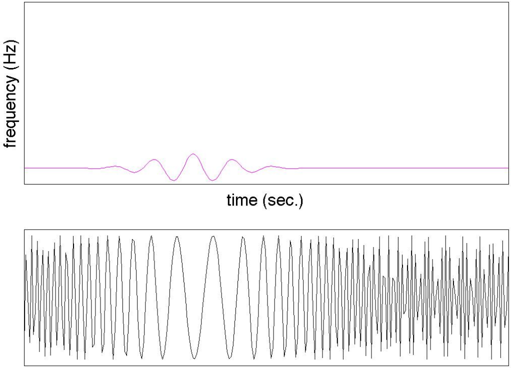

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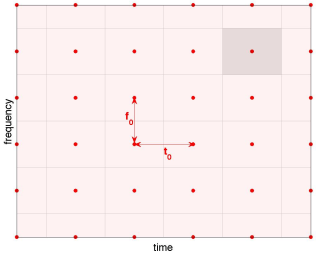

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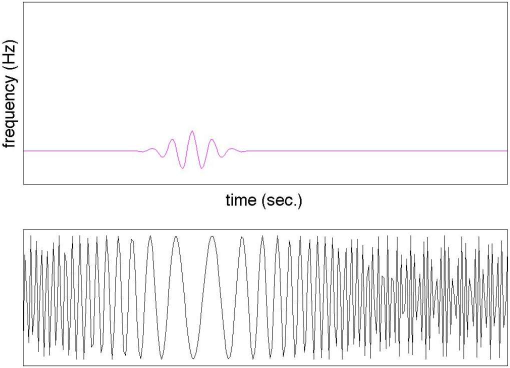

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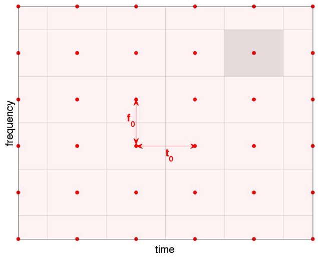

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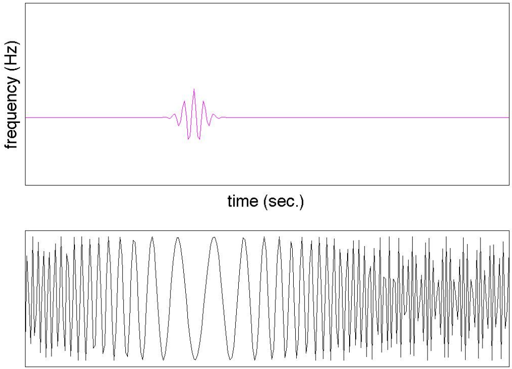

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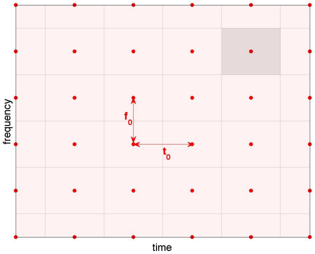
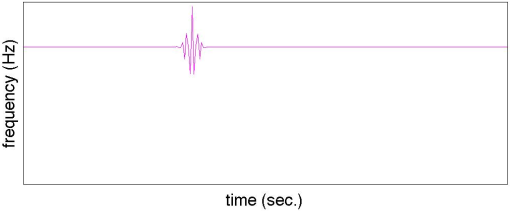
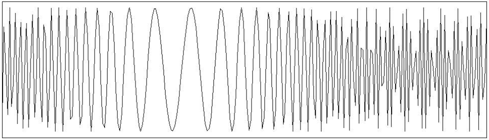


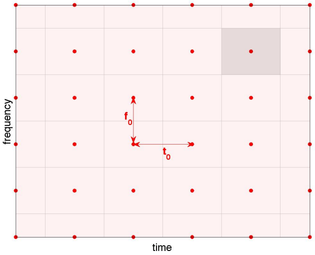
Wavelet transform





The Wavelet revolution

STWT



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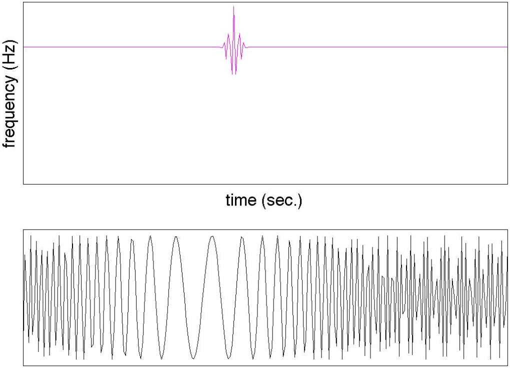

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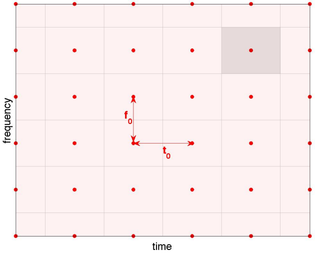

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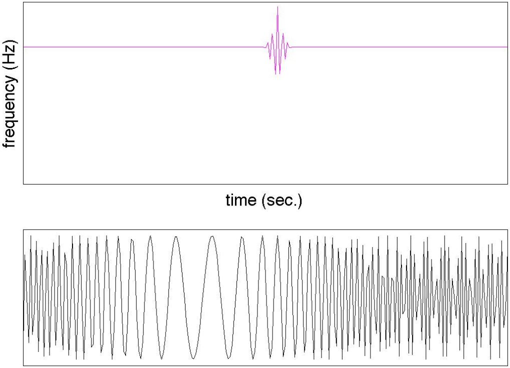

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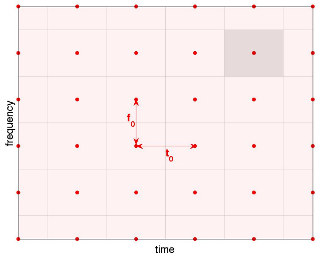

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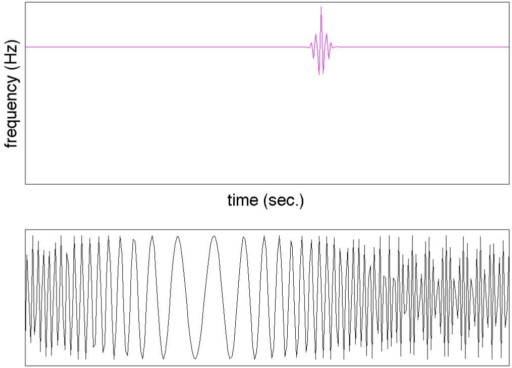

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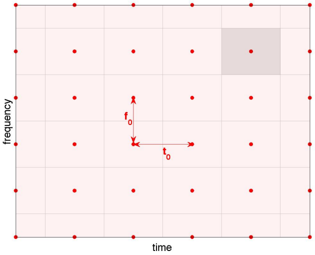

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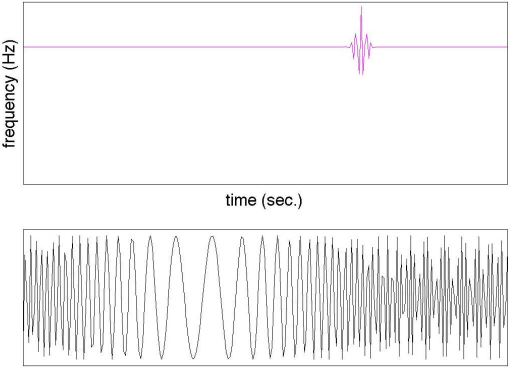

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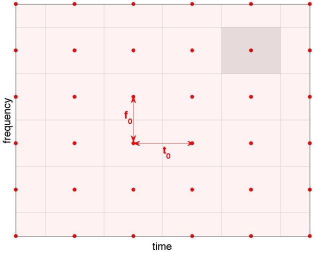

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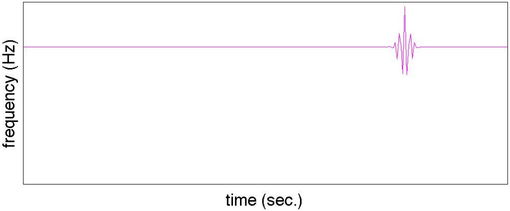

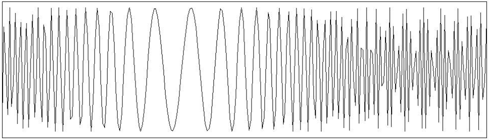

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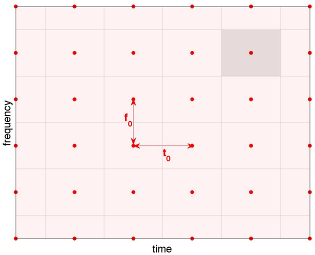

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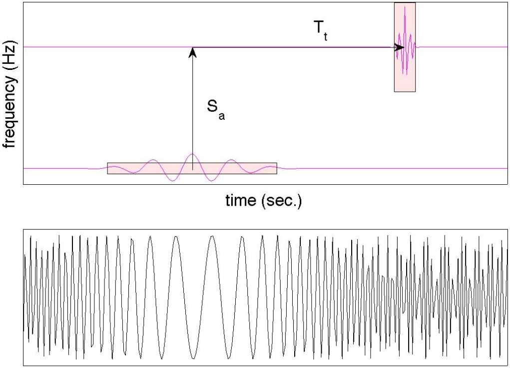

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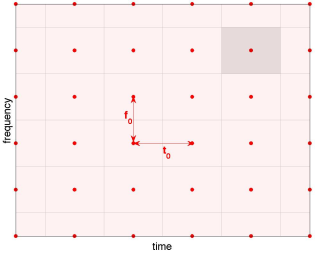

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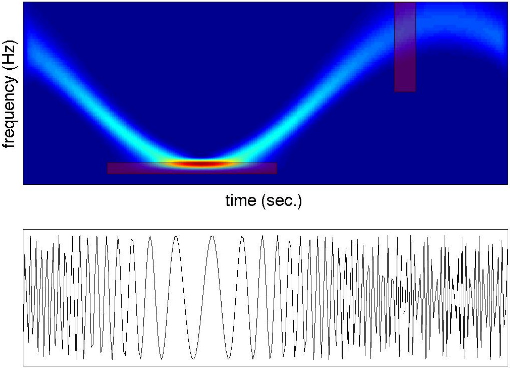

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Continuous wavelet transform (definition)

$$W_x(t,a) = \int x(u)\,\psi_{t,a}(u)\,\mathrm{d}u \quad \text{with} \quad \psi_{t,a}(u) := \frac{1}{\sqrt{a}}\,\psi\left(\frac{u-t}{a}\right)$$

Admissibility condition

$$\int_{\mathbb{R}} \frac{|\Psi(\xi)|^2}{\xi}\,\mathrm{d}\xi = 1 \;\Rightarrow\; \Psi(0) = \int \psi(t)\,\mathrm{d}t = 0$$

ψ is an oscillating function (*wavelet*)

Reconstruction formula (invertible)

$$x(t) = \int_0^\infty \int_{-\infty}^\infty W_x(u,a)\,\psi_{u,a}(t)\,\frac{\mathrm{d}u\,\mathrm{d}a}{a^2}$$

Note : the offset value of x can never be recovered. . .

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Wavelets : A mathematical breakthrough

Orthogonal bases (I. Daubechies theorems)

There exists compactly supported functions ψ that generate orthonormal wavelet bases

$$\{\psi_{j,k}(t);\ (j,k) \in \mathbb{Z}^2\} \quad \text{of} \quad L^2(\mathbb{R}) \quad \text{with} \quad \langle \psi_{j,k}, \psi_{j',k'} \rangle = \delta_{j,j'}\, \delta_{k,k'}$$

(~~*Balian-Law theorem*~~ since $\int t|\psi(t)|^2\, dt < \infty$ *and* $\int f|\Psi(f)|^2\, dt < \infty$)

Discrete wavelet transform

$$\exists (t_0, a_0) \ \text{s.t.} \ (t,a) \longmapsto \left(k\, t_0\, a_0^{-j},\ a_0^{-j}\right)_{(k,j)\in\mathbb{Z}\times\mathbb{Z}} \quad \text{at critical sampling}$$

and

$$\psi_{j,k} := a_0^{j/2}\psi\left(a_0^j t - k t_0\right) \quad : \quad \begin{cases} d^x_{j,k} = \int_{\mathbb{R}} x(u)\, \psi_{j,k}(u)\, \mathrm{d}u \\ x(u) = \sum_{j,k} d^x_{j,k}\, \psi_{j,k}(u) \end{cases}$$

Yields strict conditions on the admissible ψ's (but it turns out "*easy*" to construct localised tight frames, e.g. *Morlet* wavelets, *Mexican hat*, …)

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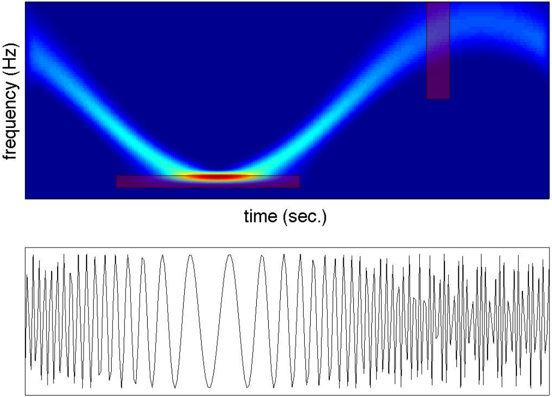
Continuous $\mapsto$ Discrete : Dyadic tiling



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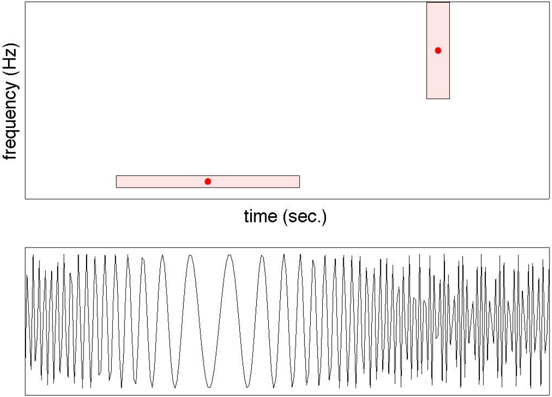

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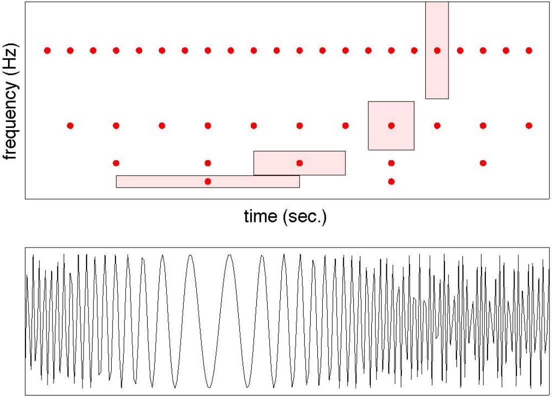

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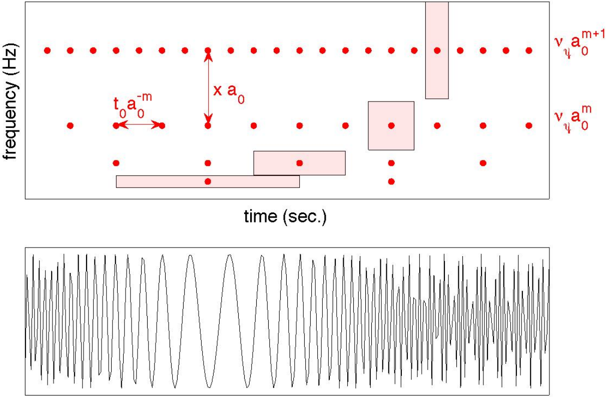

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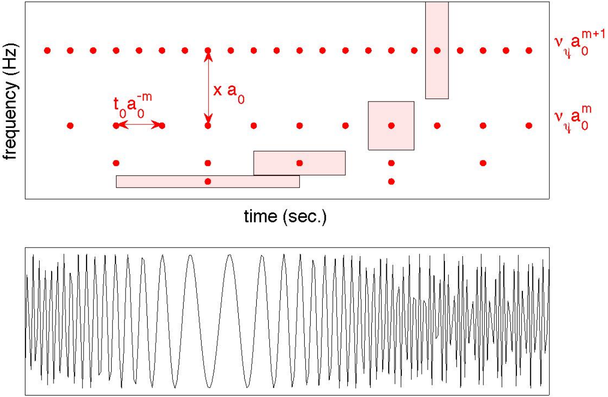
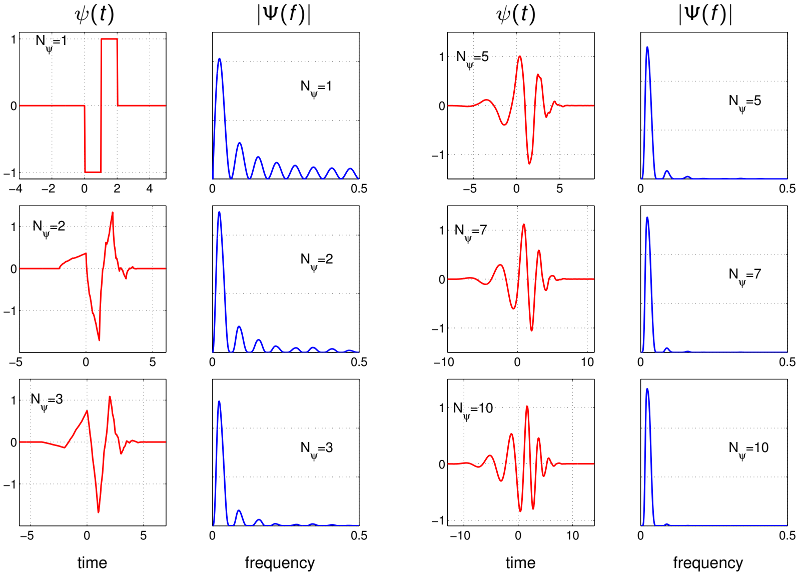


Commonly, in practice, $a_0 = 2$: dyadic tiling

Daubechies wavelet bases



Some properties of (discrete) wavelets

Orthogonal bases

For all series $\{d_{j,k}\}_{(k,j)\in\mathbb{Z}\times\mathbb{Z}}$ there exists a unique *(up to some dc)* signal $x \in L^2(\mathbb{R})$ s.t.

$$x(t) = \sum_{j,k} d_{j,k}\psi_{j,k}(t) \quad \text{and} \quad d_{j,k} = \langle x, \psi_{j,k}\rangle$$

Vanishing moments and regularity (extension of the admissibility condition)

A wavelet ψ has $N_\psi > 0$ vanishing moments iff

$$\forall n < N_\psi : \int t^n \psi(t)\,\mathrm{d}t = 0 \;\Rightarrow\; \Psi(\xi) \overset{\xi\to 0}{\sim} \mathcal{O}\left(\xi^{N_\psi}\right) \quad \left[\Psi^{(n)}(0) = 0, \text{ in Taylor expansion}\right]$$

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For a large class of signals, only a few coefficients $d_{j,k}$ are (significantly) non zero
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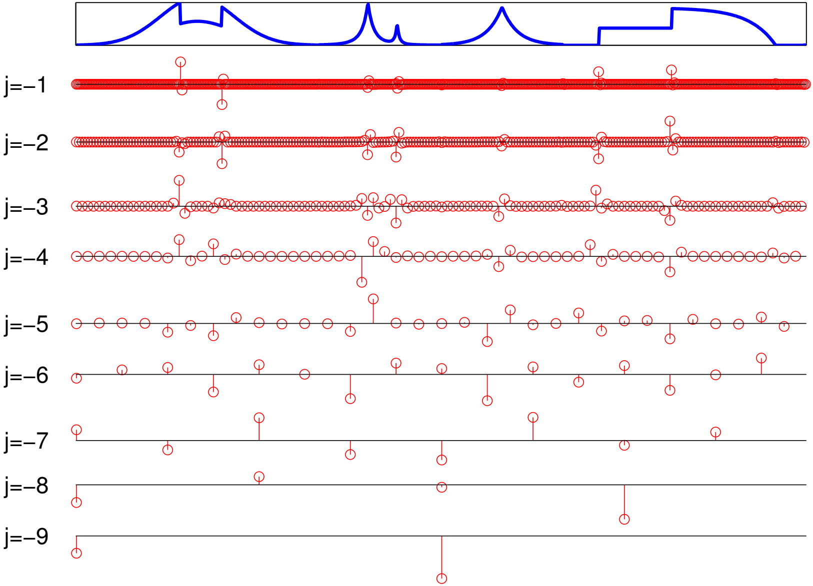
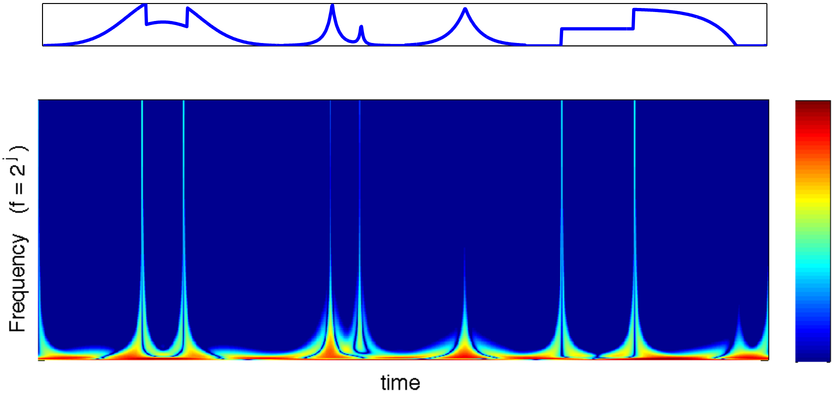
Orthogonal wavelet (Daubechies $N_\psi = 6$)



Continuous wavelet (2nd derivative of Gauss window)



Some properties of (discrete) wavelets

Multiresolution analysis

Orthogonal wavelet bases can be associated to **multiresolution analysis schemes** (S. Mallat, Y. Meyer), with efficient pyramidal filter-bank implementations

Haar system for signal approximation. . .

Multiresolution analysis (MRA)

A multiresolution analysis consists of a sequence of successive approximation spaces V_j satisfying the embedding relation

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots$$

(i) $\displaystyle\bigcap_{j=-\infty}^{j=\infty} V_j = \{0\}$

(ii) $\displaystyle\bigcup_{j=-\infty}^{j=\infty} V_j$ is dense in $L^2(\mathbb{R})$

(iii) $\mathbf{x(t) \in V_j \;\Leftrightarrow\; x(2t) \in V_{j+1}}$

(iv) **there exists a function $\phi(t)$ s.t. $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ is a basis for V_0.**

$$\forall x \in V_0, \quad \int |x(t)|^2\, dt = \sum_k \left| \int x(t)\, \phi(t-k)\, dt \right|^2$$

(iii) & *(iv)* $\Rightarrow$ $\{\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)\}_{k\in\mathbb{Z}}$ is a basis for V_j

Orthogonal projector onto V_j : $\displaystyle P_j x(t) = \sum_k \langle x, \phi_{j,k}\rangle \phi_{j,k}(t) \quad \xrightarrow{j\to\infty} \quad x$

Multiresolution analysis (MRA)

Consider the orthogonal complement space W_j of V_j in V_{j+1},

$$V_{j+1} = V_j \oplus W_j.$$

There exists a function ψ such that (basic principle of MRA)

$$P_{j+1}x = P_j x + \sum_k \langle x, \psi_{j,k} \rangle \psi_{j,k}(t)$$

and the set $\{\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)\}_{k\in\mathbb{Z}}$ is an orthogonal basis of W_j.

(i) $W_j \perp W_{j'}$ for any $j \neq j'$

(ii) $V_j = V_J + \bigoplus_{m=0}^{j=J-j+1} W_{J-m}$ and $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$

(iii) $\mathbf{x(t)} \in \mathbf{W_j} \;\Leftrightarrow\; \mathbf{x(2t)} \in \mathbf{W_{j+1}}$ and $x(t) \in W_j \;\Leftrightarrow\; x(2^{-j}t) \in W_0$

(iv) there exists a function $\psi(t)$ such that the collection

$\{2^{j/2}\psi(2^j t - k),\, j \in \mathbb{Z},\, k \in \mathbb{Z}\}$ forms a basis for $L^2(\mathbb{R})$.

Construction of a MRA system $\{\phi, \psi\}$

Let $\{\phi_{0,n}\}_{n\in\mathbb{Z}}$ be a basis of $V_0 \subset V_1$

$$h[n] = \langle \phi, \phi_{1,n} \rangle = \int \phi(t)\,\sqrt{2}\phi(2t-n)\,dt \quad ; \quad \sum_n |h[n]|^2 = 1 \ \text{(since } \phi \text{ is of norm 1)}$$

and $\phi(t) = \sqrt{2}\sum_n h[n]\phi(2t-n)$ the *two-scale relation*

By Fourier transform and posing $H(\nu) = 2^{-1/2}\sum_n h[n]e^{i2\pi\nu n}$

$$\Phi(\nu) = H\left(\frac{\nu}{2}\right)\Phi\left(\frac{\nu}{2}\right)$$

a bit of linear algebra (on 1-periodic functions) . . . leads to the central relation :

Similarly for $\psi(t) \in W_0 \subset V_1$, posing $\psi(t) = \sqrt{2}\sum_n g[n]\phi(2t-n)$

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Construction of a MRA system $\{\phi, \psi\}$

Let $\{\phi_{0,n}\}_{n\in\mathbb{Z}}$ be a basis of $V_0 \subset V_1$

$$h[n] = \langle \phi, \phi_{1,n} \rangle = \int \phi(t)\sqrt{2}\phi(2t-n)\,dt \quad ; \quad \sum_n |h[n]|^2 = 1 \text{ (since } \phi \text{ is of norm 1)}$$

and $\phi(t) = \sqrt{2}\sum_n h[n]\phi(2t-n)$ the *two-scale relation*

By Fourier transform and posing $H(\nu) = 2^{-1/2}\sum_n h[n]e^{i2\pi\nu n}$

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Solving the MRA system equations

For $\{\phi, \psi\}$ to generate a MRA, H and G must form a pair of **quadrature mirror filters** (QMF)

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Solution of this system imposes

$$G(\nu) = \lambda(\nu) H^*\left(\nu + \frac{1}{2}\right) \quad \text{where} \quad \lambda(\nu) \begin{cases} \text{is a 1-periodic function} \\ \text{verifies} \ \ \lambda(\nu) + \lambda\left(\nu + \frac{1}{2}\right) = 0. \end{cases}$$

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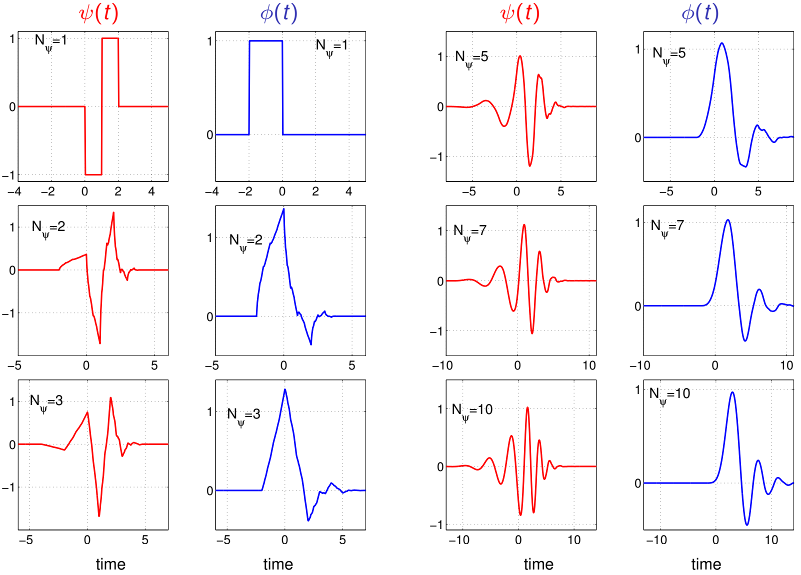
Daubechies MRA system



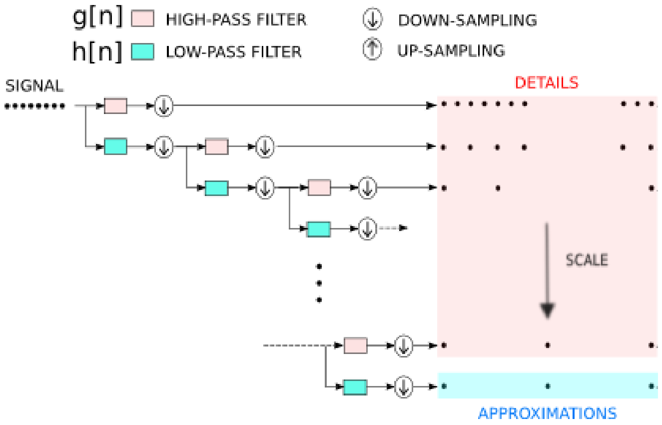
Pyramidal algorithm (S. Mallat)

$$x(t) = \underbrace{\sum_k \underbrace{\langle x, \phi_{J,k} \rangle}_{a_x[J,k]} \phi_{J,k}(t)}_{\text{approximation in } V_J} + \sum_{j \geq J} \underbrace{\sum_k \underbrace{\langle x, \psi_{j,k} \rangle}_{d_x[j,k]} \psi_{j,k}(t)}_{\text{detail in } W_j}, \quad \text{for any arbitrary } J$$

$$\begin{aligned}
a_x[j,n] &= \int x(t)\, 2^{j/2} \phi(2^j t - n)\, dt \\
&= \int x(t) 2^{j/2} \underbrace{\left[\sqrt{2} \sum_k h[k] \phi(2(2^j t - n) - k) \right]}_{\text{decomp. of } \phi_{j,n} \text{ onto } \phi_{j+1,k}} \\
&= \sum_k h[k] \int x(t)\, 2^{(j+1)/2} \phi(2^{j+1} t - (k + 2n))\, dt \\
&= \sum_k h[k]\, a_x[j+1, k+2n] \; = \sum_k h[k-2n]\, a_x[j+1, k] \\
&= a_x[j+1, \cdot] \underset{2n}{*} h[\cdot] \\
d_x[j,n] &= \int x(t)\, 2^{j/2} \psi(2^j t - n)\, dt \\
&= a_x[j+1, \cdot] \underset{2n}{*} g[\cdot]
\end{aligned}$$

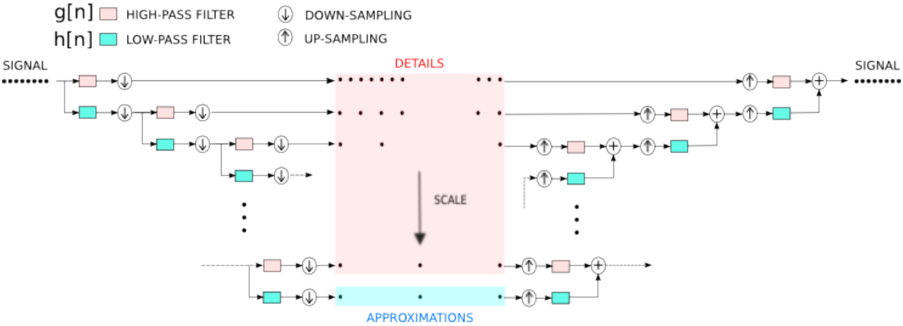
Pyramidal algorithm – decomposition

Assuming a signal $x(t) \in V_0$ (sampling resolution), its projection on $V_J \oplus W_J \oplus W_{J+1} \oplus \ldots, \oplus W_{-1}$ follows a pyramidal decomposition



Pyramidal algorithm – synthesis

The **synthesis** of $x \in V_0$ from its decomposition on $V_J \oplus W_J \oplus W_{J+1} \oplus \ldots, \oplus W_{-1}$ **is perfectly reversible**



Separable wavelet bases for images

To any wavelet orthogonal basis $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2}$ of $L^2(\mathbb{R})$,one can associate a separable wavelet orthogonal basis of $L^2(\mathbb{R}^2)$:

$$\{\psi_{j_1,n_1}(x_1)\psi_{j_2,n_2}(x_2)\}_{(j_1,j_2,n_1,n_2)\in\mathbb{Z}^4}$$

But the resulting decomposition mixes information at different scales 2^{j_1} and 2^{j_2}...

To process images at different levels of detail, we need multi resolutions approximation deriving from dilated functions at the same scale

Separable multiresolutions

Definition The approximation of an image $f(x_1, x_2)$ at resolution 2^{-j} is the orthogonal projection of f on a space V_j^2 that is included in $L^2(\mathbb{R}^2)$

The space V_j^2 is the set of all approximations at the resolution 2^{-j}

When the resolution 2^{-j} decreases, the size of V_j^2 decreases as well

Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution of $L^2(\mathbb{R})$, a separable two-dimensional multiresolution is composed of the tensor product space :

$$V_j^2 = V_j \otimes V_j$$

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Orthogonal bases of a two-dimensional multiresolution

From the theory of *tensor product spaces*. . .

Theorem

if $\{\phi_{j,n}\}_{k\in\mathbb{Z}}$ is an orthonormal bases of V_j, then, for $x = (x_1, x_2)$ and $n = (n_1, n_2)$

$$\left\{ \phi^2_{j,n}(x) = \phi_{j,n_1}(x_1)\phi_{j,n_2}(x_2) = \frac{1}{2^j}\phi\left(\frac{x_1 - 2^j n_1}{2^j}\right)\phi\left(\frac{x_2 - 2^j n_2}{2^j}\right)\right\}_{n\in\mathbb{Z}^2}$$

is an orthonormal basis of V^2_j.

Warning : scale convention changed... $\phi^2_{j,n}(x)$ obtained by scaling by 2^j the two-dimensional separable scaling function $\phi^2(x) = \phi(x_1)\phi(x_2)$ and shifting it on the two-dimensional **square grid of intervals** 2^j

Multiresolution vision

Multiresolution approximation $a_j[n_1, n_2]$ of an image at scales 2^j, for -5 (coarse !) $\geq j \geq -8$ (fine !)

Two-Dimensional wavelet bases

Let ϕ^2 be the scaling function of $\{V_j^2 = V_j \otimes V_j\}_{j\in\mathbb{Z}}$, a separable two-dimensional multiresolution. As for the 1-d case, let W_j^2 be the detail space equal to the orthogonal complement of the lower resolution approximation space $V_j^2 \subset V_{j-1}^2$:

$$V_{j-1}^2 = V_j^2 \oplus W_j^2$$

Theorem Let $\{\phi, \psi\}$ generate a wavelet orthogonal basis (MRA) of $L^2(\mathbb{R})$. We define three wavelets :

$$\psi^1(x) = \phi(x_1)\psi(x_2), \quad \psi^2(x) = \psi(x_1)\phi(x_2), \quad \psi^3(x) = \psi(x_1)\psi(x_2)$$

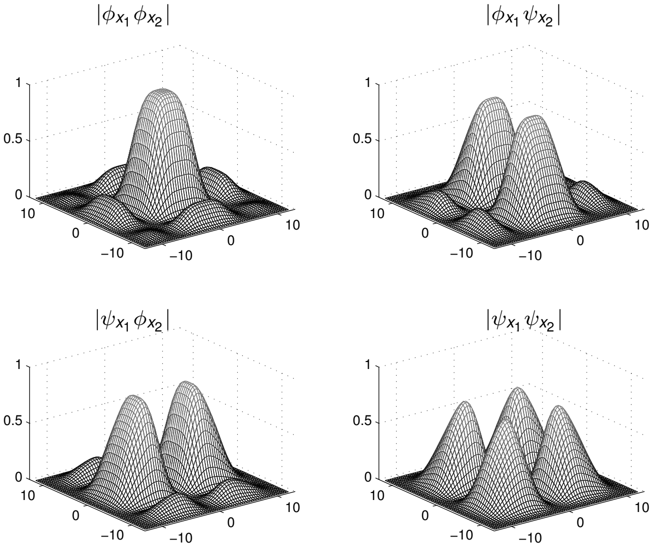
and for $1 \leq m \leq 3$

$$\psi_{j,n}^m(x) = \frac{1}{2^j}\psi^m\left(\frac{x_1 - 2^j n_1}{2^j}, \frac{x_2 - 2^j n_2}{2^j}\right)$$

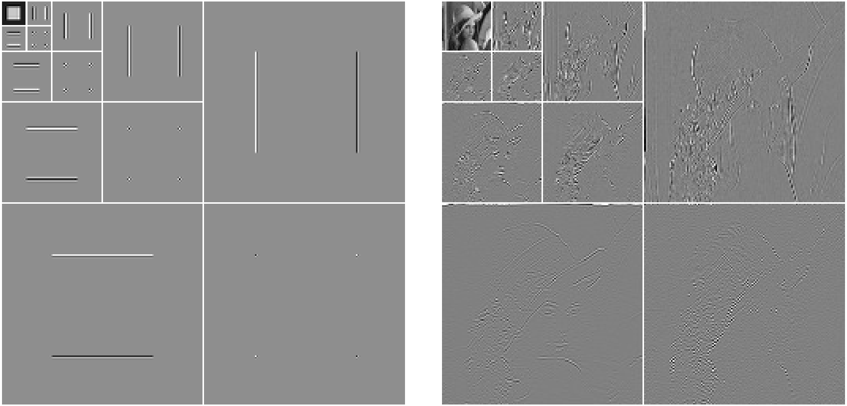
The wavelet family $\{\psi_{j,n}^1, \psi_{j,n}^2, \psi_{j,n}^3\}_{n\in\mathbb{Z}^2}$ is an orthonormal basis of W_j^2

The $\{\psi_{j,n}^1, \psi_{j,n}^2, \psi_{j,n}^3\}_{(j,n)\in\mathbb{Z}^3}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$

Two-Dimensional separable wavelets



Two-Dimensional wavelet decomposition



Separable wavelet transforms of a white square in black background and of Lena, decomposed on resp. 4 and 3 octaves.

Pyramidal Algorithm for 2-D wavelet decompositions

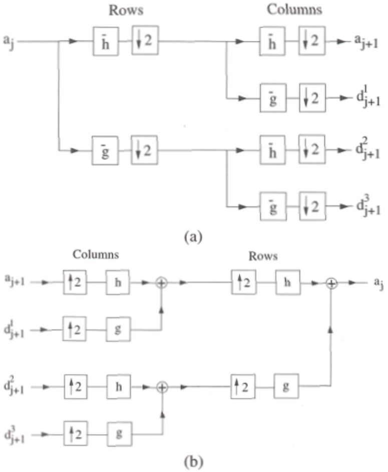


FIGURE 7.27 (a): Decomposition of a_j with 6 groups of one-dimensional convolutions and subsamplings along the image rows and columns. (b): Reconstruction of a_j by inserting zeros between the rows and columns of a_{j+1} and d^k_{j+1}, and filtering the output.

Thresholding Estimation

Let $\mathcal{B} = \{g_m\}_{0 \leq m \leq N-1}$, be a basis of some vector space $\mathcal{S}$.

A *diagonal estimator* of $f \in \mathcal{S}$ from the observation $X = f + W$ can be obtained from :

$$\widetilde{F} = DX = \sum_{m=0}^{N-1} d_m \left(X_{\mathcal{B}}[m] \right) g_m$$

where the functions $\{d_m\}_{m=0,\ldots N-1}$ form the **diagonal operator** that estimates the component $f_{\mathcal{B}}[m]$ independently from $X_{\mathcal{B}}[m]$

There exist optimality results proving that the estimation risk

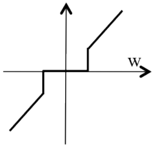
$$r_t(f) = \sum_{m=0}^{N-1} \mathbb{E}\{|f_{\mathcal{B}}[m] - d_m \left(X_{\mathcal{B}}[m] \right)|^2\}$$

is close to the Oracle risk $r_p(f)$ (the risk one would obtain if f and W were known !)

Thresholding Estimation

Hard Thresholding

$$d_m(x) = \rho_T(x) = \begin{cases} x & \text{if } |x| > T \\ 0 & \text{if } |x| \le T \end{cases}$$



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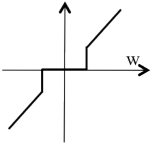
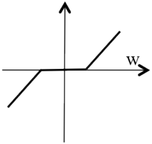


Soft Thresholding

$$d_m(x) = \rho_T(x) = \begin{cases} x - T & \text{if } x > T \\ x + T & \text{if } x \leq -T \\ 0 & \text{if } |x| \leq T \end{cases}$$



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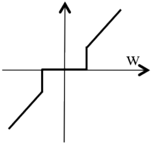

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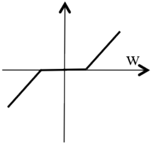


Theorem (Donoho, Johnstone) Let $T = \sigma\sqrt{2\log(N)}$, the risk $r_t(f)$ of a hard or soft thresholding estimator satisfies for all $N \geq 4$

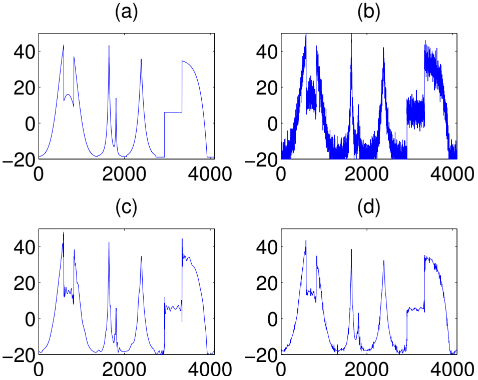
$$r_t(f) \leq (2\log N + 1)\left(\sigma^2 + r_p(f)\right).$$

Remark : the same risk bound holds true for coloured white noise $\sigma_m^2 = \mathbb{E}\{|W_{\mathcal{B}}[m]|^2\}$ and generalises to the adaptive threshold $T_m = \sigma_m\sqrt{2\log N}$

Wavelet Thresholding

$$\widetilde{F} = \sum_{j=L+1}^{J} \sum_{m=0}^{2^{-j}} \rho_T \left(\langle X, \psi_{j,m} \rangle \right) \psi_{j,m} + \sum_{m=0}^{2^{-J}} \rho_T \left(\langle X, \phi_{J,m} \rangle \right) \phi_{J,m}$$

The thresholding performs an adaptive smoothing of the observation that depends on the regularity of the signal f : at scale j, wavelet coefficients above the threshold T localise at the neighbourhood of sharp signal transitions.



(a) : Original signal.

(b) : Noisy signal obtained by adding a Gaussian white noise (SNR = 12.9db)

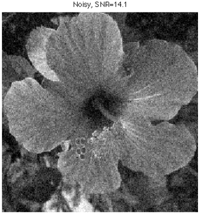
(c) : Estimation with a hard thresholding in a Symmlet 4 wavelet basis (SNR = 23.5db)

(d) : Estimation with a wavelet soft thresholding (SNR = 21.7db)

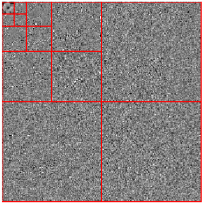
Image denoising with wavelet thresholding

Original

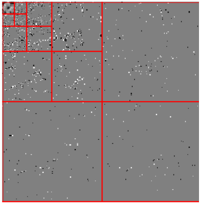
Noisy (SNR=14.1dB)



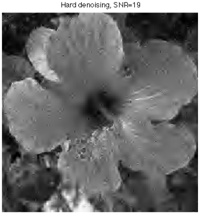
Wavelet coefficients



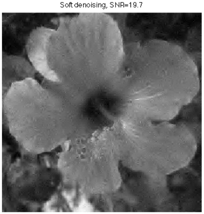
Thresholded coefficients



Hard (SNR=19)



Soft (SNR=19.7)



There exits a variety of advanced wavelet thresholding based denoising (e.g. shift invariant wavelet demonising, block thresholding. . .)

Deconvolution using a mirror wavelet basis

Frequency tiling of a mirror wavelet basis

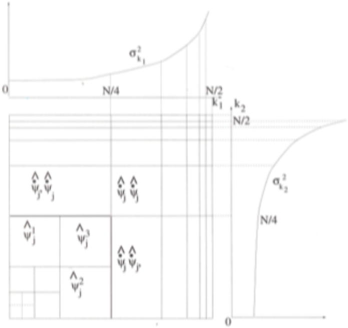


FIGURE 10.13 The mirror wavelet basis (10.167) segments the frequency plane (k_1, k_2) into rectangles over which the noise variance $\sigma^2_{k_1,k_2} = \sigma^2_{k_1}\sigma^2_{k_2}$ varies by a bounded factor. The lower frequencies are covered by separable wavelets ψ^k_j, and the higher frequencies are covered by separable mirror wavelets $\tilde{\psi}_j \tilde{\psi}_{j'}$.

Deconvolution of an airplane image

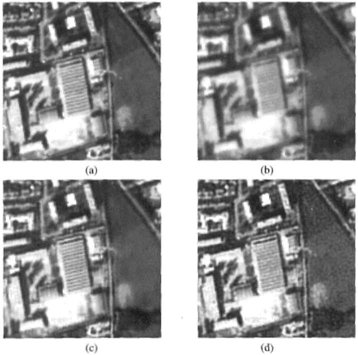


FIGURE 10.14 (a): Original airplane image. (b): Simulation of a satellite image provided by the CNES (SNR = 31.1db). (c): Deconvolution with a translation invariant thresholding in a mirror wavelet basis (SNR = 34.1db). (d): Deconvolution calculated with a circular convolution, which yields a nearly minimax risk for bounded variation images (SNR = 32.7db).

Image Compression

Shannon (theorem) Let X be a source whose symbols $\{x_x\}_{1 \le k \le K}$ occur with probabilities $\{p_k\}_{1 \le k \le K}$. The average bit rate satisfies

$$R_X \ge \mathcal{H}(X) = -\sum_k p_k \log_2 p_k$$

Wavelet image code Let f, a $N\!-\!\text{by}\!-\!N$ image and its wavelet decomposition

$$f = \sum_{m=0}^{N^2-1} f_{\mathcal{B}}[m]\psi_m$$

All wavelets coefficients are quantised with a uniform quantizer

$$\mathcal{Q}(x) = \begin{cases} 0 & \text{if } |x| < \Delta/2 \\ \text{sign}(x)\, k\, \Delta & \text{if } (k-1/2)\Delta \le |x| < (k+1/2)\Delta \end{cases}$$

and the coded image $\widetilde{f} = \sum_{m=0}^{N^2-1} \mathcal{Q}\left(f_{\mathcal{B}}[m]\right)\psi_m$ requires a bit budget (total number of bits needed to encode the N^2 coefs.) $R = N^2 R_X$.

The specific distribution of wavelet coefficients allows a small bit rate !

Image Compression

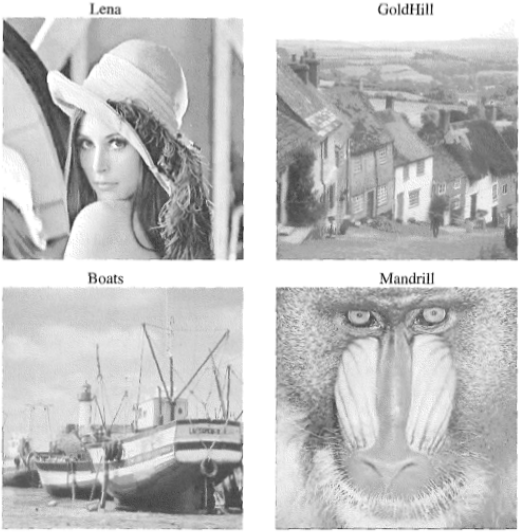


FIGURE 11.6 These images of $N^2 = 512^2$ pixels are coded with $\bar{R} = 0.5$ bit/pixel, by a wavelet transform coding.

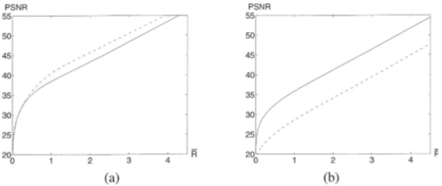


FIGURE 11.7 PSNR as a function of $\bar{R}$. (a): Lena (solid line) and Boats (dotte line). (b): GoldHill (solid line) and Mandrill (dotted line)

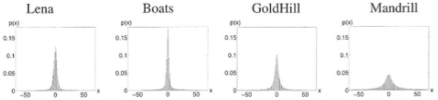


FIGURE 11.8 Normalized histograms of orthogonal wavelet coefficients for each image.

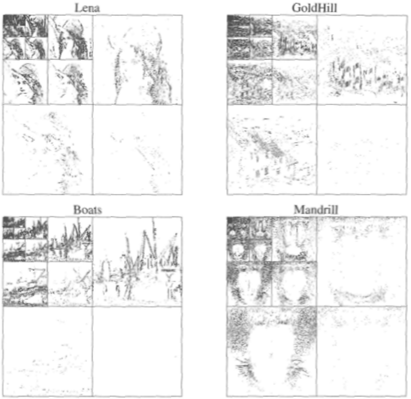


FIGURE 11.9 Significance map of quantized wavelet coefficients for images coded with $\bar{R} = 0.5$ bit/pixel.

Dyadic Wavelets (non-orthogonal)

Application of oriented (dyadic) wavelets in image processing lie in many **physiological** and **computer vision** studies : Textures can be synthesised and discriminated with oriented two-dimensional wavelet transforms.

$\rightarrow$ multiscale edge detection from the local maxima of a wavelet transform.

Oriented wavelets (definition) In 2-d, a dyadic wavelet transform is computed with several mother wavelet $\{\psi^k\}_{1\leq k\leq K}$ which have different spatial orientations. For $x = (x_1, x_2)$, we denote

$$\psi^k_{2^j}(x_1, x_2) = \frac{1}{2^j}\psi^k\left(\frac{x_1}{2^j}, \frac{x_2}{2^j}\right)$$

and the wavelet transform of $f \in L^2(\mathbb{R}^2)$ in the direction k, at position $u = (u_1, u_2)$ and scale 2^j

$$W^k f(u, 2^j) = \langle f(x), \psi^k_{2^j}(x - u)\rangle$$

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Dyadic Wavelets (non-orthogonal)

We can show that dyadic wavelet transform can generate a frame and there exist reconstruction (dual) wavelets $\left\{\widetilde{\psi}^k\right\}_{1\leq k\leq K}$ such that

$$f(x) = \sum_{j=-\infty}^{j=\infty} \frac{1}{2^{2j}} \sum_{k=1}^{K} W^k f(\cdot, 2^j) \star \widetilde{\psi}^k_{2^j}(x)$$

For example, a wavelet in the direction α may be defined as the partial derivative of order p of a window $\theta(x)$ in the direction of the vector $\vec{n} = (\cos\alpha, \sin\alpha)$

$$\begin{aligned}\psi^\alpha(x) &= \frac{\partial^p \theta(x)}{\partial \vec{n}^p} = \left(\cos\alpha \frac{\partial}{\partial x_1} + \sin\alpha \frac{\partial}{\partial x_2}\right)^p \theta(x) \\ &= \sum_{k=0}^{k=p} \binom{p}{k} (\cos\alpha)^k (\sin\alpha)^{p-k} \psi^k(x) \quad (K = p+1)\end{aligned}$$

and $\quad \psi^k(x) = \dfrac{\partial^p \theta(x)}{\partial x_1^k \partial x_2^{p-k}}, \quad$ for $\; 0 \leq k \leq p$

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&= \quad \sum_{k=0}^{k=p} \left(\begin{array}{c} p \\ k \end{array}\right) (\cos\alpha)^k (\sin\alpha)^{p-k} \psi^k(x) \quad (K = p+1)
\end{aligned}$$

$$\text{and} \qquad \psi^k(x) = \frac{\partial^p \theta(x)}{\partial x_1^k \partial x_2^{p-k}}, \quad \text{for } \; 0 \leq k \leq p$$

Dyadic Wavelets (non-orthogonal)

For appropriate windows $\theta(x)$, these $K = p + 1$ partial derivatives define a family of dyadic wavelets. In the direction α the wavelet transform $W^{\alpha} f\left(u, 2^{j}\right)$ is computed as a linear combination of the $p + 1$ components $W^{k} f\left(u, 2^{j}\right)$.

For example

$$\theta(x) = \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) \text{ and } p = 1 \Rightarrow \vec{\psi}(x) = \begin{cases} \psi^0(x) = \frac{\partial\theta(x)}{\partial x_2} = -x_2\theta(x) \\ \psi^1(x) = \frac{\partial\theta(x)}{\partial x_1} = -x_1\theta(x) \end{cases}$$

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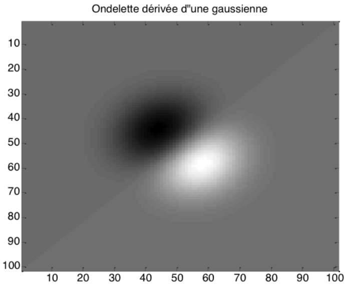
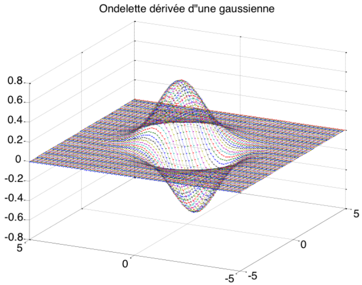


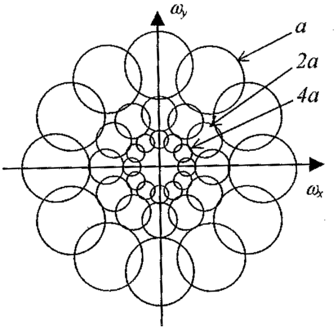


Dyadic Wavelets (non-orthogonal)

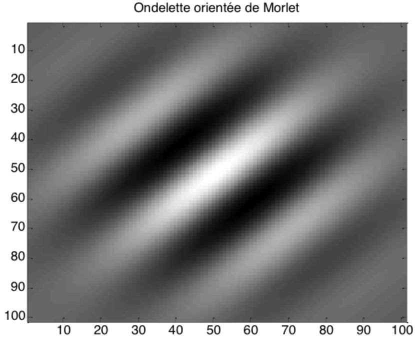
Gabor Wavelets

$$\psi^k(x) = \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) \exp[-i\eta(x_1 \cos\alpha_k + x_2 \sin\alpha_k)]$$

Gabor Dyadic wavelets



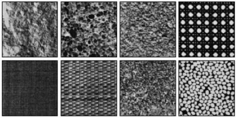
Oriented Morlet wavelet



Dyadic wavelet transform of textures

Dyadic wavelet transform of textures

classification



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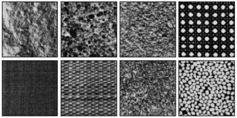
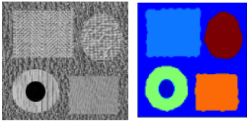


segmentation



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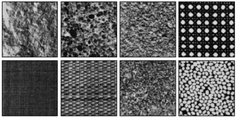

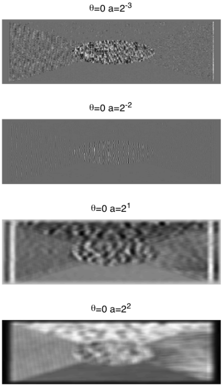

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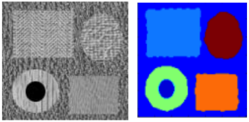

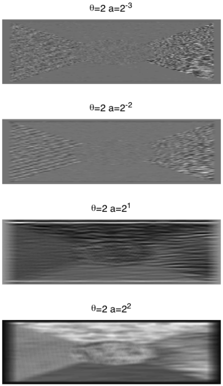

Multiscale edge detection

Goal Detect points of sharp variation in a image $f(x_1, x_2)$

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Canny Algorithm Calculate the modulus of the gradient vector $\vec{\nabla} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right)$

The partial derivative of f in the direction $\vec{n} = (\cos\alpha, \sin\alpha)$ is

$$\frac{\partial f}{\partial \vec{n}} = \vec{\nabla} \cdot \vec{n} = \frac{\partial f}{\partial x_1} \cos\alpha + \frac{\partial f}{\partial x_2} \sin\alpha$$

$\left| \frac{\partial f}{\partial \vec{n}} \right|$ is maximum when $\vec{n}$ is collinear to $\vec{\nabla} f$

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$\Rightarrow$ $\vec{\nabla} f$ **is parallel to the direction of maximum changes of the surface** $f(x)$

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$\Rightarrow$ $\vec{\nabla} f$ **is parallel to the direction of maximum changes of the surface** $f(x)$

A point $y \in \mathbb{R}^2$ is defined as an edge if $|\vec{\nabla} f(x)|$ is locally maximum at $x = y$

when $x = y + \lambda \vec{\nabla} f(y)$ in the vicinity of y (i.e. λ small)

Edge points are inflexion points of f

Multiscale edge detection

Wavelet Maxima for images

Multiscale edge detection

Wavelet Maxima for images

Let us consider the Gabor dyadic (oriented) wavelet

$$\vec{\psi}(x) = \begin{cases} \psi^0(x) = \frac{\partial\theta(x)}{\partial x_2} = -x_2\theta(x) \\ \psi^1(x) = \frac{\partial\theta(x)}{\partial x_1} = -x_1\theta(x) \end{cases}$$

and the corresponding dyadic wavelet transform

$$W^k f(u, 2^j) = \langle f(x), \psi^k_{2^j}(x-u) \rangle, \quad k = 0, 1$$

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We can show that the wavelet transform components of a image f verifies

$$\vec{W} f(u, 2^j) = \begin{pmatrix} W^0 f(u, 2^j) \\ W^1 f(u, 2^j) \end{pmatrix} = 2^j \vec{\nabla} \left(f \star \theta_{2^j}\right)(u) \qquad \text{(multiscale)}$$

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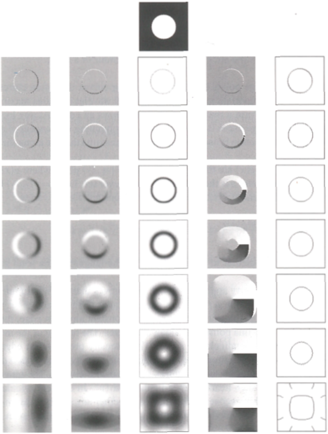
An edge point at scale 2^j is a point ν such that $\left|\vec{W} f(u, 2^j)\right|$ is **locally maximum** at $u = \nu$ when $u = \nu + \lambda$ angle $\left\{\vec{W} f(u, 2^j)\right\}$ (for λ small enough)

Wavelet Maxima for images



(a) (b) (c) (d) (e)

FIGURE 6.9 The top image has $N^2 = 128^2$ pixels. (a): Wavelet transform in the horizontal direction, with a scale 2^j that increases from top to bottom: $\{W^1 f(u,2^j)\}_{-6\leq j\leq 0}$. Black, grey and white pixels correspond respectively to negative, zero and positive values. (b): Vertical direction: $\{W^2 f(u,2^j)\}_{-6\leq j\leq 0}$. (c): Wavelet transform modulus $\{Mf(u,2^j)\}_{-6\leq j\leq 0}$. White and black pixels correspond respectively to zero and large amplitude coefficients. (d): Angles $\{Af(u,2^j)\}_{-6\leq j\leq 0}$ at points where the modulus is non-zero. (e): Wavelet modulus maxima are in black.

Wavelet Maxima for images

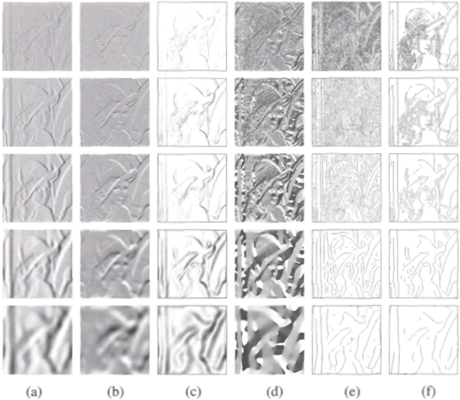


FIGURE 6.10 Multiscale edges of the Lena image shown in Figure 6.11. (a): $\{W^1 f(u, 2^j)\}_{-7 \le j \le -3}$. (b): $\{W^2 f(u, 2^j)\}_{-7 \le j \le -3}$. (c): $\{Mf(u, 2^j)\}_{-7 \le j \le -3}$. (d): $\{Af(u, 2^j)\}_{-7 \le j \le -3}$. (e): Modulus maxima. (f): Maxima whose modulus values are above a threshold.

Reconstruction from Wavelet Maxima lines

FIGURE 6.11 (a): Original Lena. (b): Reconstructed from the wavelet maxima displayed in Figure 6.10(e) and larger scale maxima. (c): Reconstructed from the thresholded wavelet maxima displayed in Figure 6.10(f) and larger scale maxima.