WAVELET ANALYSIS OF FRACTIONAL BROWNIAN MOTION IN MULTIFRACTAL TIME

Paulo Gonçalvès and *Rudolf H. Riedi* ◇

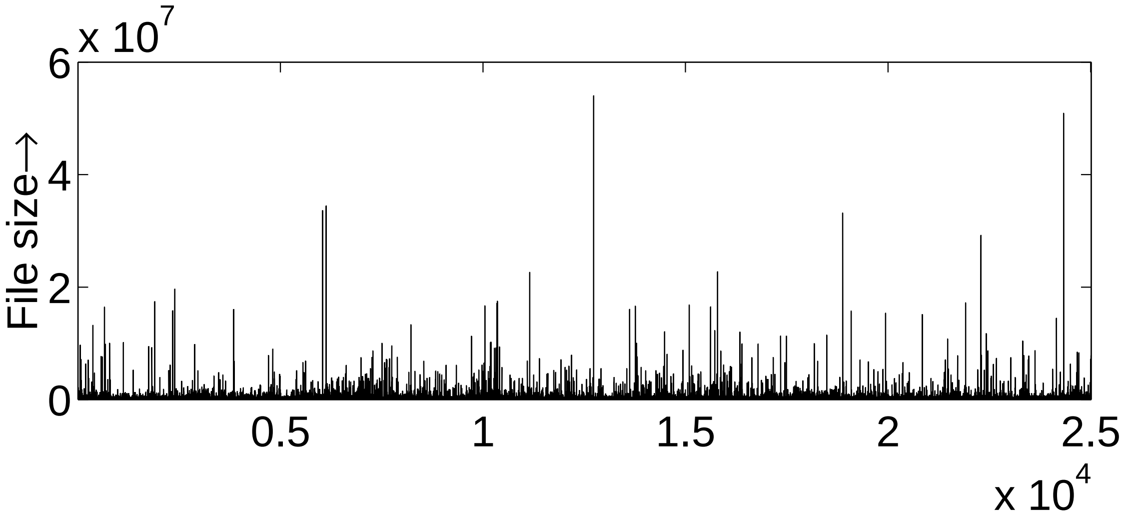
INRIA Rocquencourt, France

Rice University, Houston TX

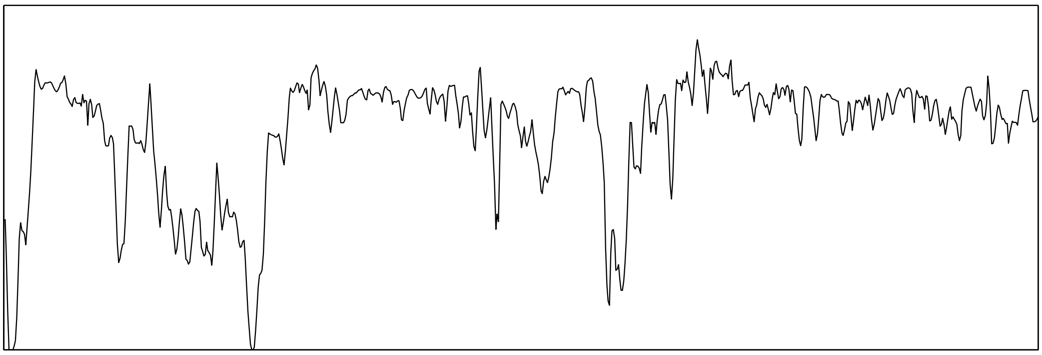
◇ Supported by
National Science Foundation
Office of Naval Research
Rice Consortium for Computational Seismic Interpretation

Multifractals in Nature

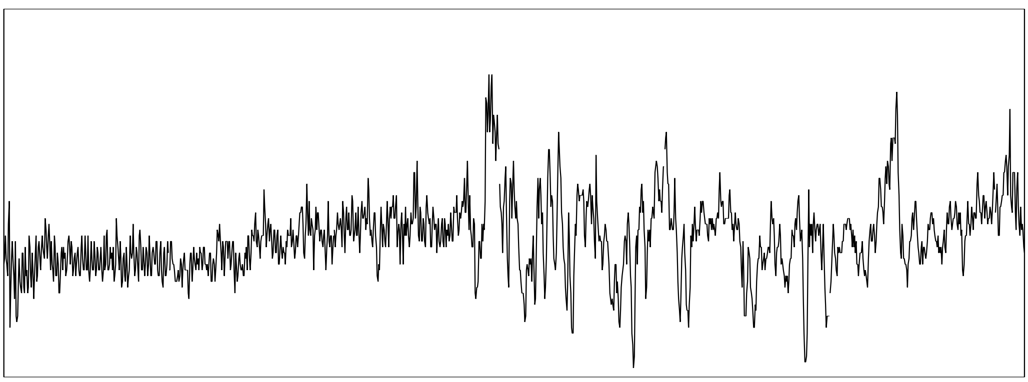
Network traffic



Well-log data



Heart bit inter-arrival time



Wavelet characterization of Hölder Exponent

Wavelet Decomposition

Definition

$$c_{j,k} \; = \; 2^{-j} \int \psi^*(2^{-j}t - k)\, x(t)\, dt, \; (j,k) \in \mathbb{Z}^2$$

Vanishing moments

$$\int t^r\, \psi(t)\, dt = 0, \; r = 0, \ldots, \mathcal{R} - 1.$$

Oscillatory behavior measure

x has local Hölder regularity $\alpha(t)$ at time t

$$\Downarrow$$

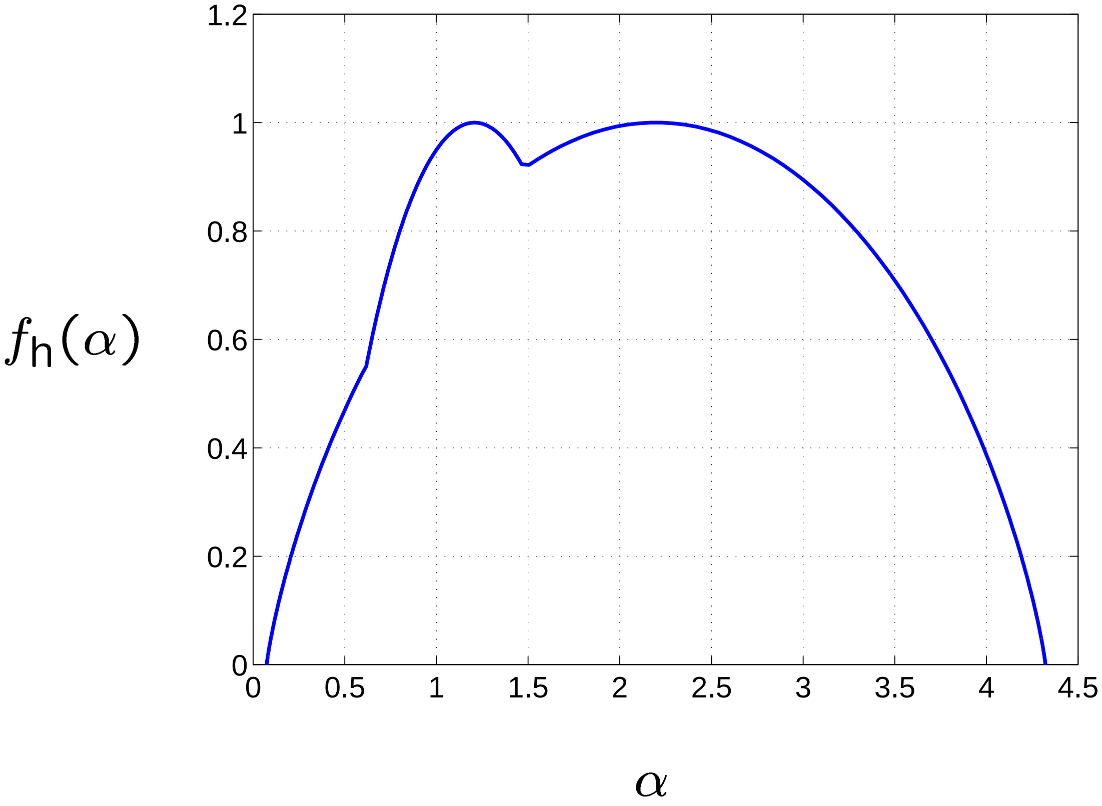
$$\alpha(t) \; : \; |c_{j,k}| = O\left(2^{j\alpha(t)}\right), \;\; k2^j \to t$$

Hölder Spectra
(Regularity Spectra)

Typically, a process will possess many different singularity strengths. The frequency (in t) of occurrence of a given singularity strength α is measured by the *multifractal spectrum*

Hausdorff spectrum (geometric spectrum)

$$f_{\mathsf{h}}(\alpha) \;=\; \dim_{\mathcal{H}}\{t : \alpha(t) = \alpha\}$$



Hölder Spectra

Grain Spectrum (Large Deviation principle)

An adaptation of the Box-counting dimension to the multifractal case

- Define a *iso-*α 2^j*-covering*

$$M^j(\alpha,\varepsilon) \;:=\; \#\left\{k \;:\; |\alpha_k^j - \alpha| < \varepsilon\right\}$$

- with "free" choice of *Grain exponent*

$$\begin{aligned}
\alpha_k^j \;&:=\; \tfrac{1}{j}\log \sup_{s,t}\{|x(t)-x(s)| : (k-1)2^j \le s \le t \le (k+2)2^j\} \\
&:=\; \frac{1}{j}\log|c_{j,k}| \\
&\quad\;\left(M^j(\alpha,\varepsilon) = \#\left\{k \;:\; 2^{j(\alpha+\varepsilon)} \le |c_{j,k}| \le 2^{j(\alpha-\varepsilon)}\right\}\right) \\
&:=\; ???
\end{aligned}$$

- The *Grain Spectrum* reads

$$f_g(\alpha) \;:=\; \lim_{\varepsilon\to 0}\lim_{j\to-\infty} -\frac{1}{j}\log_2 M^j(\alpha,\varepsilon)$$

$$\boxed{f_h(\alpha) \;\le\; f_g(\alpha)}$$

Hölder Spectra

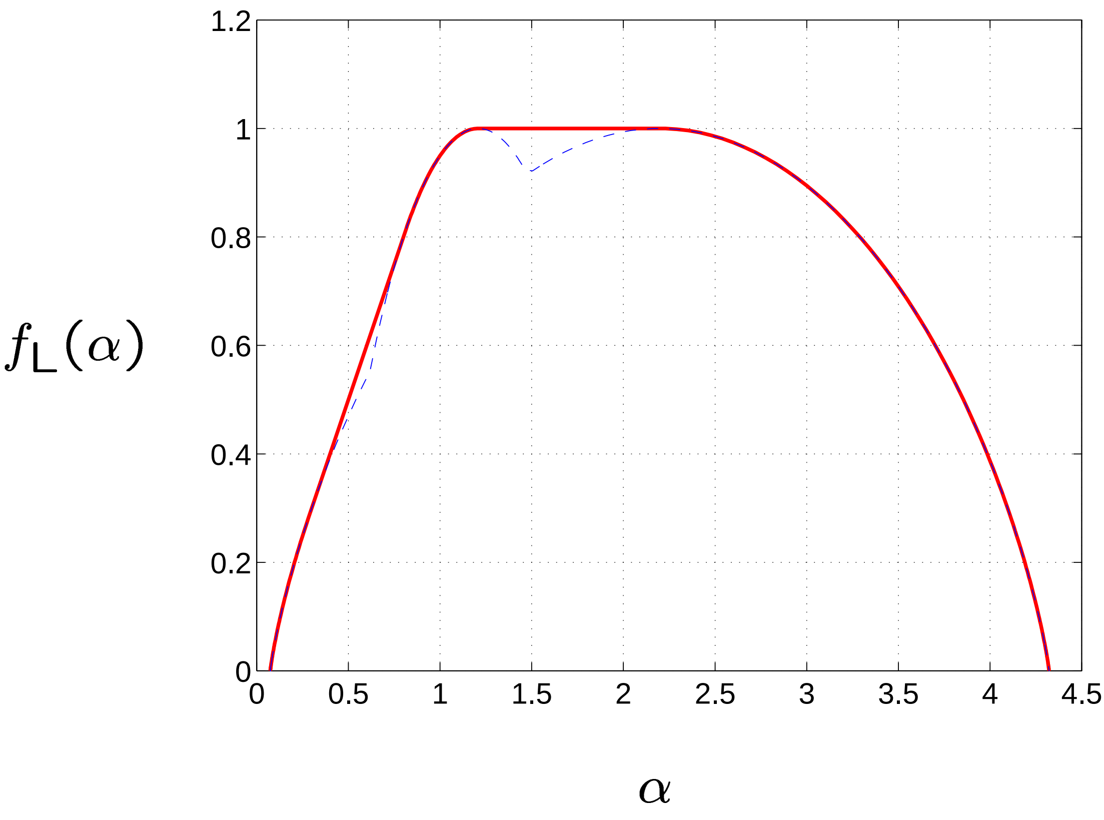
Legendre spectrum (statistical spectrum)

- Log moment generating function

$$\sum_k |c_{j,k}|^q \;=\; O\left(2^{jT(q)}\right)$$

- wavelet-based Legendre Spectrum

$$f_{\mathsf{L}}(\alpha) \;:=\; \inf_q (\alpha\, q - T(q)) \;=\; T^*(\alpha)$$



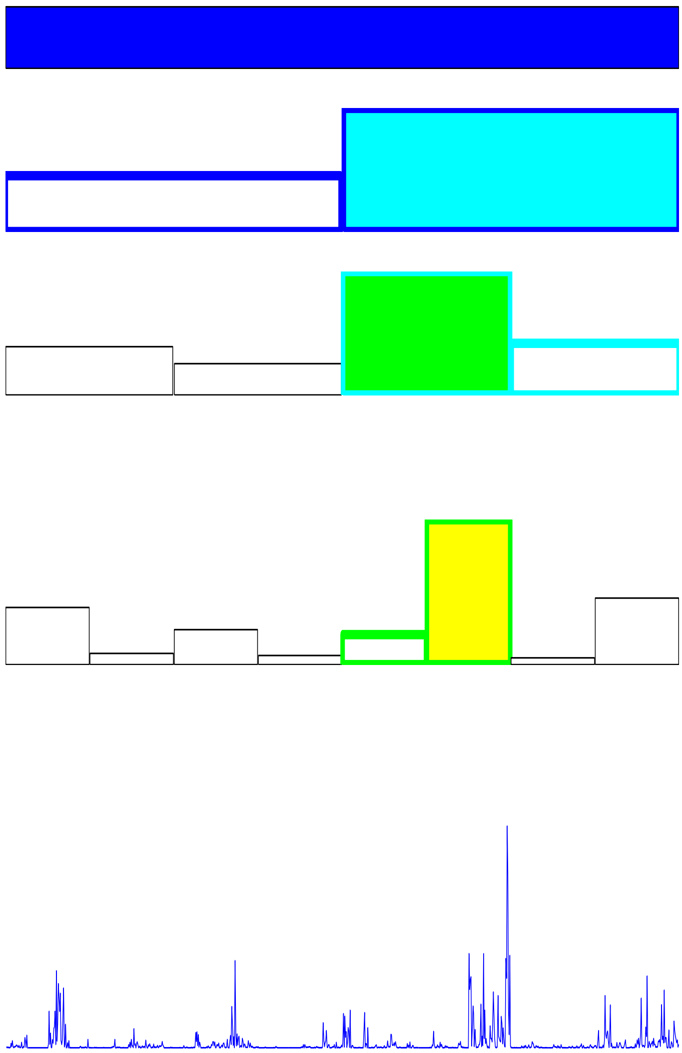
$$f_h(\alpha) \;\le\; f_g(\alpha) \;\le\; f_{\mathsf{L}}(\alpha)$$

Multiplicative Random Cascades

The most well known processes with truly multifractal properties

Scaling	$\longrightarrow$	Strong dependence
Multiplicative	$\longrightarrow$	Log-Normal
Tree	$\longrightarrow$	Fast algorithms (wavelets)

Binomial cascade : a paradigm



i.i.d. random variables

$$\left\{ \mu_k^i, \;\middle|\; \begin{array}{l} i = 0, -1, -2, \ldots \\ k = 0, \ldots, 2^{-i} - 1 \end{array} \right\}$$

Define the random measure μ on dyadic intervals

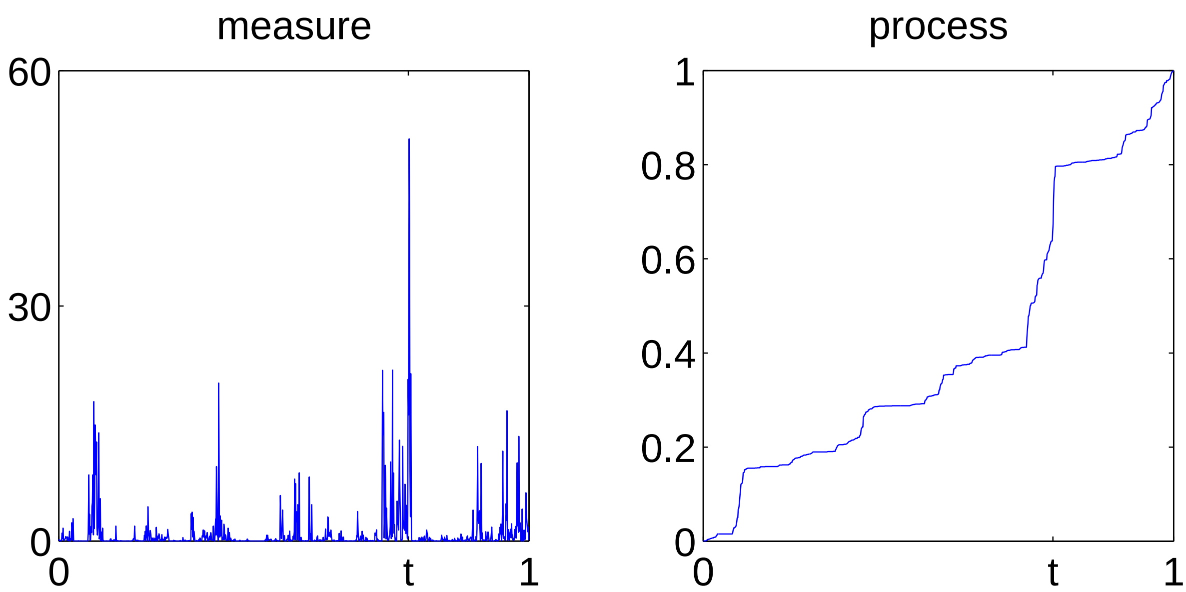
$$\mu([k_j 2^j, (k_j + 1) 2^j]) = \mu_{k_0}^0 \cdot \mu_{k_1}^1 \cdots \mu_{k_j}^j$$

with mass conservation

$$\mu_{2k}^{i+1} + \mu_{2k+1}^{i+1} = 1$$

Multifractal Cascade Process

$$\mathcal{M}(t) := \int_0^t d\mu$$



- Rescaling property (*self similarity*)

$$\mathcal{M}(2^j(k+t)) - \mathcal{M}(2^j(k+s)) \stackrel{d}{=} W_j \cdot (\mathcal{M}(t) - \mathcal{M}(s))$$

with $W_j \stackrel{d}{=} \mu^0_{k_0} \cdot \mu^1_{k_1} \cdots \mu^j_{k_j}$

- Stationarity of increments / Scaling of moments

$$\mathbb{E}|\mathcal{M}(t) - \mathcal{M}(s)|^q = |t-s|^{T(q)}$$

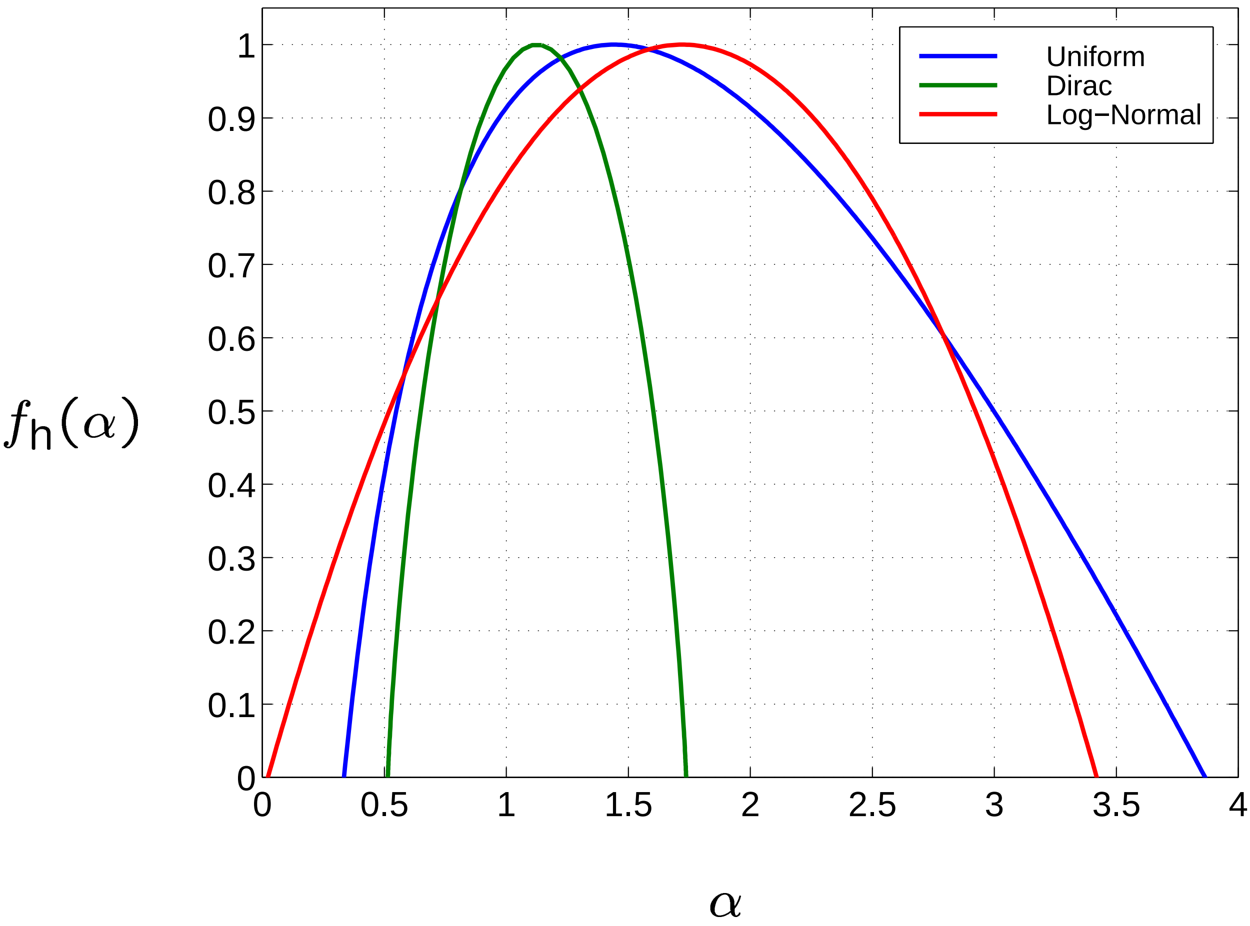
(binomial process: holds for $t = k2^j$, $s = (k+1)2^j$

and for stationary multifractals in general

e.g. Lévy motions, $T(q) = qH$)

- True multifractal spectrum - Closed form
  Depends on the distribution law of the multipliers

$$\mu_k^j \stackrel{d}{=} \begin{cases} M_0 & \text{if } k \text{ is even} \\ M_1 & \text{if } k \text{ is odd} \end{cases}$$



- Monotonous increasing process

Fractional Brownian Motion

Unique process that:

1. is Gaussian
2. is statistically self- similar

$$B_H(\lambda t) \stackrel{d}{=} \lambda^H B_H(t), \; 0 < H < 1$$

3. has stationary increments

$$G_H(t) = B_H(t+\Delta) - B_H(t)$$

- Self-similarity implies non stationarity

$$\mathbb{E} B_H(t) B_H(s) = \frac{\sigma^2}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right)$$

- Long-range dependence when $H > 1/2$

$$\mathbb{E}[G_H(t) G_H(t+\tau)] \approx \tau^{2H-2}, \, \tau \gg \Delta$$

- Mono-fractal process: $\alpha(t) = H, \forall t$
  $\rightarrow$ degenerated (trivial) singularity spectrum

Multifractional Processes

Previous attempts exist towards

“*Multifractal Processes*”

1992 Generalized Weierstrass functions
(*P. Flandrin, P. Gonçalvès*)

1998 Construction of continuous functions with prescribed local regularity
(*Y. Meyer, K. Daoudi, J. Lévy Véhel*)

1998 Generalized Multifractional Brownian Motion
(*A. Ayache, J. Lévy Véhel*)

1999 Locally self-similar processes
(*S. Cohen, J. Istas, A. Benassi*)

others...

Multifractional Motion

Let :

1. $B_H(t)$ be a fractional Brownian Motion
2. $\mathcal{M}(t)$ be a multiplicative process

and consider the compound process :

$$\mathcal{B}(t) \ := \ B_H(\mathcal{M}(t))$$

Fractional Brownian Motion in Multifractal Time

- Covariance structure

$$\mathbb{E}\mathcal{B}(t)\mathcal{B}(s) \ =$$
$$\frac{\sigma^2}{2}\mathbb{E}\left[|\mathcal{M}(t)|^{2H} + |\mathcal{M}(s)|^{2H} - |\mathcal{M}(t) - \mathcal{M}(s)|^{2H}\right]$$

- Special cases
  - $H = 1/2$

    $$\mathcal{G}(k) := \mathcal{B}((k+1)\Delta) - \mathcal{B}(k\Delta)$$

    is decorrelated, but dependent
  - $H > 1/2$

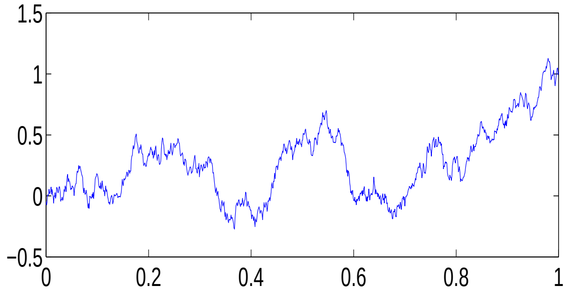
    $\mathcal{B}$ exhibits long range dependence

Multifractional Motion

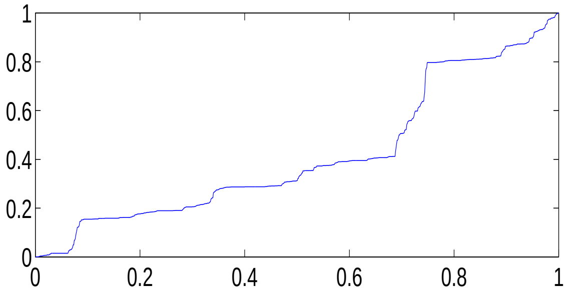
$\mathcal{B}(t)$ inherits from the rich multifractal structure of the underlying multiplicative cascade process $\mathcal{M}(t)$

$$f_h^{\mathcal{B}}(\alpha) = f_h^{\mathcal{M}}\left(\frac{\alpha}{H}\right)$$

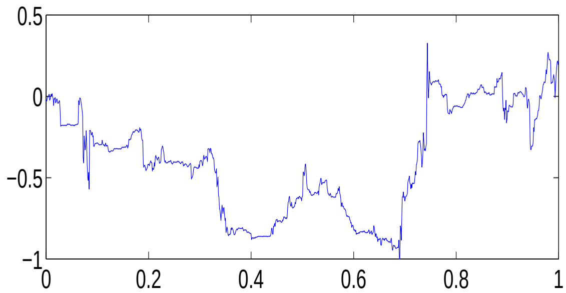
$B_H(t)$



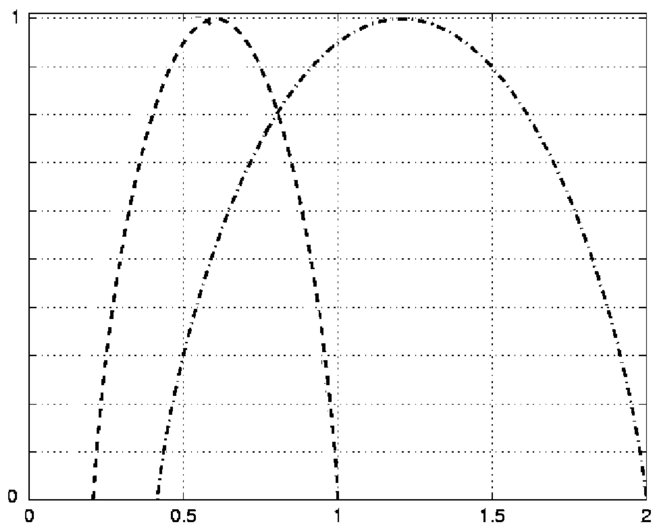
$\mathcal{M}(t)$



$\mathcal{B}(t)$



Singularity Spectra



Wavelet Analysis of Multifractional Motions

A wavelet-based multifractal spectrum estimator relies on *the empirical high order moments* of the wavelet coefficients

$$T(q) \;:=\; \lim_{j \to -\infty} \frac{1}{j} \log \sum_k |c_{j,k}|^q$$

Potential obstacles:

- non-stationarity of multifractional process $\mathcal{B}$ can carry over to the wavelet coefficient series of its decomposition

- *Long Range Dependence* intrinsic to $\mathcal{B}$, $(H > 1/2)$ may affect the convergence of the empirical estimator

Statistics of Wavelet Decomposition

- Identical distributions

$$c_{j,k} \stackrel{d}{=} (W_j)^H c_{0,0}, \quad (\text{with } W_j \stackrel{d}{=} \mu^0_{k_0} \cdot \mu^1_{k_1} \cdots \mu^j_{k_j})$$

ψ : admissible wavelet with compact support
$\mathcal{M}$: verifies rescaling property
B_H : self-similar process

- Wide sense stationarity

$$\mathbb{E} c_{j,k} c_{j,k'} =$$
$$-\frac{\sigma^2}{2} 2^{-j} \int \psi(t) \int |s|^{T(2H)} \psi(t + 2^{-j}s - (k - k'))ds\, dt$$

$\mathcal{M}$: satisfies to scaling of moments for $q = 2H$
$\mathbb{E}|\mathcal{M}(t) - \mathcal{M}(s)|^{2H} = |t - s|^{T(2H)}$
Covariance structure of $\mathcal{B}$

The wavelet decomposition of a fBm in multifractal time is stationary at each scale

Statistics of Wavelet Decomposition

- Fast decay of the correlation

$$\mathbb{E} c_{j,k} c_{j,k'} \approx O\left(|k-k'|^{T(2H)-2\mathcal{R}}\right), \; |k-k'| \to \infty$$

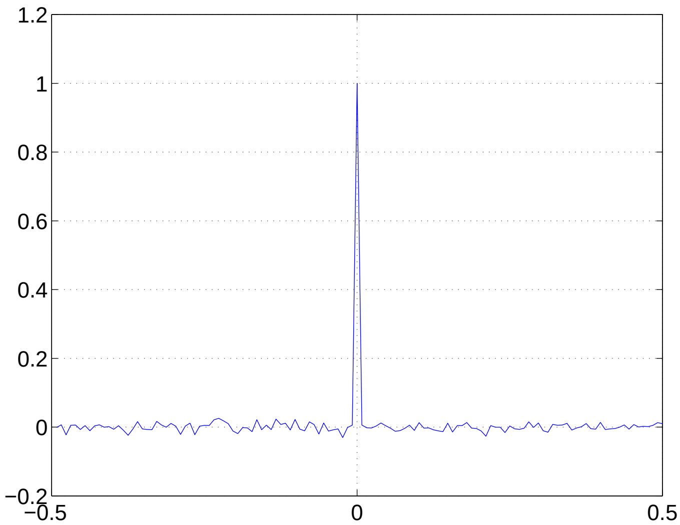
ψ : $\mathcal{R}$ vanishing moments

$\lim_{\nu \to 0} |\Psi(\nu)|^2 = |\nu|^{2\mathcal{R}}$

inc. of $\mathcal{B}$: $T(2H) - 2 > -1 \rightarrow$ LRD

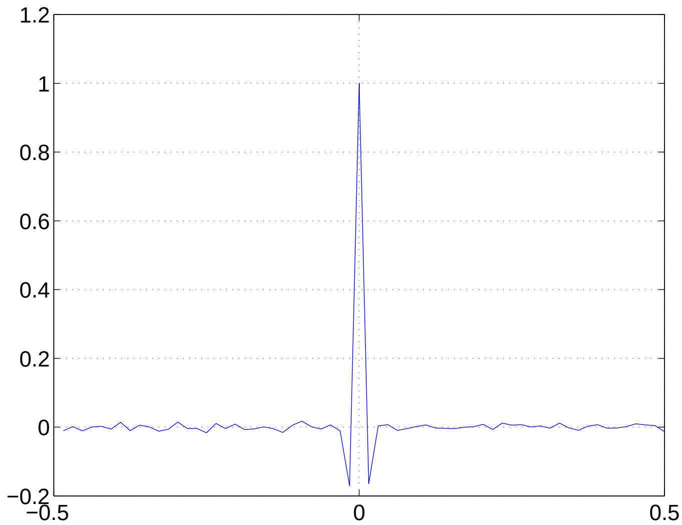
$c_{j,k}$: $T(2H) - 2\mathcal{R} < -1 \rightarrow$ no LRD

$\mathcal{R} = 1$ (Haar) - $H = 0.5$



time lag τ

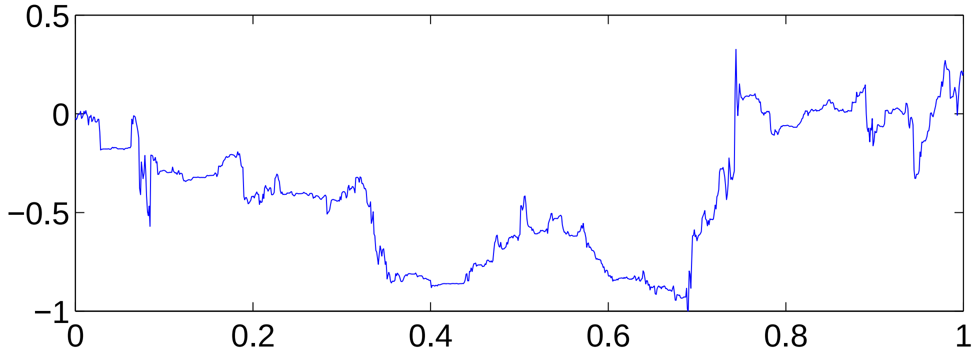
$\mathcal{R} = 2$ - $H = 0.8$



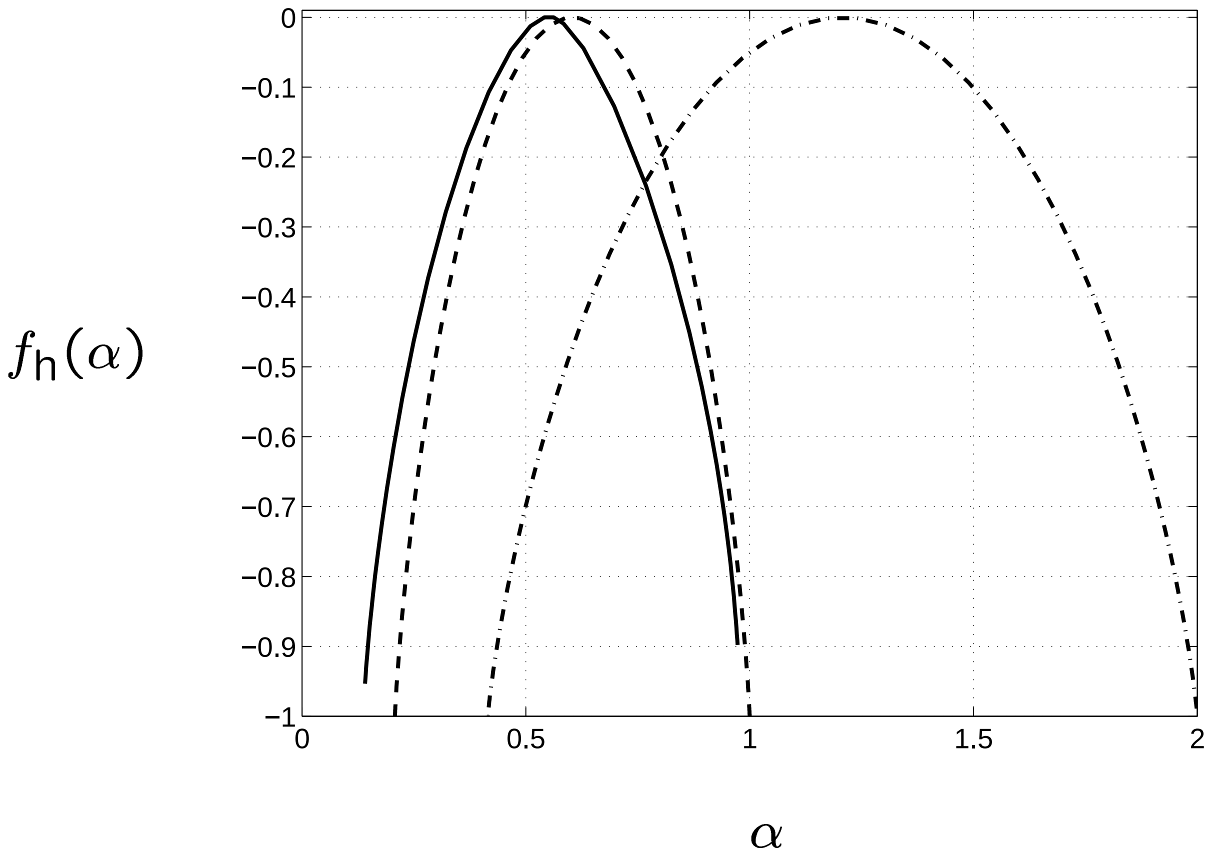
time lag τ

Legendre Spectrum Estimation

Multifractional Motion $\mathcal{B}(t)$



Wavelet based Legendre Spectrum estimate



Conclusion

- multifractals: varying degree of local variability (volatility in economics, variability in biomedical,...)

- wavelets: powerful estimation tool with favorable statistical properties

- fBm in multifractal time: versatile process with multifractal structure and LRD

Wavelet characterization of Hölder Exponent

$x(t)$: a real time process

$0 < \alpha(t) < 1$: local regularity exponent (Hölder)

$$\forall s \quad |x(s) - x(t)| \; < \; c\, |s - t|^{\alpha(t)}$$

Wavelet Decomposition

Definition

$$c_{j,k} \; = \; 2^{-j} \int \psi^*(2^{-j}t - k)\, x(t)\, dt, \;\; (j,k) \in \mathbb{Z}^2$$

Vanishing moments

$$\int t^r \, \psi(t)\, dt = 0, \;\; r = 0, \ldots, \mathcal{R} - 1.$$

Oscillatory behavior measure

x has local Hölder regularity $\alpha(t)$ at time t

$$\Downarrow$$

$$\alpha(t) \; = \; \sup_{\alpha} \left\{ \alpha \; : \; |c_{j,k}| = O\left(2^{j\alpha}\right), \; k2^j \to t \right\}$$