ICT 2000 - Acapulco, Mexico

EXISTENCE TEST OF MOMENTS:

APPLICATION TO

FRACTAL ANALYSIS

Paulo Gonçalvs ◇

INRIA Rhône-Alpes, France
Paulo.Goncalves@inria.fr

◇ In collaboration with Rudolf Riedi from Rice University, Houston TX.

Outline

1. *The Framework* : Fractal analysis

2. *The Problem* : Existence of moments

3. *A Solution* : Existence test of moments

4. *Some Applications* :
   - fractal analysis
   - estimation

Fractal analysis

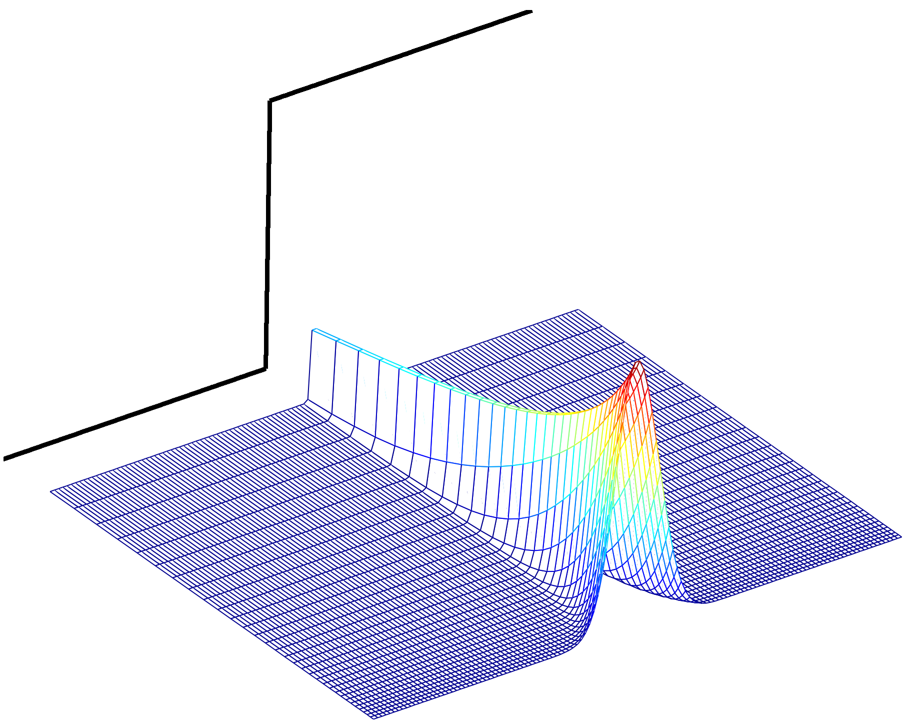
To characterize locally or globally the singularity content of a function, a measure, the graph of a process, . . .

Hölder regularity : a signal x has local Hölder regularity $\alpha(t_0)$ at time t_0 if its wavelet decomposition behaves like

$$W[x](t,a) \quad := \quad \int x(u) \overbrace{\left(\frac{1}{|a|}\psi\left(\frac{u-t}{a}\right)\right)}^{\psi_{t,a}(u)} du$$

$$\approx \quad \mathcal{O}\left(a^{\alpha(t_0)}\right), \quad t \to t_0$$

provided $\displaystyle\int t^r \psi(t)\, dt \equiv 0, \quad r = 0, 1, \cdots, n_\psi \geq \alpha(t_0)$



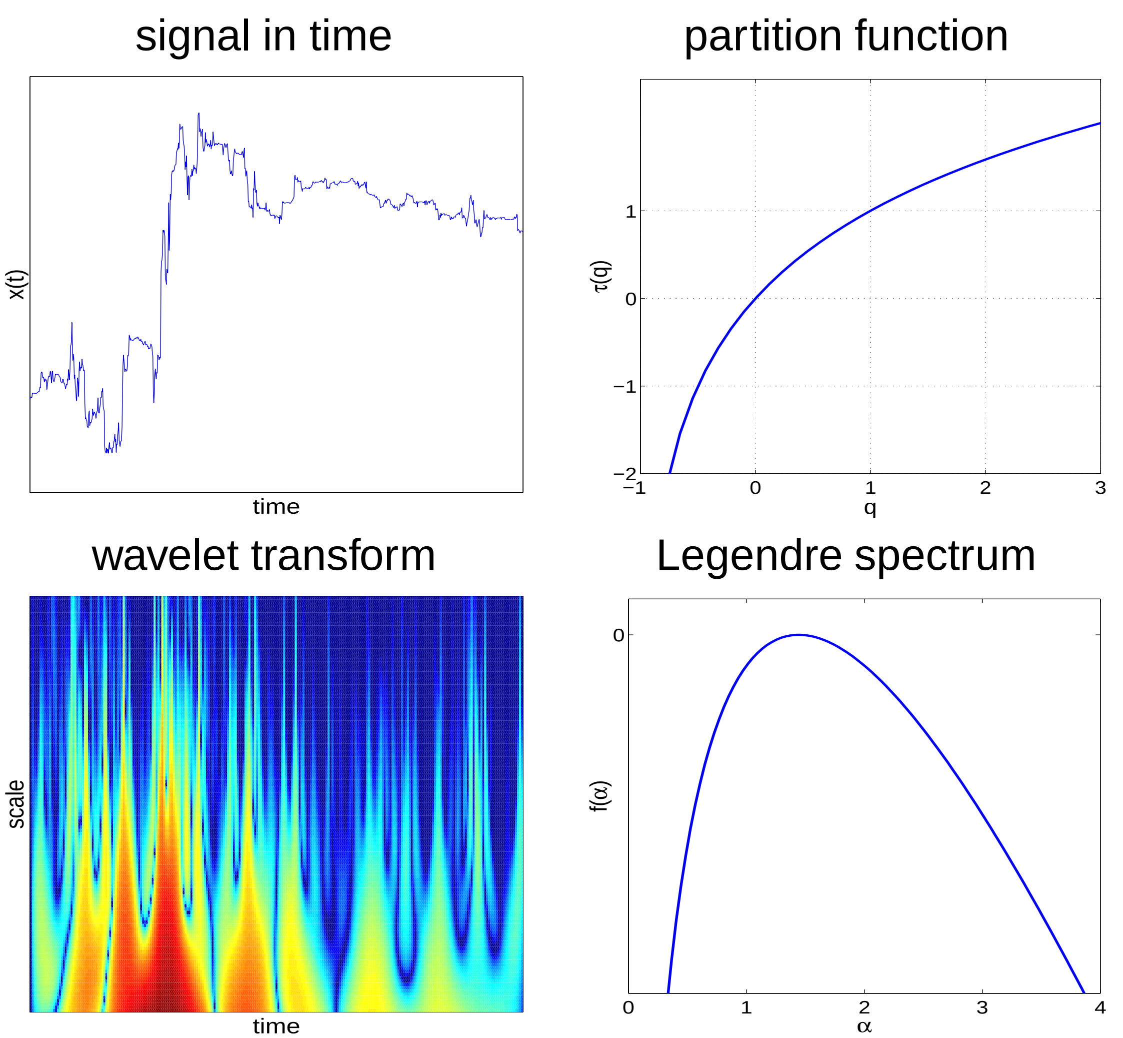
Fractal analysis

Typically, a process will possess many different singularity strengths. Multifractal spectra *measure the frequency (in t) of occurrence of a given α*

Legendre spectrum: Based upon a "Large Deviation Principle", Legendre spectrum provides us with a statistical description of the singularities distribution

partition function $\tau(q)$ s. t.: $\mathbb{E}|W[x](t,a)|^q \underset{a \to 0^+}{\sim} a^{\tau(q)}$

Legendre spectrum: $f_L(\alpha) := \inf_q [q\alpha - \tau(q)]$



Existence of moments

THEORY

Legendre spectrum relies on finite moments of random variable $W[x](t,a)$

$$\mathbb{E}|W[x](t,a)|^q \; < \; \infty, \quad \textit{for} \;\; q \in (q_{min}, q_{max})$$

PRACTICE

For finite length data sets, the sample mean estimator used to compute the partition function

$$\widehat{\tau}(q) \;=\; \lim_{a \to 0^+} \frac{\log \sum_{k=1}^{N} |W[x](t_k,a)|^q / N}{\log a}$$

does not account for the divergence of $\tau(q)$

Example of a self-similar γ-stable process (Abry et al. ICASSP'2000)

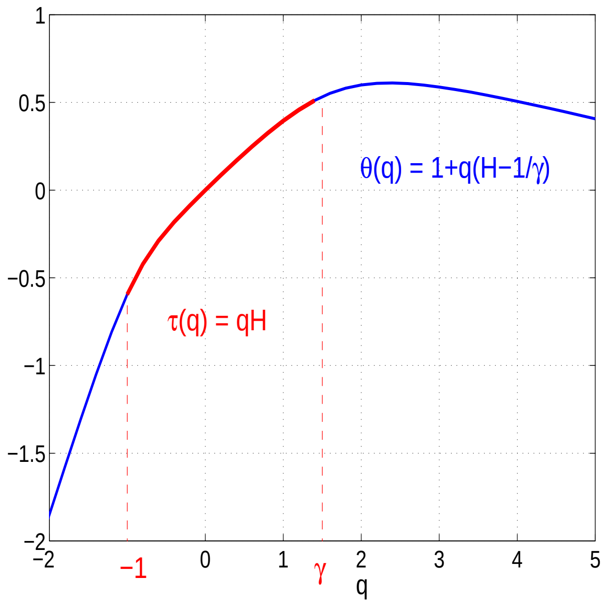
- γ-stable distributed wavelet coefficients

$$\mathbb{E}|W[x](t,a)|^q = c_q \, a^{qH}, \quad -1 < q < \gamma$$

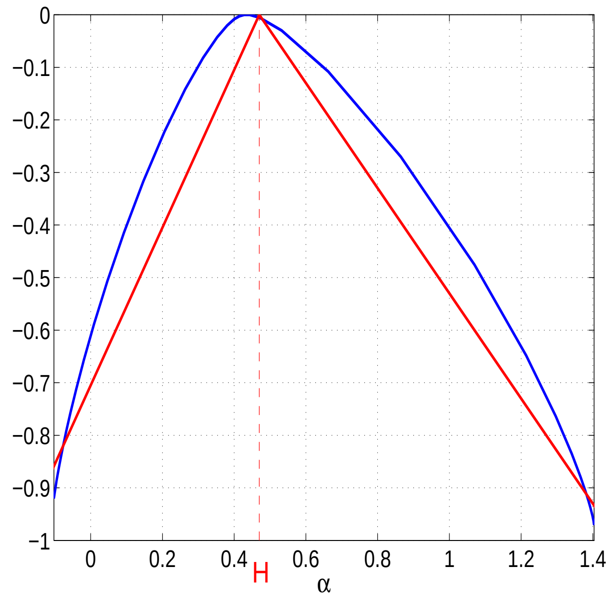
- monofractal process

$$\alpha(t) = H, \, \forall t$$

Estimated partition function



Estimated Legendre spectrum



Existence of moments

Problem: Given a finite length data set of i.i.d. random variables

$x_i,\ i = 1, \cdots, N,$

with unknown distribution $f(x)$,

find an empirical test asserting finiteness of

$$\boxed{\mathbb{E}\{x^q\} = \int x^q \, df(x) < +\infty \quad \text{for } q_{\text{min}} < q < q_{\text{max}}}$$

Existence test of moments

Characteristic function of a R.V. x

$$\Phi(s) \;=\; \mathbb{E}\{e^{isx}\} \;=\; \int e^{isx}\, df(x)$$

calculated using the sample mean estimator:

$$\widehat{\Phi}(s) \;=\; \tfrac{1}{N} \textstyle\sum_{i=1}^{N} e^{isx_i}$$

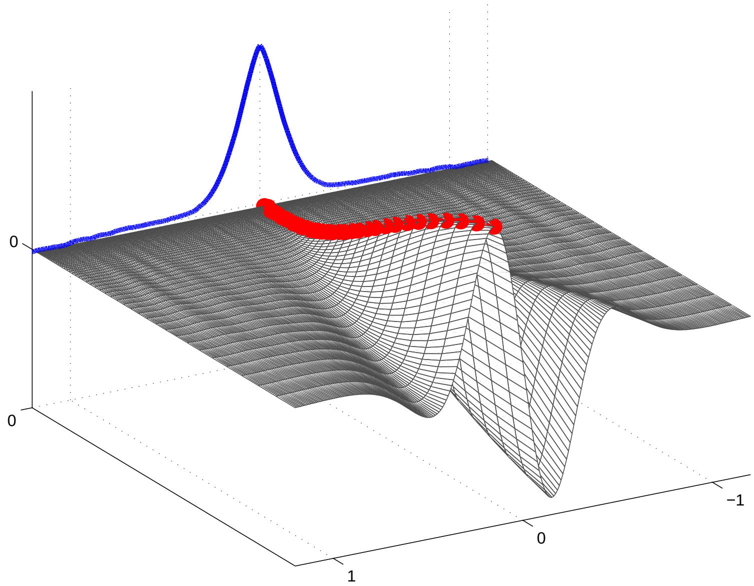
Lemma: 1) Moment generating function

$$\mathbb{E}\{x^q\} \;=\; \Phi^{(q)}(s)\Big|_{s=0}$$

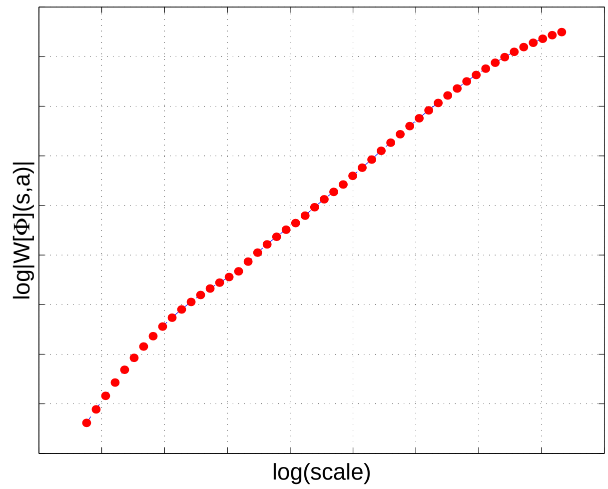
2) under technical assumptions

$$\boxed{\begin{array}{l}\mathbb{E}\{x^q\} \text{ exists for } q < q_{\text{max}} \\ \quad\Leftrightarrow \quad W[\Phi](s,a) \;\sim\; O(a^{q_{\text{max}}})\end{array}}$$

Wavelet decomposition of $\Phi(s)$



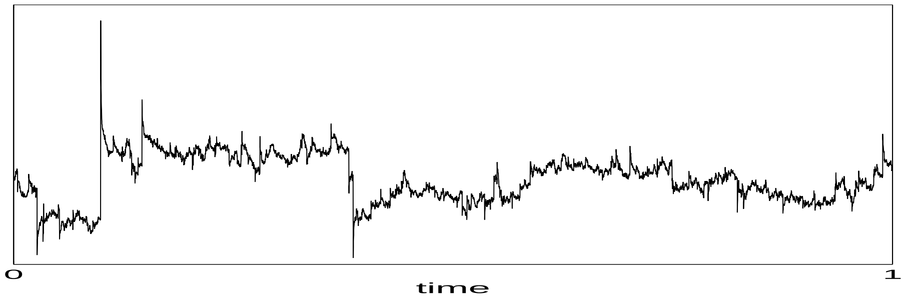
Estimation of q_{max}



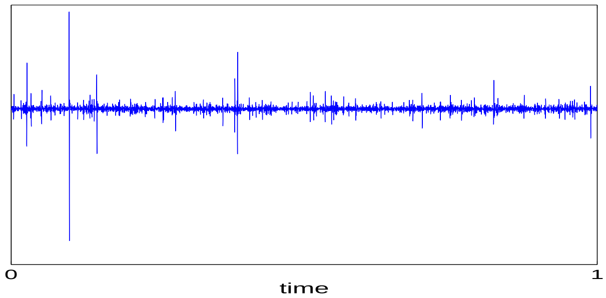
Existence test of moments

Fractal analysis : Allows for determining the range of the partition function $\tau(q)$

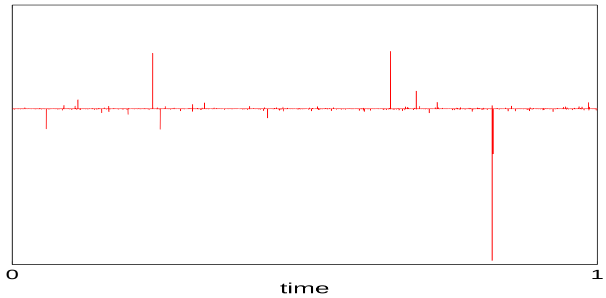
Signal x (self-similar γ-stable process)



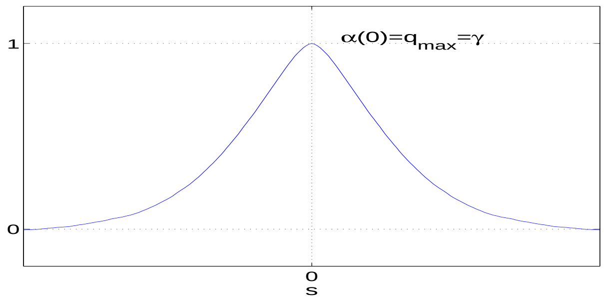
$W[x](t, a_0)$



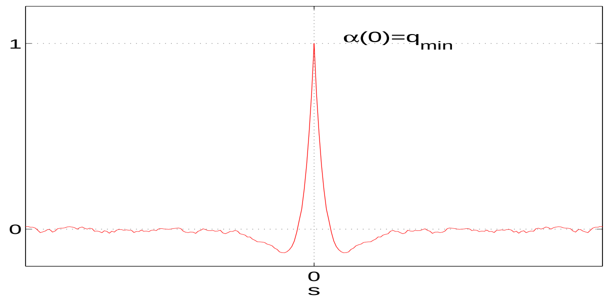
$(W[x](t, a_0))^{-1}$



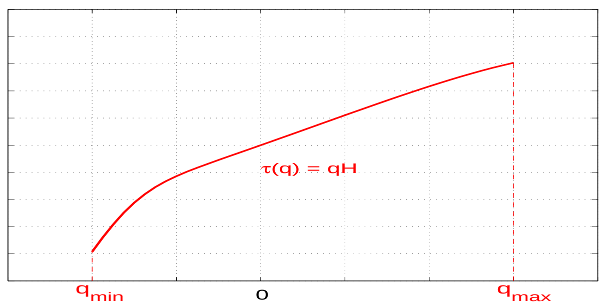
$\Phi_W(s)$



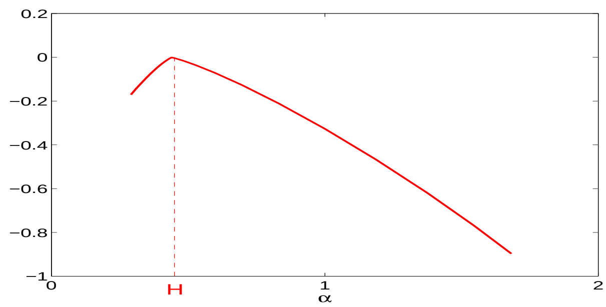
$\Phi_{W^{-1}}(s)$



Partition function $\tau(q)$



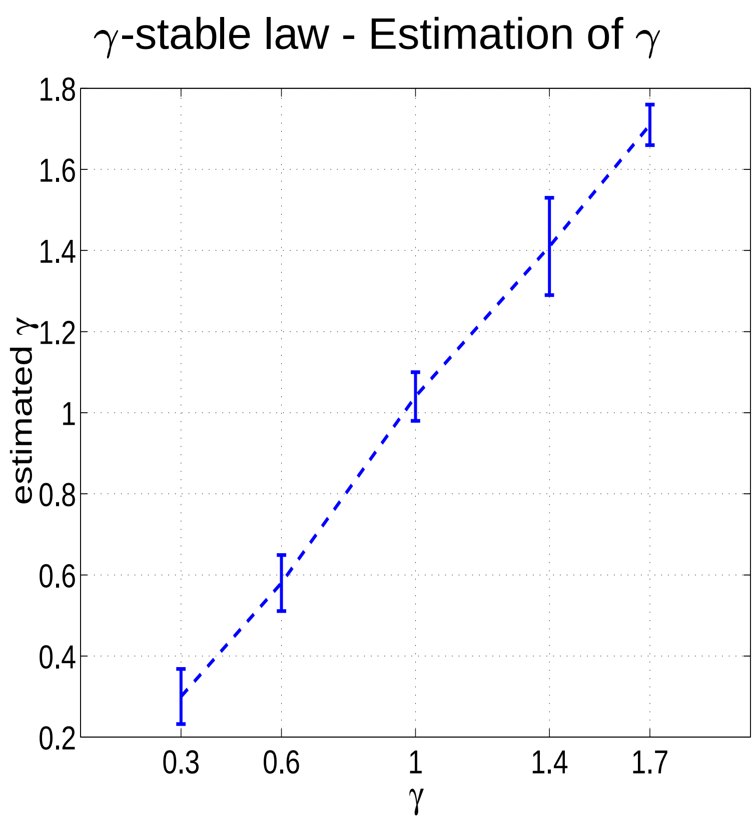
Legendre spectrum



Existence test of moments

Estimation : Allows for estimating certain indices of some probability density functions :

- γ-stable laws : $\mathbb{E}\{x^q\}$ exist for $q < \gamma$
- β-symmetric distributions : $\mathbb{E}\{x^q\}$ exist for $q > -p$
- parameterized heavy-tail distributions, . . .



Statistical performances still need to be evaluated

Conclusions

- Existence of moments : a widespread issue
- A simple test : conceptually and experimentally
- Efficient procedure when ran on small data sets
- Yields an estimate for certain stability indices