Time-frequency methods

in

time-series data analysis

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Time versus Frequency Representations

$$x \in L^2(\mathbb{R}, dt) \; : \left\{ x \; : \; \int_{\mathbb{R}} |x(t)|^2 \, dt \; < \; +\infty \right\}$$

Time Representation (Shannon)

$$x(t) \;=\; \int_{-\infty}^{+\infty} x(u)\, \delta(u-t)\, du$$

$$x(t) \;=\; \langle x, \delta_t \rangle$$

- "natural description" of the signal in the observation space (waveforms)
- perfectly localized on the time axis

Frequency Representation (Fourier)

$$X(f) \;=\; \int_{-\infty}^{+\infty} x(u)\, e^{-i2\pi f u}\, du$$

$$X(f) \;=\; \langle x, e_f \rangle$$

- harmonic description (waves, periodicity)
- perfectly localized on the frequency axis
- invertible

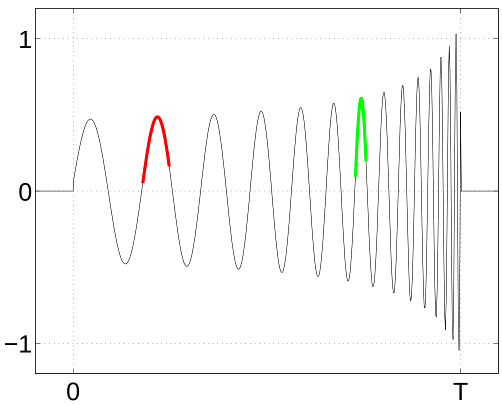
$$x(t) \;=\; \int_{-\infty}^{+\infty} X(f)\, e^{+i2\pi f t}\, df$$

Time versus Frequency Representations

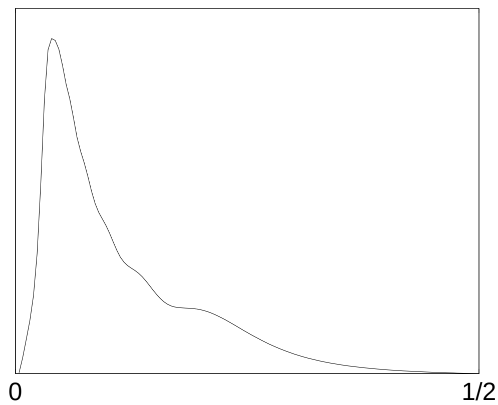
$$\text{Signal model: } X_{r,k}(f) = C f^{-(r+1)}\, e^{i\Psi_k(f)}\, U(f)$$

$$\Psi_k(f) = -2\pi \left(c f^k + t_0 f + \gamma \right)$$

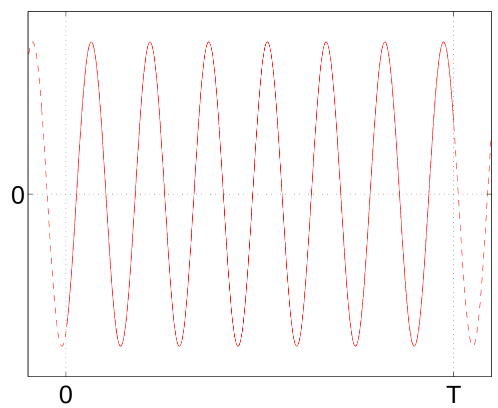
signal in time (real part)



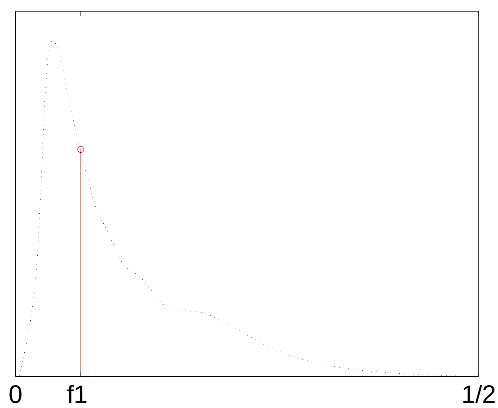
signal in frequency (spectrum)

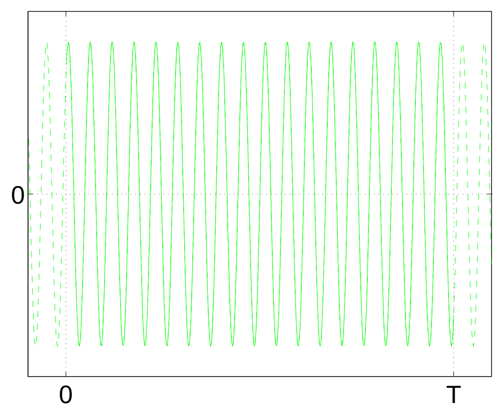


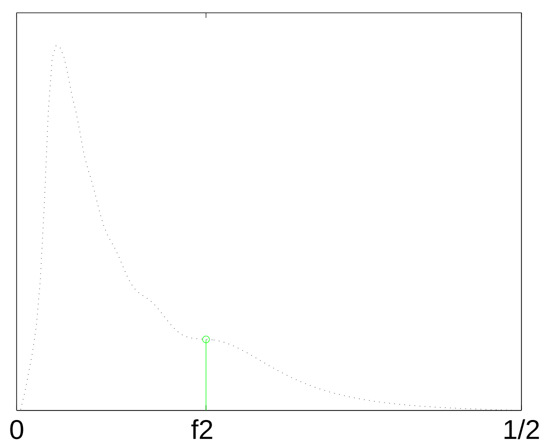
harmonic components in time



harmonic components in frequency







Instantaneous Frequency and Group Delay

Instantaneous frequency (analytic signals)

$$x(t) \;=\; a(t)\, e^{i\varphi(t)} \qquad \Rightarrow \qquad f_x(t) = \frac{1}{2\pi}\frac{d\varphi(t)}{dt}$$

Group delay (analytic signals)

$$X(f) = B(f)\, e^{i\Psi(f)}\, U(f) \qquad \Rightarrow \qquad t_x(f) = -\frac{1}{2\pi}\frac{d\Psi(f)}{df}$$

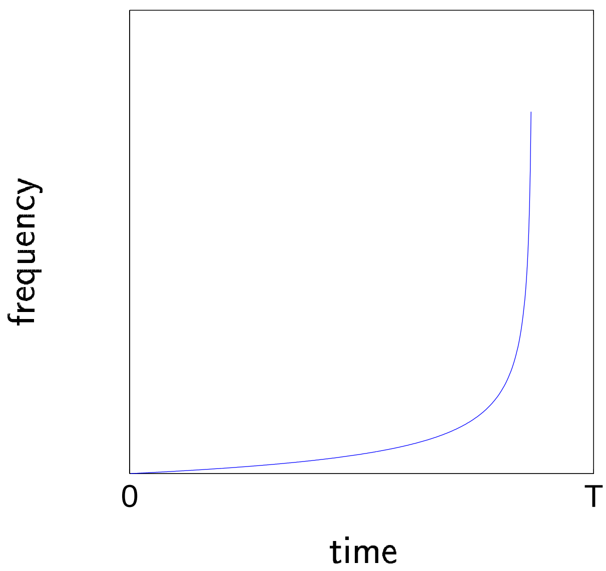
Reciprocity

For large time bandwidth product signals :

$$t_x\left(f_x(t)\right) \;\sim\; t$$

Signal model:

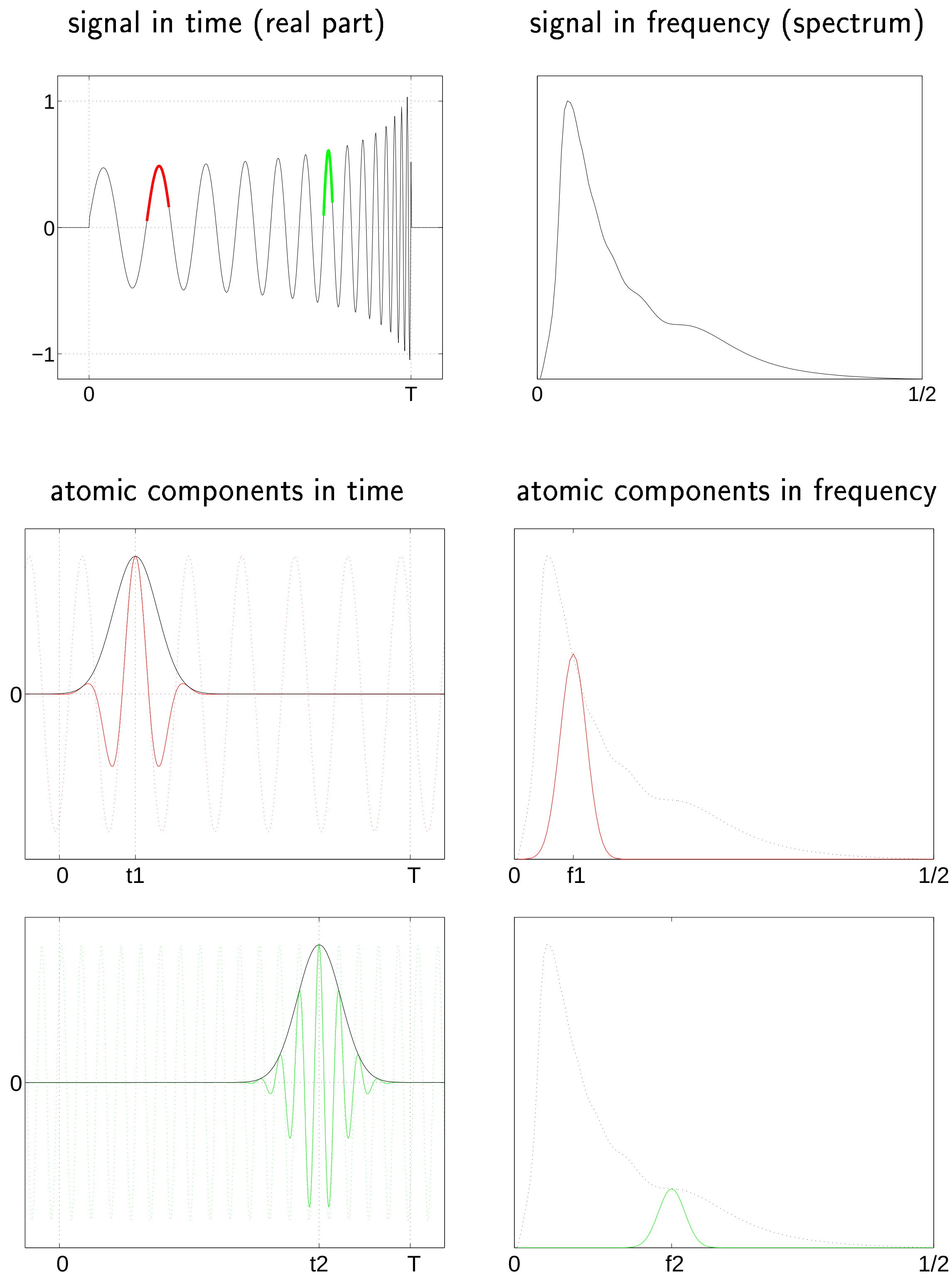
$$X_{r,k}(f) = C f^{-(r+1)}\, e^{i\Psi_k(f)}\, U(f) \quad \text{with} \quad \begin{aligned} \Psi_k(f) \;&=\; -\,2\pi\left(cf^k + t_0 f + \gamma\right) \\ t_x(f) \;&=\; ckf^{k-1} + t_0 \end{aligned}$$



Atomic Joint Time – Frequency Representations

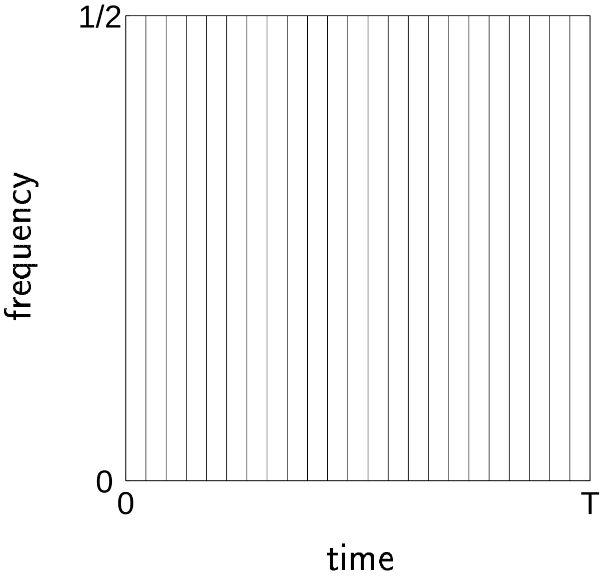
$$\text{Signal model: } X_{r,k}(f) = C f^{-(r+1)}\, e^{i\Psi_k(f)}\, U(f)$$

$$\Psi_k(f) = -2\pi\left(c f^k + t_0 f + \gamma\right)$$



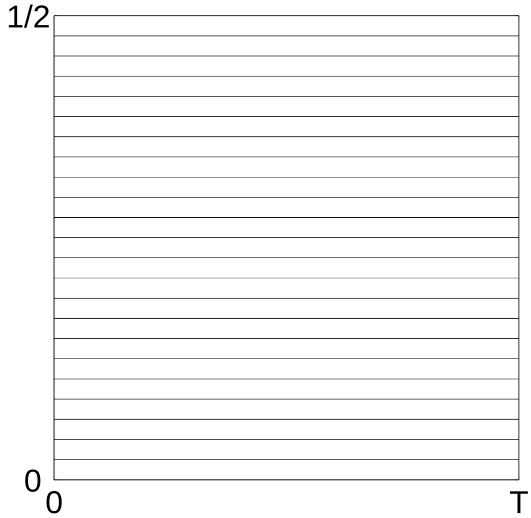
Atomic Joint Time – Frequency Representations

Shannon



$$x(t) \;=\; \langle x, \delta_t \rangle$$

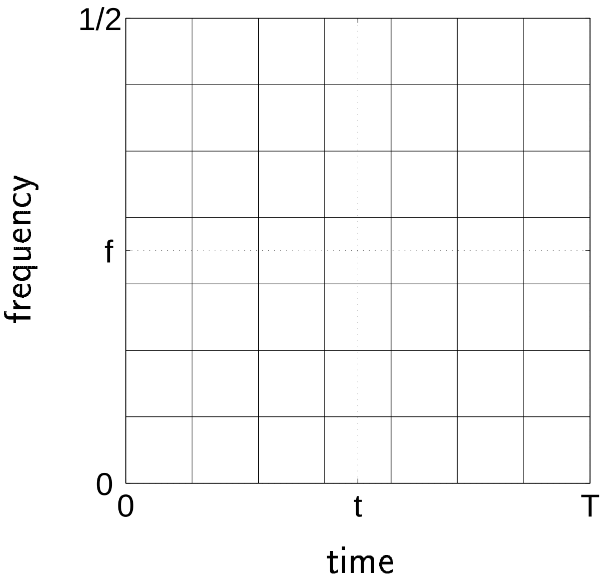
Fourier



$$X(f) \;=\; \langle x, e_f \rangle$$

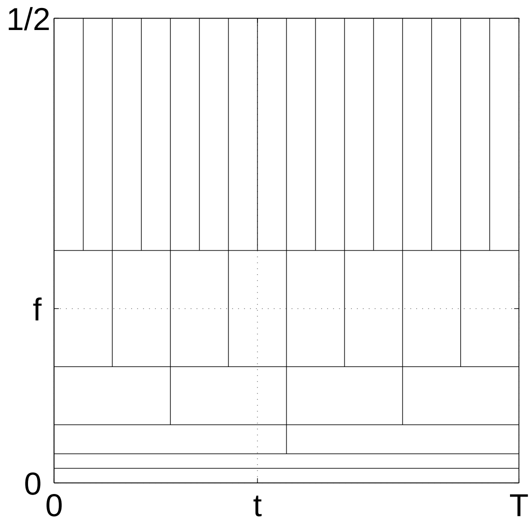
$$\Gamma_x(t, f\,;\,g) \;=\; \langle x, g_{t,f} \rangle$$

Gabor



$$g_{t,f}(u) \;=\; g(u-t)\, e^{i2\pi f u}$$

Wavelets



$$g_{t,f}(u) \;=\; \left(\frac{f}{f_0}\right)^{1/2} g_0\left(\frac{f}{f_0}(u-t)\right)$$

Atomic – Based Energetic Representations

Some Properties

Energy distributions

$$\int |x(t)|^2\, dt \quad = \; E_x \; = \quad \int |X(f)|^2\, df$$

$$\|$$

$$\iint |\Gamma_x(t, f\,;\,g)|^2\, df\, dt$$

$$\rho_x(t, f) \;\equiv\; |\Gamma_x(t, f\,;\,g)|^2$$

Covariance properties

Spectrogram (Weyl-Heisenberg group)

$$\rho_x(t, f) = \left| \int x(u)\, g^*(u-t)\, e^{-i2\pi f u}\, du \right|^2$$

$$\begin{array}{ccc} x(t) & \longrightarrow & x(t-t_0)\, e^{i2\pi f_0 t} \\ \downarrow & & \downarrow \\ \rho_x(t, f) & \longrightarrow & \rho_x(t-t_0, f-f_0) \end{array}$$

Scalogram (Affine group)

$$\rho_x(t, f) = \tfrac{f}{f_0} \left| \int x(u)\, g_0^* \left(\frac{f}{f_0}(u-t) \right) du \right|^2$$

$$\begin{array}{ccc} x(t) & \longrightarrow & |a|^{-1/2}\, x\left(\frac{t-t_0}{a}\right) \\ \downarrow & & \downarrow \\ \rho_x(t, f) & \longrightarrow & \rho_x\left(\frac{t-t_0}{a}, af\right) \end{array}$$

Localization

Atoms' shape *plus* Heisenberg uncertainty principle precludes a *perfect localization* on group delays trajectories

Energetic Bilinear Representations

A Generalization of *atomic – based energetic representations*:

$$|\Gamma_x(t,f\,;\,g)|^2 \;=\; \iint x(u)\,x^*(v)\,g^*_{t,f}(u)\,g_{t,f}(v)\,du\,dv$$

$$\Downarrow$$

$$\rho_x(t,f\,;\,K) \;=\; \iint x(u)\,x^*(v)\,K(u,v\,;\,t,f)\,du\,dv$$

such that $$\iint \rho_x(t,f\,;\,K)\,dt\,df \;=\; E_x$$

most general formulation for bilinear time-frequency representations

•

theoretical properties of ρ $\Longleftrightarrow$ structural properties of K

•

imposing displacement covariance properties on ρ

yields different classes of solutions

Cohen's class

Weyl – Heisenberg : time shifts + frequency shifts

$$\begin{array}{ccc} x(t) & \longrightarrow & x(t-t_0)\, e^{i2\pi f_0 t} \\ \downarrow & & \downarrow \\ \rho_x(t,f\,;K) & \longrightarrow & \rho_x(t-t_0, f-f_0\,;K) \end{array}$$

yields the Cohen's class of time-frequency representations :

$$C_x(t,f) \;=\; \iint W_x(\tau,\xi)\,\Pi\,(\tau-t,\xi-f)\;d\tau\,d\xi$$

with the Wigner-Ville distribution defined as :

$$W_x(t,f) \;\triangleq\; \int x\left(t+\frac{\tau}{2}\right)\,x^*\left(t-\frac{\tau}{2}\right)\,e^{-i2\pi f\tau}\,d\tau$$

- Wigner-Ville localizes on linear chirps :

$$\left\{\begin{array}{lcl} x(t) & = & \exp(i\varphi_x(t)) \\ f_x(t) & = & f_0 \;+\; \beta t \end{array}\right. \quad\Longrightarrow\quad W_x(t,f) \;=\; \delta(f-f_x(t))$$

- Wigner-Ville is unitary :

$$|\langle x,y\rangle|^2 \;=\; \langle\langle W_x, W_y\rangle\rangle$$

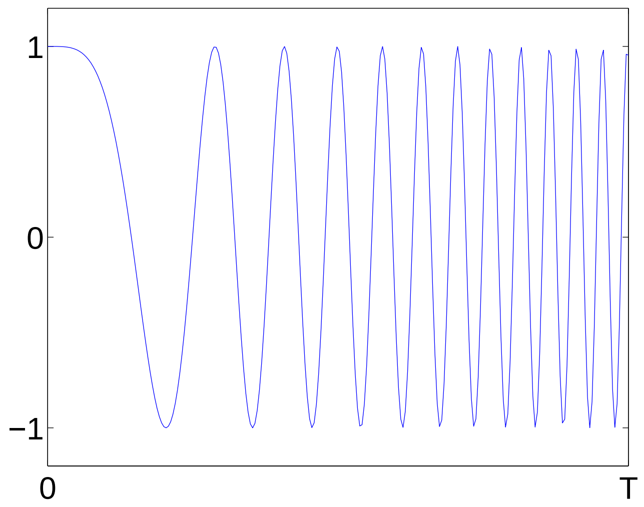
an illustration : $|\Gamma_x(t,f\,;g)|^2 \;=\; \left|\int x(u)\,g^*(u-t)\,e^{-i2\pi fu}\,du\right|^2$

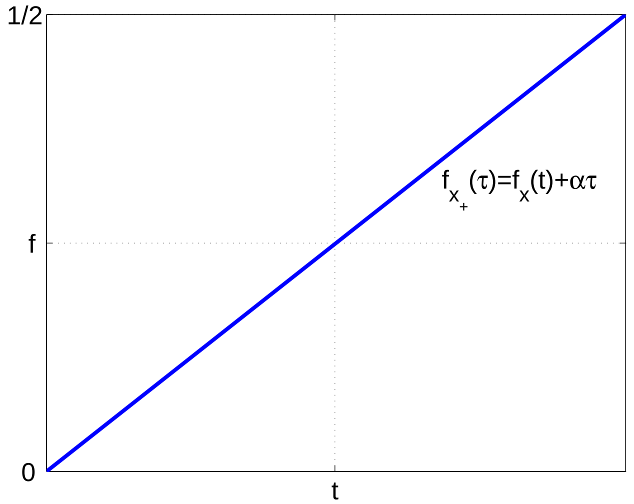
$$= \iint W_x(\tau,\xi)\,W_g(\tau-t,\xi-f)\,d\tau\,d\xi$$

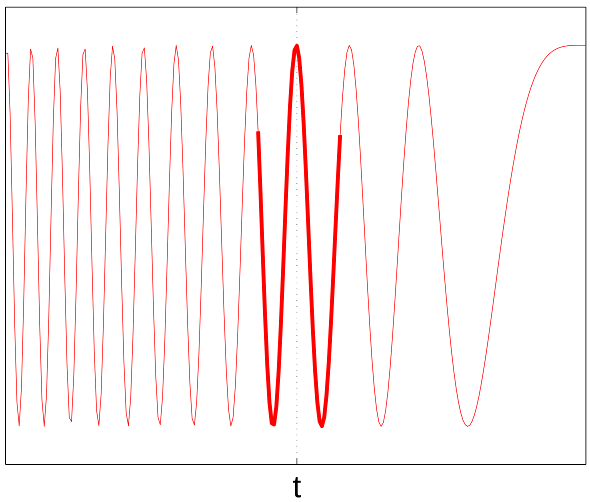
Localization of the Wigner-Ville Distribution

$$W_x(t,f) \;=\; \int x\left(t+\frac{\tau}{2}\right)\, x^*\left(t-\frac{\tau}{2}\right)\, e^{-i2\pi f\tau}\, d\tau$$

linear chirp



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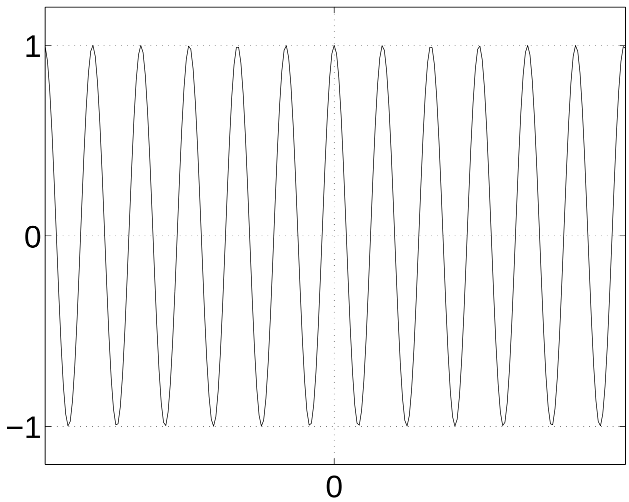
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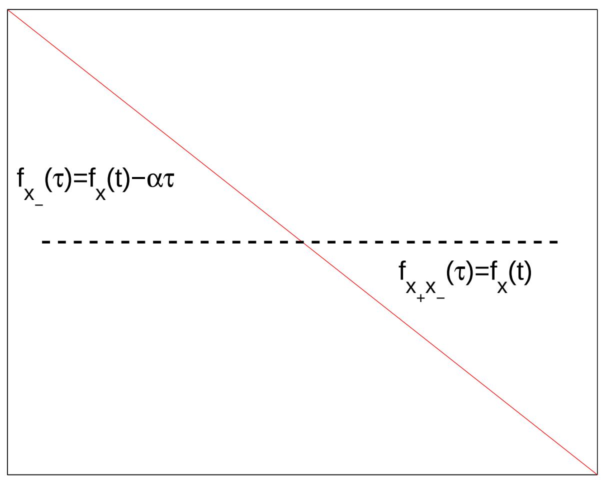
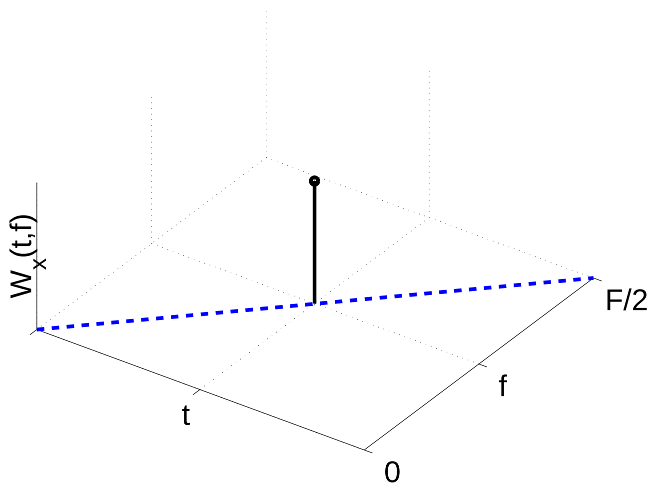
0


$f_{x_-}(\tau)=f_x(t)-\alpha\tau$
$f_{x_+x_-}(\tau)=f_x(t)$


$W_x(t,f)$
t
0
f
F/2

Affine class

<u>Affine group</u> : time shifts + scale changes

$$\begin{array}{ccc} x(t) & \longrightarrow & |a|^{-1/2}\, x\left(\frac{t-t_0}{a}\right) \\ \downarrow & & \downarrow \\ \rho_x(t,f\,;K) & \longrightarrow & \rho_x\left(\frac{t-t_0}{a}, af\,;K\right) \end{array}$$

yields the Affine class of time-frequency representations (J. & P. Bertrand) :

$$P_x^{(k)}(t,f) \;=\; f^{2(r+1)-q} \int \mu_k(u)\, X(f\lambda_k(u))\, X^*(f\lambda_k(-u))\, e^{i2\pi t f \zeta_k(u)}\, du$$

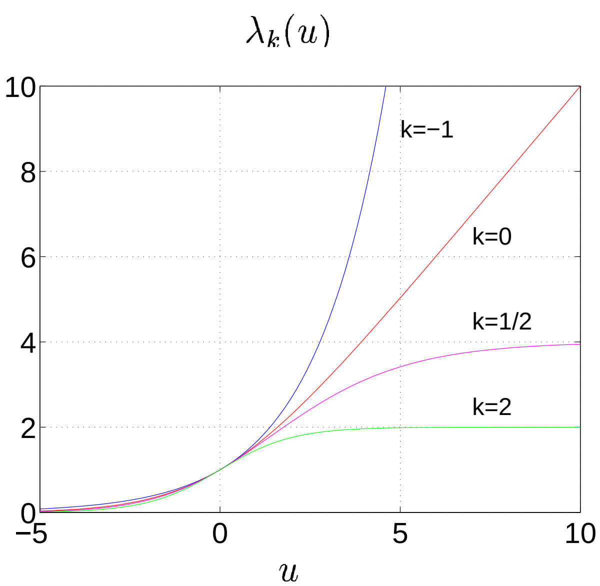
$$\left(W_x(t,f) = \int X\left(f-\xi/2\right)\, X^*\left(f+\xi/2\right)\, e^{-i2\pi t\xi}\, d\xi \right)$$

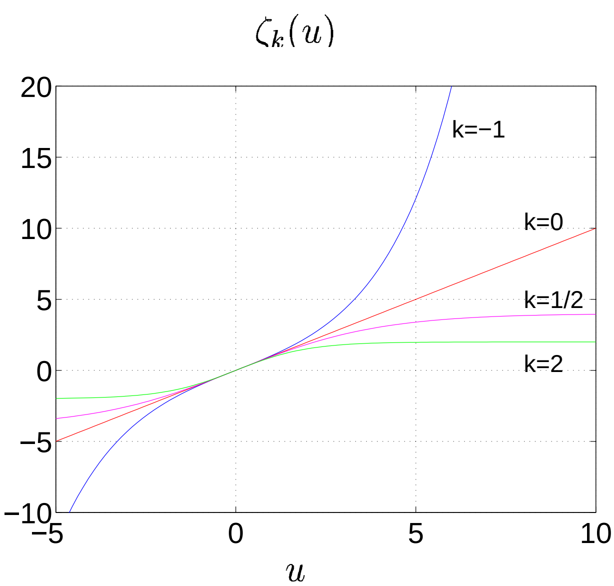
with the parametrizing functions :

$$\lambda_k(u) \;=\; \left(k\, \frac{e^{-u}-1}{e^{-ku}-1}\right)^{\frac{1}{k-1}}, \quad k \in \mathbb{R} \setminus \{0,1\}$$

$$\lambda_0(u) \;=\; \frac{u}{1-e^{-u}} \;\; ; \;\; \lambda_1(u) \;=\; \exp\left(1+\frac{u\,e^{-u}}{e^{-u}-1}\right)$$

$$\zeta_k(u) \;=\; \lambda_k(u) - \lambda_k(-u)$$





Affine Wigner Distributions

- Unitarity $(k \in \mathbb{R})$

$$|\langle x, y\rangle|^2 = \left|\int_0^{+\infty} X(f)\, Y^*(f)\, f^{2r+1}\, df\right|^2$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}_+} P_x^{(k)}(t,f)\, P_y^{(k)}(t,f)\, f^{2q}\, df\, dt$$

$$\Longleftrightarrow \quad \mu_k(u) = \left(\frac{d\zeta_k(u)}{du}\right)^{\frac{1}{2}} \left(\lambda_k(u)\,\lambda_k(-u)\right)^{r+1}$$

(Moyal formula)

- Localization $(k \leq 0)$

for $\left\{X_{r,k}(f) = C\, f^{-(r+1)}\, e^{i\Psi_k(f)}\, U(f) \;:\; t_x(f) = t_0 + ck\, f^{k-1}\right\}$

$$P_x^{(k)}(t,f) = C^2\, f^{-(q+1)}\, \delta(t - t_x(f))$$

$$\Longleftrightarrow \quad \mu_k(u) = \frac{d\zeta_k(u)}{du} \left(\lambda_k(u)\,\lambda_k(-u)\right)^{r+1}$$

Possible matching between these $P^{(k)}$'s distributions and a class of infinitely extended signals exhibiting a power law modulation law in order to obtain a perfect localization in the time-frequency plane

Affine Wigner Distributions : Exemples

- $k \;=\; 0$: unitary Bertrand distribution

$$\lambda_0(u) \;=\; \frac{u}{1-e^{-u}} \quad ; \quad \mu_0(u) \;=\; \left(\frac{u}{2\sinh(u/2)}\right)^{2(r+1)}$$

√ localization on hyperbolic group delay paths

√ unitarity

- $k \;=\; -1$: active Unterberger distribution

$$\lambda_{-1}(u) \;=\; e^{u/2} \quad ; \quad \mu_{-1}(u) \;=\; \cosh(u/2)$$

√ localization on time-frequency paths of the form : $t_0 \;+\; \beta\, f^{-2}$

√ admits an isometry-like relation :

$$\left|\int_0^{+\infty} X(f)\, Y^*(f)\, f^{2r+1}\, df\right|^2 \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}_+} \widetilde{P}_x^{(-1)}(t,f)\, P_y^{(-1)}(t,f)\, f^{2q}\, df\, dt$$

with the passive form $\widetilde{P}_x^{(-1)}(t,f)$,

resulting from an affine filtering of the active form $P_x^{(-1)}(t,f)$

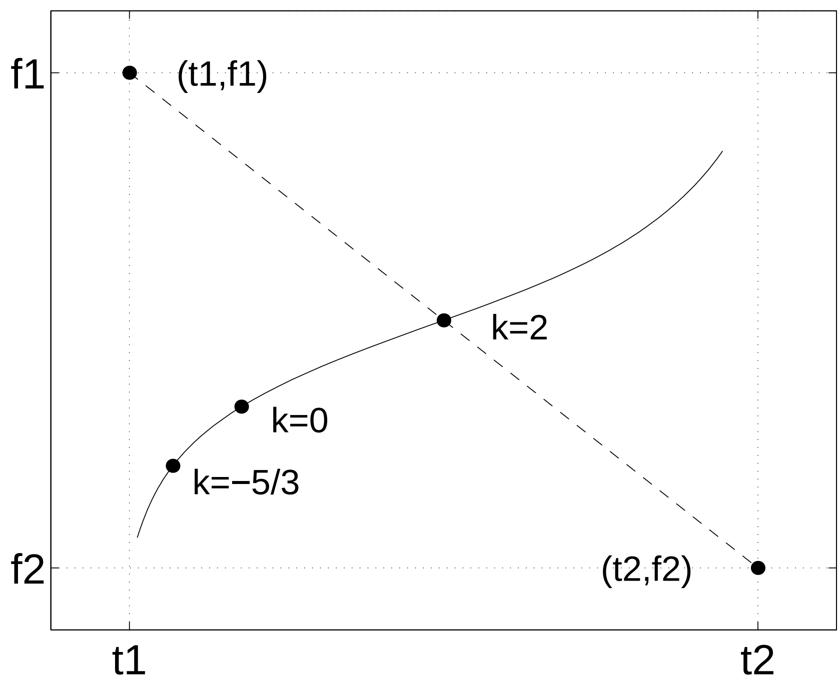
Localization : a by-product of interferences

2 *signal components* centered around (t_1, f_1) and (t_2, f_2)

will interfere at location (t_i, f_i) determined by :

$$\begin{cases} f_i = \Theta^{(k)}(f_1, f_2) \\ t_i^{\frac{1}{k-1}} = \Theta^{(k)}\left(t_1^{\frac{1}{k-1}}, t_2^{\frac{1}{k-1}}\right) \end{cases}$$

with the generalized Stolarsky mean : $\Theta^{(k)}(x,y) = \left(\frac{1}{k}\frac{y^k - x^k}{y - x}\right)^{\frac{1}{k-1}}$



Localization of $P^{(k)}$ along $t_x(f) = t_0 + ck\, f^{k-1}$, the "globally invariant" structure with respect to $\Theta^{(k)}$

Limitations

$$P_x^{(k)}(t,f) \; = \; f^{2(r+1)-q} \int \mu_k(u)\, X(f\lambda_k(u))\, X^*(f\lambda_k(-u))\, e^{i2\pi t f \zeta_k(u)}\, du$$

√ $P^{(k)}$'s distributions are difficult to compute :

- the entire signal enters their definition
- in general (for arbitrary k's) the function ζ_k is not analytically invertible

√ in the class of chirp signals, how to approximate affine Wigner distributions?

- pseudo affine Wigner distributions (P. Gonçalvès & R. Baraniuk)

$$\widetilde{P}_x^{(k)}(t,f) \triangleq \int \frac{\mu_k(u)}{\sqrt{\lambda_k(u)\lambda_k(-u)}}\, \Gamma_x(t,\, \lambda_k(u)f\,;\,\psi)\, \Gamma_x^*(t,\, \lambda_k(-u)f\,;\,\psi)\, du,$$

- reassignment methods (K. Kodera, R. Gendrin & C. de Villedary
  F. Auger, P. Flandrin. & E. Chassande-Mottin)

Reassignment methods

The spectrogram: a local time-frequency smoothing of the Wigner distribution

$$
\begin{aligned}
|\,\Gamma_x(t,f\,;\,g)\,|^2 &= \left|\int x(u)\,g^*(u-t)\,e^{-i2\pi f u}\,du\right|^2 \\
&= \iint W_x(u,\xi)\,W_g(u-t,\xi-f)\,du\,d\xi
\end{aligned}
$$

The reassigned spectrogram: principle

To move the *spectrogram coefficients* from the geometrical center of the kernel W_g to the local centroid of the Wigner distribution W_x

The reassigned spectrogram: implementation

The reassigned spectrogram supports an efficient online implementation

$$
\left\{
\begin{aligned}
\widehat{t}(t,f) &= t + \mathrm{Re}\left\{\frac{\Gamma_x(t,f\,;\,t.g)}{\Gamma_x(t,f\,;\,g)}\right\} \\
\widehat{f}(t,f) &= f - \mathrm{Im}\left\{\frac{\Gamma_x(t,f\,;\,dg/dt)}{\Gamma_x(t,f\,;\,g)}\right\}
\end{aligned}
\right.
$$

Beyond Analysis

Time-frequency representations :

√ are usefull at identifying time-varying frequency contents

√ are matched to chirp signals

√ allow a time-frequency formulation of *standard signal processing issues* such as :

- detection
- estimation
- identification

Made possible by combining the theoretical properties of both distributions theory and optimal detection theory

Exemple of estimation / detection :

$$\begin{aligned}
\Lambda(y\,;\,\theta) &= |\langle Y, X_\theta \rangle|^2 && \text{matched filter} \\
&= \langle\langle \widetilde{P}_y^{(k)}, P_{x_\theta}^{(k)} \rangle\rangle && \text{unitarity} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \widetilde{P}_y^{(k)}(t,f)\, \delta(t - t_{x_\theta}(f)) f^{2q}\, df\, dt && \text{localization} \\
&= \int_{\mathcal{L}(\theta)} \widetilde{P}_y^{(k)}(t,f) && \text{path integration}
\end{aligned}$$

→ Estimation $\quad \widehat{\theta_0} = \text{Arg}\max_\theta \Lambda(y\,;\,\theta)$

→ Detection $\quad \max \Lambda(y\,;\,\widehat{\theta_0}) \geq \eta$

Conclusions

- √ Time frequency analysis offers a natural language for describing non stationary signals and chirp signals in particular

- √ There exist a host of time frequency representations with well defined theoretical properties

- √ It is possible to adapt time frequency methods to signals for the purpose of their analysis (e.g. localization)

- √ *Standard signal processing operations* can be revisited through the time frequency formalism

- √ Efficient algorithms exist (Time Frequency Toolbox for Matlab)