On the complexity of the Maker-Breaker Happy Vertex Game

Joint work with Mathieu Hilaire and Nacim Oijid

Perig Montfort

P-GASE Workshop, April 2, 2026

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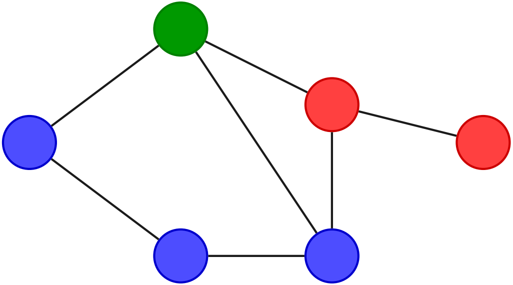
Definition (Zhang and Li – 2015)

Let $G = (V, E)$ be a graph with a vertex-coloring $c : V \to C$. A vertex $v \in V$ is called *happy* if for every neighbor $u \in N(v)$, we have $c(u) = c(v)$.



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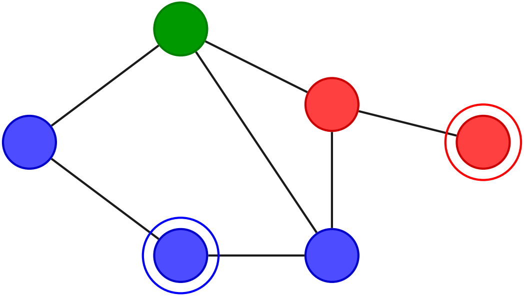

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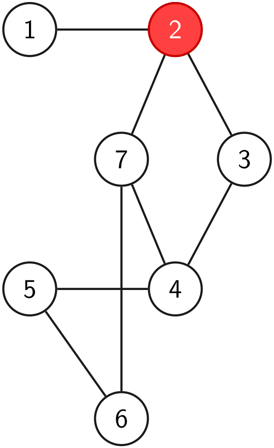

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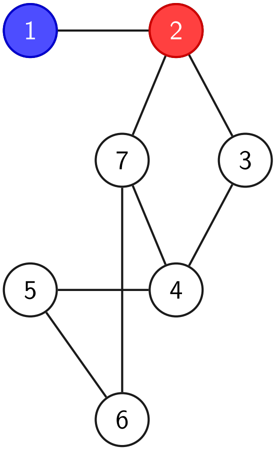

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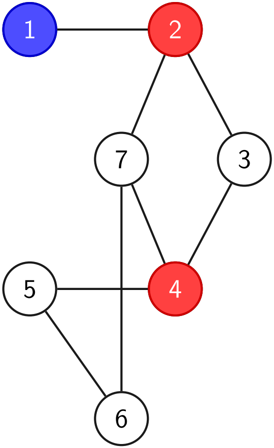
- Maker aims to maximize her number of happy vertices.



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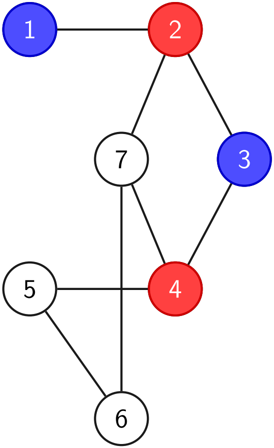

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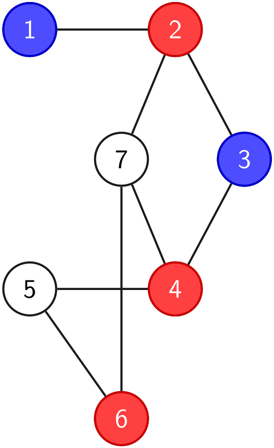

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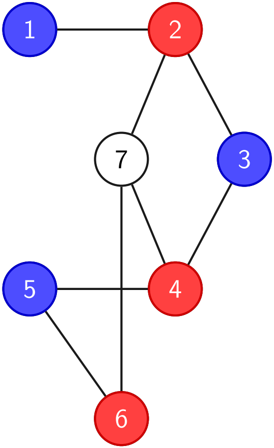

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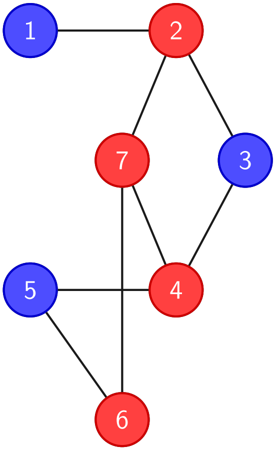

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The score is the number of red happy vertices.

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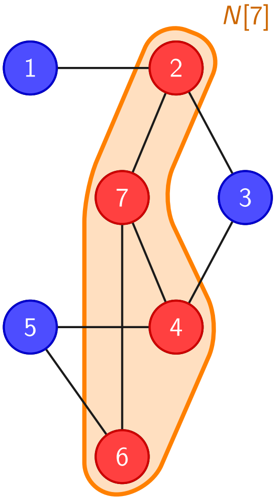

Presentation of the problem

Definition (SCORING HAPPY VERTEX GAME)

Input: a graph G, an integer s, and a first player $X \in \{M, B\}$.

Question: do we have

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  - FPT parameterized by neighborhood diversity.

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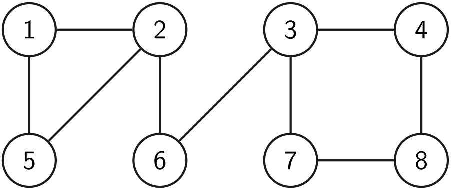
A dominating set is a subset of vertices $D \subseteq V$ such that $N[D] = V$.



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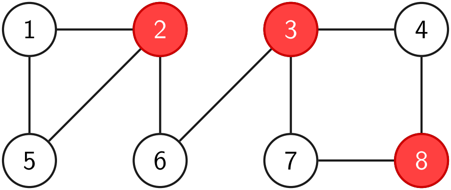

Example

$D = \{2, 3, 8\}$ is a dominating set.

Maker-Breaker Domination game

Definition (Duchene, Gledel, Parreau, Renault – 2020)

Two players, *Dominator* and *Staller*, alternately claim vertices of a graph G. Dominator wins if she claims a dominating set of G; otherwise, Staller wins.

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- *Polynomial on:*
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Link with the Happy Vertex Game

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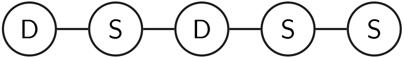
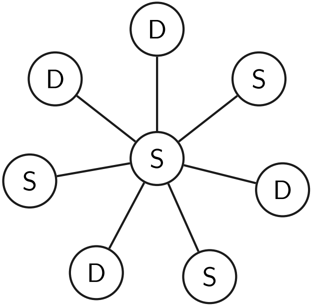
Motivation





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Milnor's universe is the class of scoring games that are both *dicotic* and *non-zugzwang*:

- *dicotic*: For any position P,

$$\text{Maker can move in } P \iff \text{Breaker can move in } P.$$

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Theorem (Bagan et al – 2024)

Every scoring positional games belong to Milnor's Universe.

Boundaries for union and neutral

Theorem (Milnor)

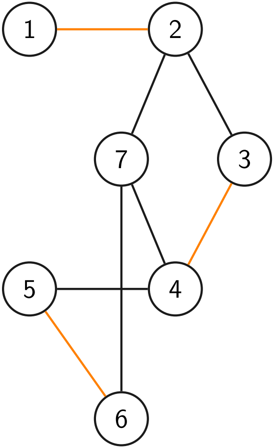
Let P_1 and P_2 be two positions in the Happy vertex game. Then

$$Ms(P_1) + Bs(P_2) \leq Ms(P_1 + P_2) \leq Ms(P_1) + Ms(P_2)$$

Corollary

The neutrals for the union on the Maker-Breaker Happy Vertex Game are exactly the games where *Dominator* wins in the Maker-Breaker Domination Game.

Pairing strategy



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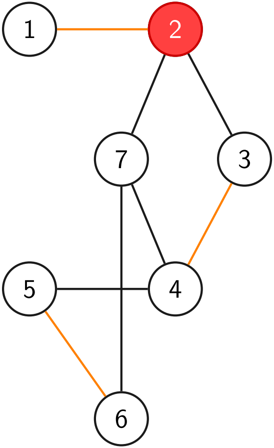

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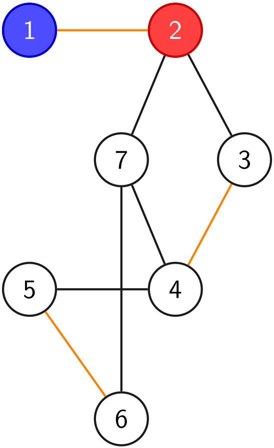

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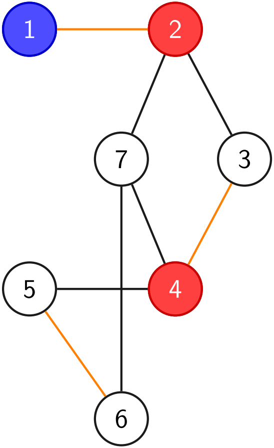

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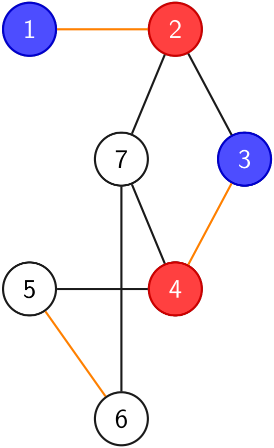

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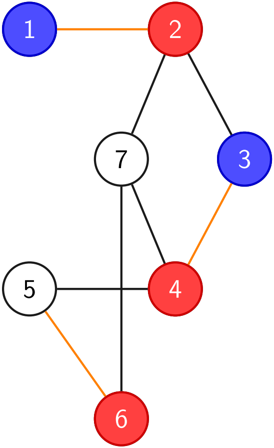

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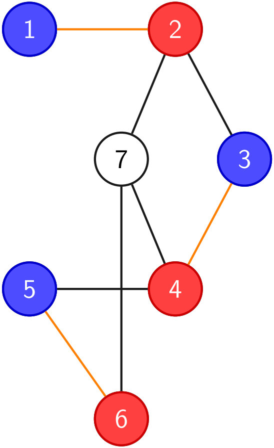
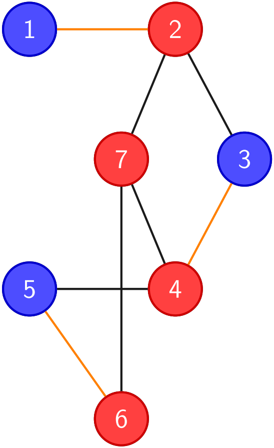


Pairing strategy

Theorem

Let G be a graph and M a matching of G:

$$s(G) \leq |V| - 2 \cdot |M|.$$



The Super Lemma

Theorem

Let (G, M, B) be a position of the game on a graph G, and let $u, v \in V_F$ be two distinct vertices such that for all $X \subseteq V_F \setminus \{u, v\}$,

$$|\{w \in V \mid N[w] \subseteq X \cup \{u\} \cup M\}| = |\{w \in V \mid N[w] \subseteq X \cup \{v\} \cup M\}| .$$

Then

$$s(G, M, B) = s(G,\ M \cup \{u\},\ B \cup \{v\}).$$

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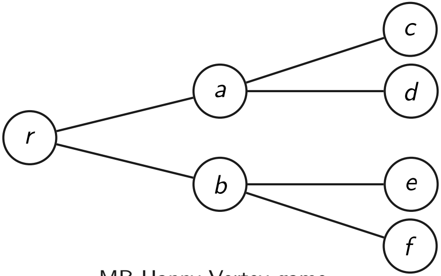

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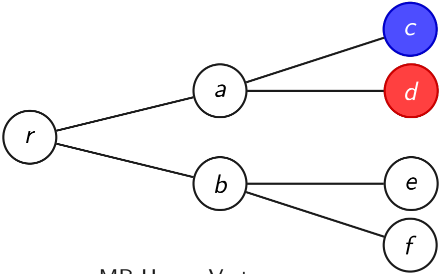

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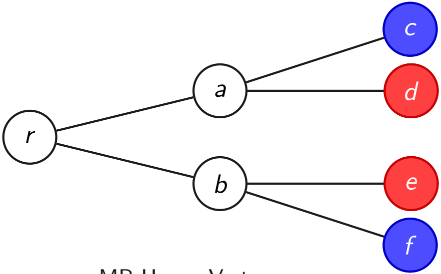

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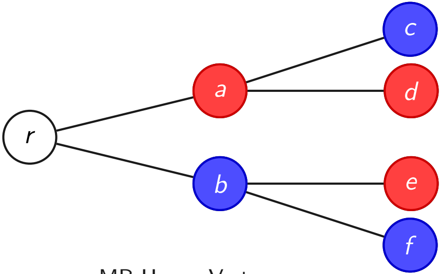

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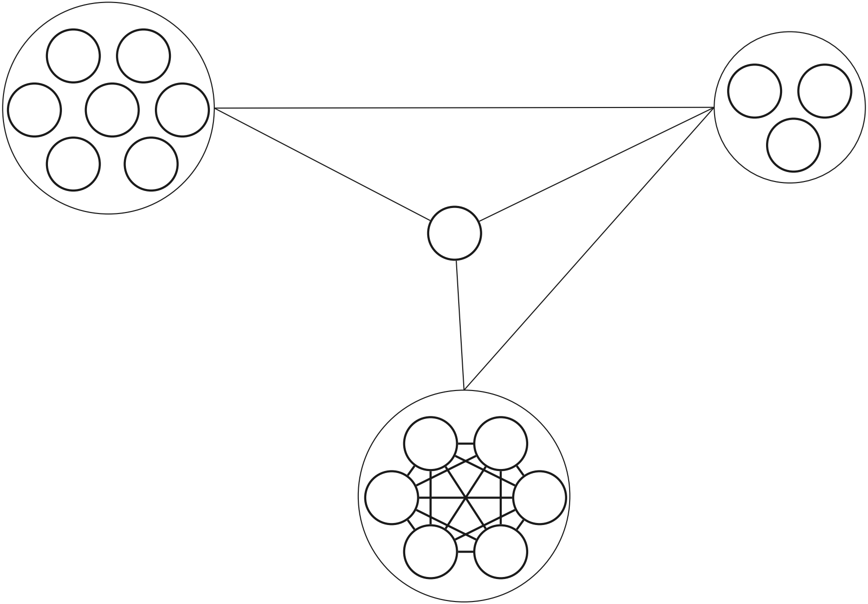

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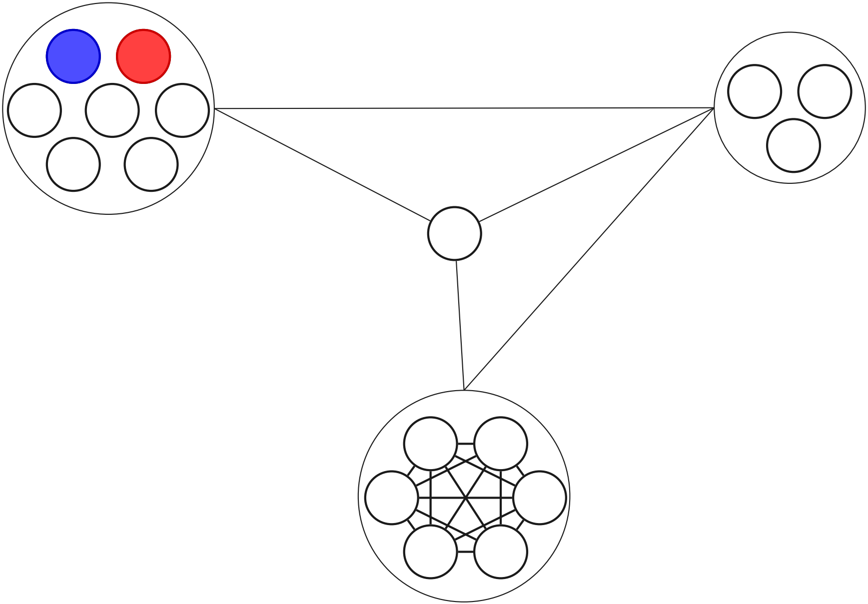

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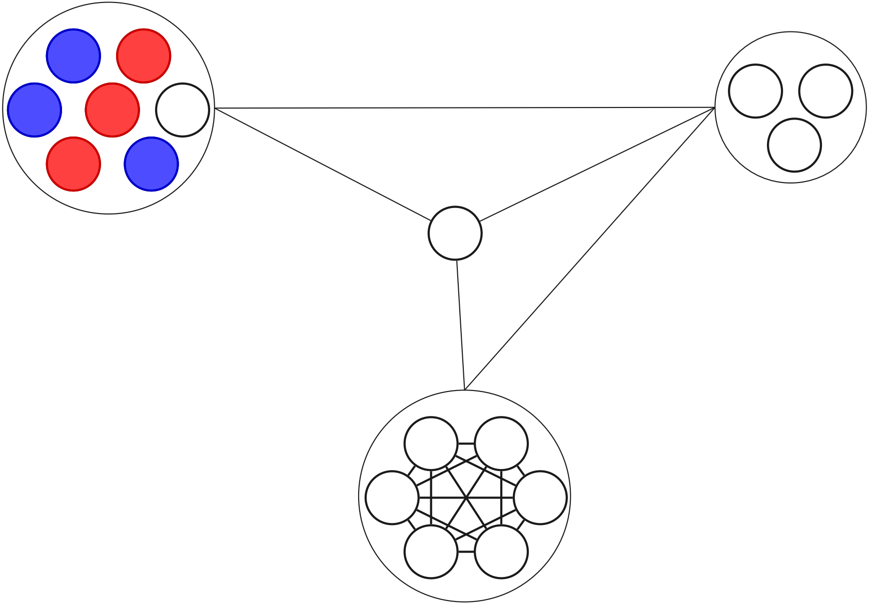

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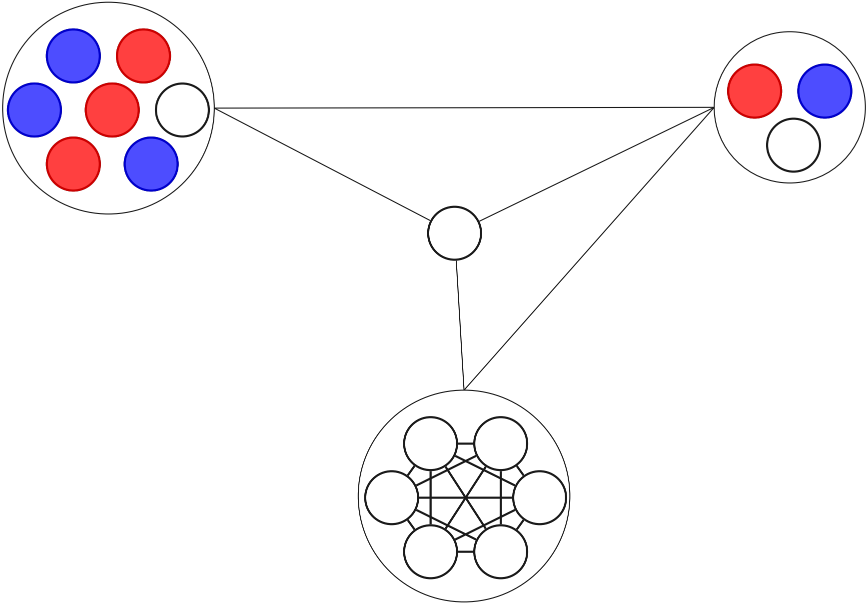

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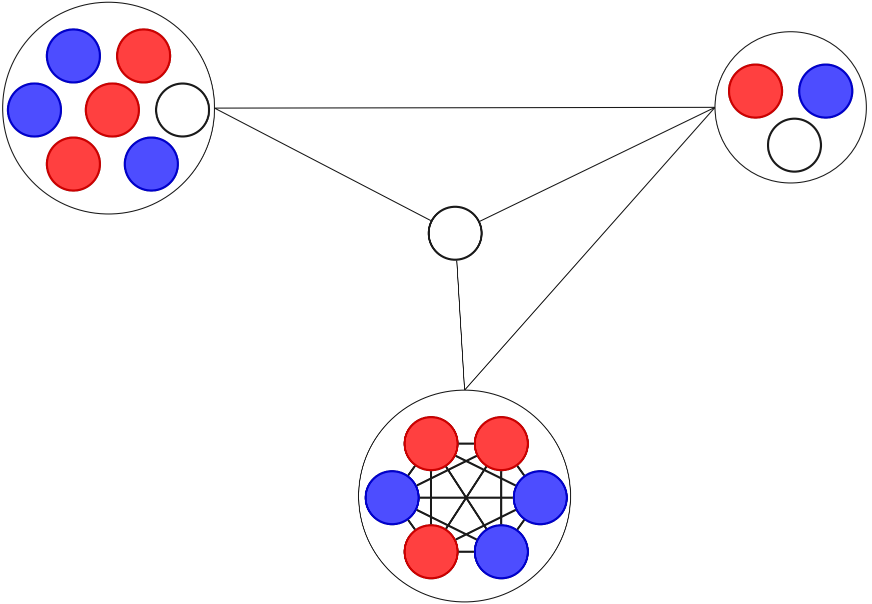

Definition (Quantified Max 2-SAT)

Let

$$\psi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \varphi$$

be a quantified Boolean formula where $Q_i \in \{\forall, \exists\}$ and φ is a quantifier-free 2-CNF formula over variables $\{x_1, \ldots, x_n\}$.

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Theorem (Bagan et al – 2024)

QUANTIFIED MAX 2-SAT *is* PSPACE-*complete.*

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Quantified Max 2-SAT-2-2 *is* NP*-hard.*

Literal–clause incidence graph

The literal–clause incidence graph G_φ with

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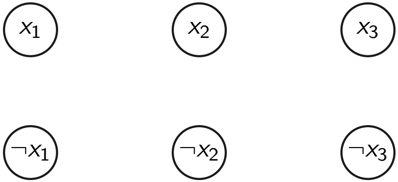

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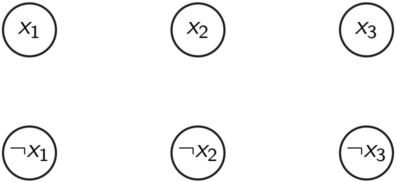

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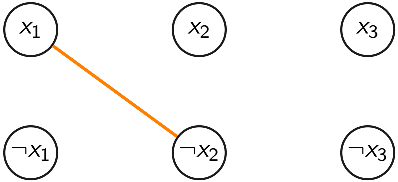
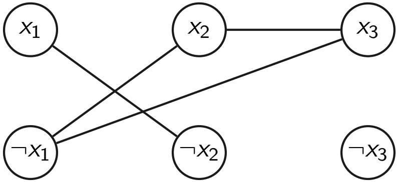


Literal–clause incidence graph

The literal–clause incidence graph G_φ with

$$\varphi = (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_1 \vee x_2)$$

is constructed as follows.



Restricted variants

Definition

- Acyclic Quantified Max 2-SAT: the graph G_φ is acyclic.
- Acyclic Quantified Max 2-SAT-2-2: the graph G_φ is a disjoint union of paths.

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Theorem

ACYCLIC QUANTIFIED MAX 2-SAT *is* PSPACE*-complete.*

Restricted variants

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Theorem

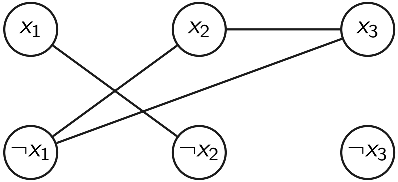
Acyclic Quantified Max 2-SAT *is* PSPACE-*complete.*

Theorem

Acyclic Quantified Max 2-SAT-2-2 *is* NP-*hard.*

Cycle-breaking gadget

$$\varphi = (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_3 \vee \neg x_1)$$



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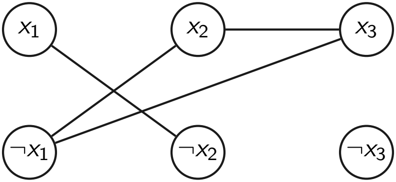
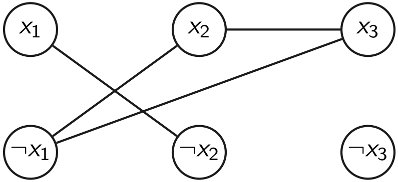


Add the gadget clauses

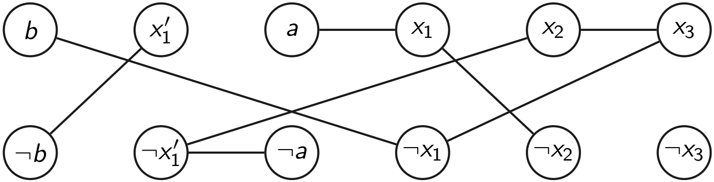
$$(x_1 \vee a), \qquad (\neg x_1 \vee b), \qquad (\neg a \vee \neg x_1'), \qquad (\neg b \vee x_1').$$

Cycle-breaking gadget

$$\varphi = (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_3 \vee \neg x_1)$$



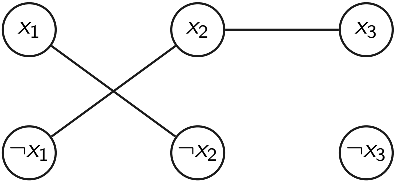
Becomes



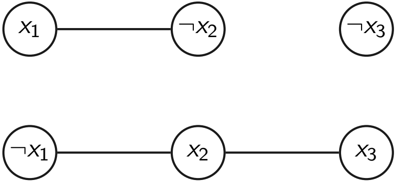
Reduction to SHVG in trees/caterpillars

$$\varphi = (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (x_2 \lor x_3)$$

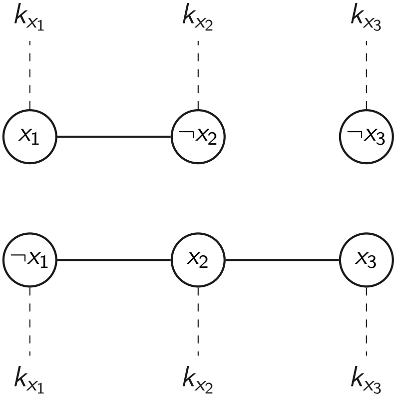
Reduction to SHVG in trees/caterpillars



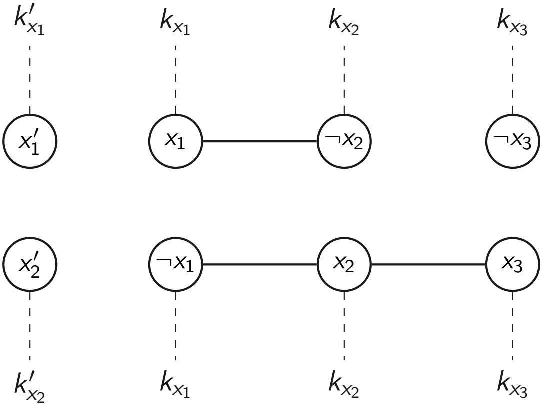
Reduction to SHVG in trees/caterpillars



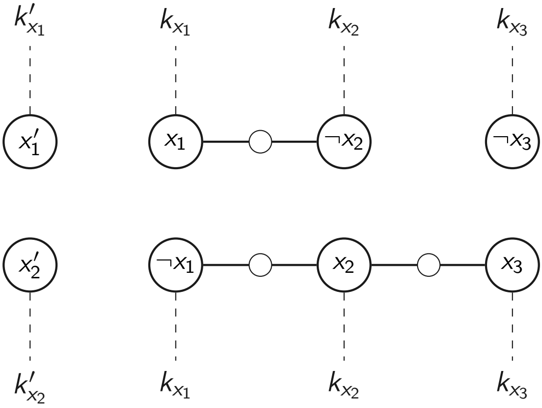
Reduction to SHVG in trees/caterpillars



Reduction to SHVG in trees/caterpillars

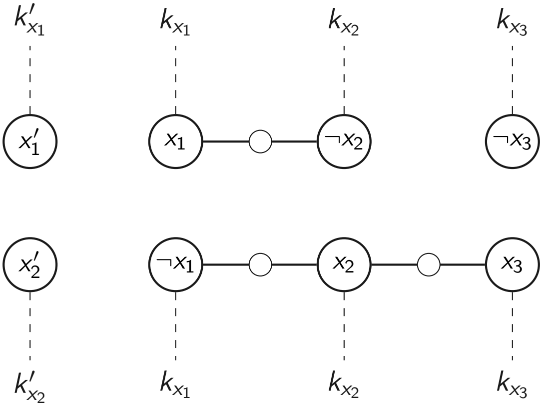


Reduction to SHVG in trees/caterpillars



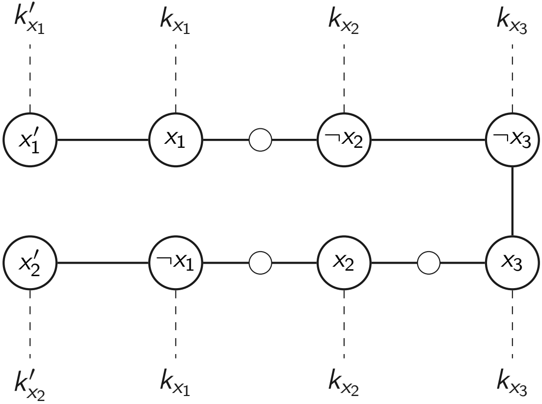
Reduction to SHVG in trees/caterpillars

$2\times$



Reduction to SHVG in trees/caterpillars

$2\times$



Open problems

Caterpillars

Is the problem on caterpillars NP-complete or already PSPACE-complete?

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Larger tractable graph classes

Can the positive results be extended beyond neighborhood diversity, for instance through decomposition parameters?

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Cographs and modular decompositions

Can decomposition techniques be pushed far enough to obtain exact score computations on cographs or related classes?