Stochastic Discrete Scale Invariance

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Abstract— **A definition of stochastic discrete scale invariance (DSI) is proposed and its properties studied. It is shown how the Lamperti transformation, which transforms stationarity in self-similarity, is also a means to connect processes deviating from stationarity and processes which are not exactly scale invariant: in particular we interpret DSI as the image of cyclostationarity. This theoretical result is employed to introduce a multiplicative spectral representation of DSI processes based on the Mellin transform, and preliminar remarks are given about estimation issues.**

Keywords— **Discrete scale invariance, Lamperti's theorem, Cyclostationary processes, Mellin transformation.**

Introduction

The notion of scale invariance, or self-similarity, is a largely used paradigm to interpret many natural and man-made phenomena (landscape structure and texture, turbulence, network traffic,...). The idea is that a function is scale invariant if it is identical to any of its rescaled function, up to some suitable renormalization of its amplitude. The proper mathematical statement is:

Definition 1: $\{X(t), t \in \mathbb{R}\}$ is scale invariant with scaling exponent H, or H-ss for self-similar with exponent H, if for any $k \in \mathbb{R}$:

$$X(kt) = k^H X(t). \tag{1}$$

This definition holds for deterministic signals. The concept is extended with much profit to stochastic processes, in which case the equality has to be understood in a probabilistic sense [1].

Scale invariance, valid for any scale factor k, is however a strong statement. It may be useful to study classes of processes which obey a weakened version of scale invariance, which might be more realistic. Among different proposals, one is of special interest: it is to require the invariance by dilation for certain prefered scaling factors only. A simple example of a deterministic fractal set, the middle third Cantor set, is in fact only invariant for scaling factors which are powers of 3. This is known as *discrete scale invariance* (DSI), a concept which has been introduced by Sornette and Saleur [2] [3] to model with some efficiency a number of extreme phenomena (critical phenomena, fracture, growth problems, earthquakes).

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I. Discrete scale invariance

We propose to extend the concept of DSI to stochastic processes, in the same way that scale invariance is used in a stochastic context:

Definition 2: $\{X(t),\ t \in \mathbb{R}^+\}$ has discrete scale invariance with scaling exponent H and scale λ if:

$$X(\lambda t) \stackrel{d}{=} \lambda^H X(t),\ t \in \mathbb{R}^+. \tag{2}$$

We will refer to this property as (H,λ)-DSI. It follows from this definition that a (H,λ)-DSI process is also scale invariant for any scaling factor of the form $\{k = \lambda^n, n \in \mathbb{Z}\}$.

The probabilistic equality in law, noted as $\stackrel{d}{=}$ in (2), has to be understood as the equality of any finite-dimensional distributions of both sides of the equation. We can however be less strict, and think of this equality in wide-sense, that is for correlation functions only. All the results discussed in the following will be valid for both interpretations.

A simple example of a DSI function is the so-called Mandelbrot-Weierstrass function [4]:

$$W(t) = \sum_{n=-\infty}^{+\infty} \lambda^{-nH}(1 - \cos(\lambda^n t + \phi_n)), \tag{3}$$

with $\lambda > 1$ and $0 < H < 1$. If the phases ϕ_n are i.i.d. random variables, W is a (H,λ)-DSI process.

Note that if one is interested in deterministic DSI, a general solution of (2) is not necessarily fractal as is the W function. A basis of deterministic DSI functions is given by the hyperbolic "chirps" of the form:

$$X(t) = t^H e^{i2\pi(m(\ln t/\ln\lambda)+\phi)}, \tag{4}$$

where $m \in \mathbb{Z}$. These functions are known as Mellin functions [5] and we will see that they are central in the study of DSI processes.

II. The Lamperti transformation

A. Lamperti's theorem

The purpose of this section is to establish general results connecting processes deviating from self-similarity and processes deviating from stationarity. The starting point is that a connection can be established between self-similarity and stationarity: this result is referred to as *Lamperti's theorem* and can be stated as follows:

Theorem 1 ([6]) If $\{X(t),\ t \in \mathbb{R}^+\}$ is H-ss, then

$$Y(t) = e^{-Ht} X(e^t),\ t \in \mathbb{R}, \tag{5}$$

is stationary. Conversely, if $\{Y(t),\ t \in \mathbb{R}\}$ is stationary, then:

$$X(t) = t^H Y(\ln t),\ t \in \mathbb{R}^+, \tag{6}$$

is H-ss.

A first illustration of Lamperti's theorem is given by the fractional Brownian motion $B_H(t)$, which is the only Gaussian H-ss process with stationary increments. Let $R_X(t,s) \hat{=} \mathbb{E}\{X(t)X(s)\}$ be the covariance function of the stochastic process $X(t)$. The covariance function of the fBm is

$$R_{B_H}(t,s) = \frac{\sigma^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \qquad (7)$$

Using (5), the fBm is transformed into a stationary process $Y(t)$ whose stationary covariance expresses as [7], [8]:

$$R_Y(t,t+\tau) = \sigma^2\{\cosh(H|\tau|) - 2^{2H-1}\sinh^{2H}(|\tau|/2)\}, \qquad (8)$$

thus generalizing the Ornstein-Uhlenbeck process, obtained in the specific case $H = 1/2$ (Lamperti image of the ordinary Brownian motion).

B. *Lamperti transformation and variations*

It is easy to develop variations on Theorem 1 by relaxing in some way the strict notions of scale invariance on one side, and of stationarity on the other side. The main point is to understand that the Lamperti transformation (6) is a means to connect a dilation operator and a time-shift operator, as stated by the following result:

Theorem 2: Let $(\mathcal{L}Y)(t) \hat{=} t^H Y(\ln t)$ be the Lamperti transformation, and $\left(\mathcal{L}^{-1}X\right)(t) \hat{=} e^{-Ht}X(e^t)$ its inverse. Let also $\left(\mathcal{D}_\lambda^H X\right)(t) \hat{=} \lambda^{-H}X(\lambda t)$ be the dilation operator and $(\mathcal{S}_\tau Y)(t) \hat{=} Y(t+\tau)$ the time-shift operator. Then for any process $\{Y(t),\, t \in \mathbb{R}\}$, and any $\lambda > 0$

$$\left(\mathcal{L}^{-1}\, \mathcal{D}_\lambda^H\, \mathcal{L}\, Y\right)(t) = \left(\mathcal{S}_{\ln\lambda} Y\right)(t). \qquad (9)$$

Proof: It is immediate to check that:

$$\begin{aligned}
\left(\mathcal{L}^{-1}\, \mathcal{D}_\lambda^H\, \mathcal{L}\, Y\right)(t) &\stackrel{d}{=} \mathcal{L}^{-1}\, \mathcal{D}_\lambda^H\, \left(t^H Y(\ln t)\right) \\
&\stackrel{d}{=} \mathcal{L}^{-1}\, \left(\lambda^{-H}(\lambda t)^H Y(\ln \lambda t)\right) \\
&\stackrel{d}{=} \mathcal{L}^{-1}\, \left(t^H Y(\ln t + \ln\lambda)\right) \\
&\stackrel{d}{=} Y(t + \ln\lambda) = \left(\mathcal{S}_{\ln\lambda} Y\right)(t)
\end{aligned}$$

■

The spirit of Lamperti's theorem is thus the correspondence by means of the $\mathcal{L}$ transformation between processes which are invariant in law by a time-shift, and processes which are invariant in law by a dilation, *i.e.*, H-ss.

We can examine processes whose behaviour under the action of the translation group or dilation group is not strict and find the class of corresponding processes obtained by $\mathcal{L}$ or $\mathcal{L}^{-1}$. Some possibilities are given as follows:

- Given (6), the correlation function of $X(t) = (\mathcal{L}Y)(t)$ follows from the correlation function of $Y(t)$ as:

$$R_X(t,s) = (st)^H R_Y(\ln t, \ln s). \qquad (10)$$

- Locally stationary processes [9] are defined via their covariance function, which reads:

$$\mathbb{E}\{Y(t)Y(s)\} = m\left(\frac{t+s}{2}\right)\gamma(s-t), \qquad (11)$$

with $m(\cdot) > 0$ and $\gamma(\cdot)$ some non-negative definite function. The Lamperti transformation of this class of processes is a class of locally self-similar processes, described by the property:

$$\mathbb{E}\{X(t)X(s)\} = (st)^H m\left(\sqrt{st}\right)\gamma(s/t). \qquad (12)$$

- Applying the Lamperti transformation to continuous-time ARMA(p,q) processes, we obtain a parametric model for self-similar processes [7] which is a generalization of the Euler-Cauchy system driven by a Gaussian white noise with nonstationary variance proportional to t.

III. DSI and cyclostationary processes

Given Theorem 2, it becomes natural to think that invariance by a time-shift of a certain period T is in correspondence with some invariance by dilation by a certain preferred factor. We introduced the latter as DSI. The former is the defining property of periodically-correlated [10], [11] or cyclostationary [12] processes.

A. *Cyclostationarity*

We will briefly summarize some results on cyclostationary processes. A process is cyclostationary if its statistical properties are periodic in time. More precisely, if T is given:

Definition 3: $\{Y(t),\, t \in \mathbb{R}\}$ is T-cyclostationary if for any time t: $Y(t+T) \stackrel{d}{=} Y(t)$.

An immediate consequence is that, if $\{Y(t), t \in \mathbb{R}\}$ is T-cyclostationary, its correlation function must satisfy

$$\mathbb{E}\{Y(t+T)Y(s+T)\} = \mathbb{E}\{Y(t)Y(s)\}, \qquad (13)$$

whence, because $R_Y(t, t+\tau)$ is periodic in t of period T, one can decompose R_Y in a Fourier series:

$$R_Y(t,t+\tau) = \sum_{n=-\infty}^{+\infty} C_n(\tau)e^{i2\pi nt/T}. \qquad (14)$$

B. *DSI and cyclostationary processes*

We can now prove the following result concerning DSI:

Theorem 3: If $\{X(t),\, t \in \mathbb{R}^+\}$ is (H, e^T)-DSI, then

$$Y(t) = e^{-Ht}X(e^t),\ t \in \mathbb{R}, \qquad (15)$$

is T-cyclostationary. Conversely, if $\{Y(t),\, t \in \mathbb{R}\}$ is T-cyclostationary, then

$$X(t) = t^H Y(\ln t),\ t \in \mathbb{R}^+, \qquad (16)$$

is (H, e^T)-DSI.

Proof: If $X(t)$ is (H, e^T)-DSI, then

$$X(e^T t) \stackrel{d}{=} e^{HT}X(t), \qquad (17)$$

and it follows that

$$\begin{aligned}
Y(t+T) &= e^{-Ht-HT}X\left(e^T e^t\right) \\
&\stackrel{d}{=} e^{-Ht}X\left(e^t\right) = Y(t), \qquad (18)
\end{aligned}$$

and $Y(t)$ is therefore T-cyclostationary. Conversely, if $Y(t)$ is T-cyclostationary, then

$$Y(t+T) \stackrel{d}{=} Y(t), \tag{19}$$

and we have therefore

$$\begin{aligned} X(e^T t) &= e^{HT} e^{H \ln t} Y(\ln t + T) \\ &\stackrel{d}{=} e^{HT} t^H Y(\ln t) = \left(e^T\right)^H X(t), \end{aligned} \tag{20}$$

thus proving that $X(t)$ is (H, e^T)-DSI. ■

A first immediate consequence, using (14), is the general form for the correlation function of (H, λ)-DSI processes:

$$R_X(t, kt) = k^H t^{2H} \sum_{n=-\infty}^{+\infty} C_n(k) t^{i2\pi n/\ln\lambda}. \tag{21}$$

This function is therefore naturally expressed on a Mellin basis [5]. This is not a surprise: in changing the time-shift operator for the dilation operator, the Lamperti transformation changes also the Fourier basis (invariant up to a phase by time-shift) for the Mellin basis (invariant up to a phase by dilation).

In the same way, the Lamperti transformation changes substractions or additions of the time variable in divisions or multiplications: we will then use the "multiplicative" adjective to qualify properties of the processes after Lamperti transformation, from the terminology of [13] which studied some results for the $H = 0$ case.

IV. Multiplicative harmonizability

The processes which we study, Lamperti images of nonstationary processes, are not stationary. They often have nonstationary increments. We thus have trouble using classical results for spectral representation of signals or theorems funding the estimation of a correlation function. As a remedy, we can introduce a "multiplicative" spectrum by means of the Lamperti transformation.

It is known that a natural description of some nonstationary processes is Loève's decomposition [14]. A process is called harmonizable when its correlation function admits a Fourier transform which reads:

$$R_Y(t,s) = \iint e^{i2\pi(sf - t\nu)} \Phi(\nu, f) \mathrm{d}\nu \mathrm{d}f, \tag{22}$$

where the spectral distribution function $\Phi(\nu, f)$ is related to the correlation of the spectral increments of $Y(t)$.

The corresponding notion for processes after a Lamperti transformation introduces a new representation for a class of processes deviating from self-similarity, which we will call multiplicative harmonizability. A process has this property if it verifies:

$$R_X(t,s) = \iint t^{H+i2\pi\beta} s^{H-i2\pi\sigma} \Phi(\sigma,\beta) \mathrm{d}\sigma \mathrm{d}\beta. \tag{23}$$

A necessary and sufficient condition for this equality to hold is adapted from Loève's condition for harmonizability. For (23) to be written, Φ must verify:

$$\iint |\Phi(\sigma,\beta)| \mathrm{d}\sigma \mathrm{d}\beta < +\infty. \tag{24}$$

It can be shown that if a process can be written as (23), the realization $X(t)$ admits a "spectral" representation on a Mellin basis (that is a multiplicative harmonizability for the process):

$$X(t) = \int t^{H+i2\pi\beta} \mathrm{d}\tilde{X}(\beta), \tag{25}$$

where the corresponding multiplicative spectral increments $\mathrm{d}\tilde{X}(\beta)$ are not orthogonal. Their correlation is given by the spectral distribution function :

$$\mathbb{E}\left\{ \mathrm{d}\tilde{X}(\sigma) \mathrm{d}\tilde{X}^*(\beta) \right\} = \Phi(\sigma,\beta) \mathrm{d}\sigma \mathrm{d}\beta. \tag{26}$$

The inverse Mellin transformation gives the expression of the spectral function if the correlation is known:

$$\Phi(\sigma,\beta) = \iint t^{-H-i2\pi\beta} s^{-H+i2\pi\sigma} R_X(t,s) \frac{\mathrm{d}t}{t} \frac{\mathrm{d}s}{s}. \tag{27}$$

Note that the Mellin variable β has the property of a scale [15]. In fact this is the notion of scale if one considers invariance by dilation as the main property to define a scale. The proposed decompositions are thus decompositions in scale.

If X is H-ss, then the spectral function is a distribution non-zero only on the diagonal, $\sigma = \beta$, *i.e.*, the multiplicative spectral increments are uncorrelated. This is the equivalent property of harmonizability of stationary processes, which are known to have uncorrelated spectral increments. For H-ss processes, this property is that there is no correlation between different Mellin scales.

The usefulness of the multiplicative harmonizability is apparent when one considers the multiplicative spectral function of a DSI process. Harmonizable, T-cyclostationary processes have a spectral function non-zero only on parallel lines verifying $\nu - f = n/T$, where $n \in \mathbb{Z}$ [11]. In the same way, (H, λ)-DSI processes have a characteristic structure for their multiplicative spectral function.

Given (21) and (23), the spectral function reads for DSI:

$$\Phi(\beta,\sigma) = \sum_n \tilde{C}_n(\sigma) \delta\left(\beta - \sigma - \frac{n}{\ln\lambda}\right), \tag{28}$$

where $\tilde{C}_n(\sigma)$ is the Mellin transform of the $C_n(k)$ in (21).

V. Towards Mellin-based tools for estimation

A practical study of the class of processes introduced herein is to estimate the correlation function or the spectral function and, in the case of DSI processes, the two parameters H and λ. Since the increment process is not necessarily stationary, the wavelet methods [16] may fail to estimate H and the estimation issue can then be adressed in two ways.

First a direct use of the inverse Lamperti transformation on (H, λ)-DSI processes converts the estimation problem in an equivalent one for cyclostationary processes. Classical methods can be used (see for instance [17]). This possibility, which is under practical consideration, is analogous to the way of studying H-ss processes in [8].

Another way is to convert estimators directly in a Mellin formalism. Many variations are possible, depending on the method for nonstationary processes which is adapted:

• The multiplicative spectral function can be associated to time-Mellin scale decompositions in the same way that time-frequency decompositions [18] are related to the usual spectrum. For instance, the Scale Invariant Wigner Spectrum (SIWS) [19] is a partial Mellin transform of the correlation function:

$$\underline{W}_X(t,\beta) = \int R_X(k^{1/2}t, k^{-1/2}t)k^{-i2\pi\beta-1}\mathrm{d}k. \quad (29)$$

A general class of time-Mellin scale representations may then give rise to different estimations of the SIWS, in the same way that the Cohen class gives methods to estimate the usual Wigner-Ville spectrum. The multiplicative spectrum is then related to the SIWS by:

$$\Phi(\sigma,\beta) = \int t^{-2H-i2\pi(\beta-\sigma)}\underline{W}_X\left(t,\frac{\beta+\sigma}{2}\right)\frac{\mathrm{d}t}{t}. \quad (30)$$

• For (H,λ)-DSI processes the problem is particular: a singular density has to be estimated. An easier way is to find first λ. The best method is to use an estimation of the multiplicative spectrum, and then compute the marginal in cyclic scale $\beta_c = \beta - \sigma$:

$$M(\beta_c) = \int \Phi(\nu - \beta_c/2, \nu + \beta_c/2)\mathrm{d}\nu. \quad (31)$$

If the process is (H,λ)-DSI, the marginal has peaks disposed in $\beta_c = n/\ln\lambda$ for some $n \in \mathbb{Z}$. A non-parametric estimation of λ can be characterized in this manner.

• For (H,λ)-DSI, when λ is known, estimators for $\tilde{C}_n(\sigma)$ or $C_n(k)$ adapted from those in the cyclostationary case [20] can be constructed. This will generalize the estimators for self-similar processes, given in [13], to the DSI problem.

• One can object that the representation, and then the proposed tools, use explicitely the unknown parameter H. That may be a problem but the Mellin transform:

$$\mathcal{M}(f;\beta) = \int_0^{+\infty} f(t)t^{-r-1-i2\pi\beta}\mathrm{d}t, \quad (32)$$

is weakly sensitive to the amplitude factor r. If f is a Mellin function (4), the Mellin transform with r not equal to H will be, instead of a Dirac distribution, a peak of width $\sqrt{3}|H-r|/\pi$ at half-height. In usual cases for which $0 < H < 1$, one can choose $r = 1/2$ as a substitution for H in the analysis tools to estimate λ, and then use this new information to correctly estimate H.

The main conclusion of this work is, to summarize briefly, that DSI can be envisaged directly on stochastic processes and studied by means of classical tools when one works on the corresponding cyclostationnary processes obtained with the inverse Lamperti transformation, or by means of new tools constructed on the formalism induced by this transformation and equality (9). The next step is the investigation of the proposed framework for estimation on real data. First attempts, proposing definitions and analysis of simple sequences with DSI were reported in [21]. These methods will now enable to re-issue the relevance of DSI [3] (as found, *e.g.*, in geophysics where evidence of DSI in earthquakes were given, or in the DLA model for growth phenomena), by examining directly in the time domain the underlying DSI processes attached to the problems under consideration.

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