Event Structures, Stable Families and Games

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Chapter 1

Event structures

Event structures are a fundamental model of concurrent computation and, along with their extension to stable families, provide a mathematical foundation for the course.

1.1 Event structures

Event structures are a model of computational processes. They represent a process, or system, as a set of event occurrences with relations to express how events causally depend on others, or exclude other events from occurring. In one of their simpler forms they consist of a set of events on which there is a consistency relation expressing when events can occur together in a history and a partial order of causal dependency—writing $e' \leq e$ if the occurrence of e depends on the previous occurrence of e'.

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$
\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con} \ \&\ e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.
\end{aligned}
$$

The events are to be thought of as event occurrences without significant duration; in any history an event is to appear at most once. We say that events e, e' are *concurrent*, and write $e \; co \; e'$ if $\{e, e'\} \in \mathrm{Con} \ \&\ e \nleq e' \ \&\ e' \nleq e$. Concurrent events can occur together, independently of each other. The relation of *immediate* dependency $e \rightarrowtriangle e'$ means e and e' are distinct with $e \leq e'$ and no event in between. Clearly $\leq$ is the reflexive transitive closure of $\rightarrowtriangle$.

An event structure represents a process. A configuration is the set of all events which may have occurred by some stage, or history, in the evolution of the process. According to our understanding of the consistency relation and causal dependency relations a configuration should be consistent and such that if an event appears in a configuration then so do all the events on which it causally depends.

The *configurations* of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \text{Con}$, and

Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

We shall largely work with *finite* configurations, written $\mathcal{C}(E)$. Write $\mathcal{C}^\infty(E)$ for the set of *finite and infinite* configurations of the event structure E.

The configurations of an event structure are ordered by inclusion, where $x \subseteq x'$, *i.e.* x is a sub-configuration of x', means that x is a sub-history of x'. Note that an individual configuration inherits an order of causal dependency on its events from the event structure so that the history of a process is captured through a partial order of events. The finite configurations correspond to those events which have occurred by some finite stage in the evolution of the process, and so describe the possible (finite) states of the process.

For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X. The axioms on the consistency relation ensure that the down-closure of any finite set in the consistency relation s a finite configuration, and that any event appears in a configuration: given $X \in \text{Con}$ its down-closure $\{e' \in E \mid \exists e \in X.\ e' \leq e\}$ is a finite configuration; in particular, for an event e, the set $[e] =_{\text{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e. We shall sometimes write $[e) =_{\text{def}} \{e' \in E \mid e' < e\}$.

When the consistency relation is determined by the pairwise consistency of events we can replace it by a binary relation or, as is more usual, by a complementary binary conflict relation on events (written as $\#$ or $\smile$).

Remark on the use of "cause." In an event structure $(E, \leq, \text{Con})$ the relation $e' \leq e$ means that the occurrence of e depends on the previous occurrence of the event e'; if the event e has occurred then the event e' must have occurred previously. In informal speech cause is also used in the forward-lookciaing sense of one thing arising because of another. Often when used in this way the history of events is understood beforehand. According to the history around my life, the meeting of my parents caused my birth. But the history might have been very different: in an alternative world the meeting of my parents might not have led to my birth. More formally, w.r.t. a configuration x in which an event e occurs while it seems sensible to talk about the events $[e)$ causing e, it is so only by virtue of the understood configuration x.

We also encounter events which in a history may have been caused in more than one way. There are generalisations of the current event structures which do this, and support "disjunctive causes." We will work with the simple definition above in which an event, or really an event occurrence, e is causally dependent on a unique set of events $[e)$. Much of the mathematics we develop around these simpler forms of event structures (sometimes called prime event structures in the literature) is reusable when it comes to considering events with several causes. Roughly the simpler event structures will suffice in considering nondeterministic strategies. Where their limitations first show up is in a treatment of probabilistic strategies; even there closely related structures come into play.

1.2 Maps of event structures

Let E and E' be event structures. A *(partial) map* of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and

$$\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2) \text{ (with both defined), then } e_1 = e_2.$$

The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. The map f respects the instantaneous nature of events: two distinct event occurrences which are consistent with each other cannot both coincide with the occurrence of a common event in the image. Partial maps of event structures compose as partial functions, with identity maps given by identity functions.

We will say the map is *total* if the function f is total. Notice that for a total map f the condition on maps now says it is *locally injective*, in the sense that w.r.t. any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective. Say a total map of event structures is *rigid* when it preserves causal dependency.

Maps preserve the concurrency relation, when defined.

1.2.1 Partial-total factorisation

Let $(E, \leq, \text{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\text{def}} (V, \leq_V, \text{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \text{Con}_V$ iff $X \in \text{Con}$ & $X \subseteq V$.

Consider a partial map of event structures $f : E \to E'$. Let

$$V =_{\text{def}} \{e \in E \mid f(e) \text{ is defined}\}.$$

Then f clearly factors into the composition

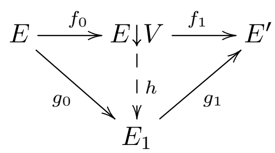
$$E \xrightarrow{f_0} E{\downarrow}V \xrightarrow{f_1} E'$$

of f_0, a partial map of event structures taking $e \in E$ to itself if $e \in V$ and undefined otherwise, and f_1, a total map of event structures acting like f on V. We call f_1 the *defined part* of the partial map f. We say a map $f : E \to E'$ is a *projection* if its defined part is an isomorphism.

The factorisation is characterised to within isomorphism by the following universal characterisation: for any factorisation

$$E \xrightarrow{g_0} E_1 \xrightarrow{g_1} E'$$

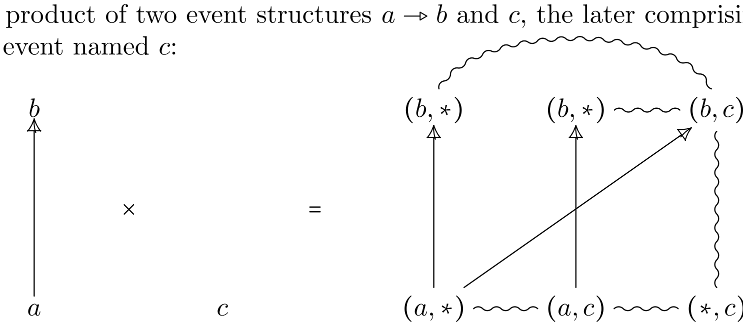
where g_0 is partial and g_1 is total there is a (necessarily total) unique map $h : E{\downarrow}V \to E_1$ such that



commutes.

1.3 Products of event structures

The category of event structures has products, which essentially allow arbitrary synchronizations between their components. For example, here is an illustration of the product of two event structures $a \rightarrowtriangle b$ and c, the later comprising just a single event named c:



The original event b has split into three events, one a synchronization with c, another b occurring unsynchronized after an unsynchronized a, and the third b occurring unsynchronized after a synchronizes with c. The splittings correspond to the different histories of the event.

It can be awkward to describe operations such as products, pullbacks and synchronized parallel compositions directly on the simple event structures here, essentially because an event determines its whole causal history. One closely related and more versatile, though perhaps less intuitive and familiar, model is that of stable families. Stable families will play an important technical role in establishing and reasoning about constructions on event structures.

Chapter 2

Stable families

Stable families support a form of disjunctive causes in which an event may be enabled in several different but incompatible ways. Stable families, their basic properties and relations to event structures are developed.[1]

2.1 Stable families

The notion of stable family extends that of finite configurations of an event structure to allow an event can occur in several incompatible ways.

Notation 2.1. *Let $\mathcal{F}$ be a family of subsets. Let $X \subseteq \mathcal{F}$. We write $X\uparrow$ for $\exists y \in \mathcal{F}.\ \forall x \in X. x \subseteq y$ and say X is* compatible. *When $x, y \in \mathcal{F}$ we write $x \uparrow y$ for $\{x, y\}\uparrow$.*

A *stable family* comprises $\mathcal{F}$, a nonempty family of finite subsets, satisfying:
Completeness: $\forall Z \subseteq \mathcal{F}.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}$;
Stability: $\forall Z \subseteq \mathcal{F}.\ Z \neq \varnothing \ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}$;
Coincidence-freeness: For all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,

$$\exists y \in \mathcal{F}.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y)\,.$$

We call the elements of $\bigcup \mathcal{F}$ of a stable family $\mathcal{F}$ its *events*.

An alternative characterisation of stable families:

Proposition 2.2. *A* stable family *comprises $\mathcal{F}$, a family of finite subsets, satisfying:*
Completeness: $\varnothing \in \mathcal{F} \ \&\ \forall x, y \in \mathcal{F}.\ x \uparrow y \implies x \cup y \in \mathcal{F}$*;*
Stability: $\forall x, y \in \mathcal{F}.\ x \uparrow y \implies x \cap y \in \mathcal{F}$*;*
Coincidence-freeness: *For all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,*

$$\exists y \in \mathcal{F}.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y)\,.$$

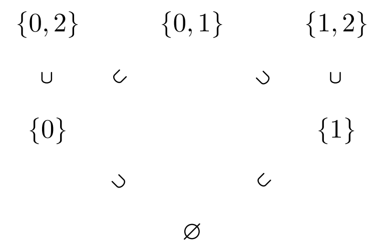
[1] A useful reference for stable families is the report "Event structure semantics for CCS and related languages," a full version of the ICALP'82 article, available from `www.cl.cam.ac.uk/~gw104`, though its terminology can differ from that here.

Proof. Simple inductions show that the reformulations of "Completeness" and "Stability" are equivalent to their original formulations. □

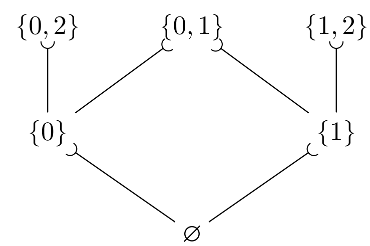
Proposition 2.3. *The family of finite configurations of an event structure forms a stable family.*

On the other hand stable families are more general than finite configurations of an event structure, as the following example shows.

Example 2.4. Let $\mathcal{F}$ be the stable family, with events $E = \{0, 1, 2\}$,



or equivalently



where $-\!\!\!\subset$ is the covering relation representing an occurrence of one event. The events 0 and 1 are concurrent, neither depends on the occurrence or non-occurrence of the other to occur. The event 2 can occur in two incompatible ways, either through event 0 having occurred or event 1 having occurred. This possibility can make stable families more flexible to work with than event structures.

A (partial) map of stable families $f : \mathcal{F} \to \mathcal{G}$ is a partial function f from the events of $\mathcal{F}$ to the events of $\mathcal{G}$ such that for all $x \in \mathcal{F}$,

$$fx \in \mathcal{G} \;\&\; (\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2)\,.$$

Maps of stable families compose as partial functions, with identity maps given by identity functions. We call a map $f : \mathcal{F} \to \mathcal{G}$ of stable families *total* when it is total as a function; the f restricts to a bijection $x \cong fx$ for all $x \in \mathcal{F}$.

Definition 2.5. Let $\mathcal{F}$ be a stable family. We use $x -\!\!\!\subset y$ to mean y covers x in $\mathcal{F}$, *i.e.* $x \subset y$ in $\mathcal{F}$ with nothing in between, and $x \overset{e}{-\!\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{F}$ and event $e \notin x$. We sometimes use $x \overset{e}{-\!\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \overset{e}{-\!\!\!\subset} y$ for some y.

2.2 Stable families and event structures

Finite configurations of an event structure form a stable family. Conversely, a stable family determines an event structure:

Proposition 2.6. *Let x be a configuration of a stable family $\mathcal{F}$. For $e, e' \in x$ define*

$$e' \leq_x e \text{ iff } \forall y \in \mathcal{F}.\ y \subseteq x \ \&\ e \in y \implies e' \in y.$$

When $e \in x$ define the prime *configuration*

$$[e]_x = \bigcap \{y \in \mathcal{F} \mid y \subseteq x \ \&\ e \in y\}\ .$$

Then $\leq_x$ is a partial order and $[e]_x$ is a configuration such that

$$[e]_x = \{e' \in x \mid e' \leq_x e\}.$$

Moreover the configurations $y \subseteq x$ are exactly the down-closed subsets of $\leq_x$.

Lemma 2.7. *Let $\mathcal{F}$ be a stable family. Then,*

$$[e]_x \subseteq z \iff [e]_x = [e]_z$$

whenever $e \in x$ and z in $\mathcal{F}$.

Proof. "⇒" From $e \in [e]_x \subseteq z$ we get $[e]_z \subseteq [e]_x$. Hence $e \in [e]_z \subseteq x$ ensuring the converse inclusion $[e]_x \subseteq [e]_z$, so $[e]_x = [e]_z$. "⇐" Trivial. □

Proposition 2.8. *Let $\mathcal{F}$ be a stable family. Then,* $\mathrm{Pr}(\mathcal{F}) =_{\mathrm{def}} (P, \mathrm{Con}, \leq)$ *is an event structure where:*

$$P = \{[e]_x \mid e \in x \ \&\ x \in \mathcal{F}\}\ ,$$
$$Z \in \mathrm{Con} \text{ iff } Z \subseteq P \ \&\ \bigcup Z \in \mathcal{F} \quad \text{and,}$$
$$p \leq p' \text{ iff } p, p' \in P \ \&\ p \subseteq p'\ .$$

There is an order isomorphism

$$\theta : (\mathcal{C}(\mathrm{Pr}(\mathcal{F})), \subseteq) \cong (\mathcal{F}, \subseteq)$$

where $\theta(y) = \bigcup y$ for $y \in \mathcal{C}(\mathrm{Pr}(\mathcal{F}))$; its mutual inverse is ϕ where $\phi(x) = \{[e]_x \mid e \in x\}$ for $x \in \mathcal{F}$.

Proof. It is easy to check that $\mathrm{Pr}(\mathcal{F})$ is an event structure. Clearly, both θ and ϕ preserve $\subseteq$.

Firstly, $\theta\phi(x) = \bigcup\{[e]_x \mid e \in x\} = x$, for all $x \in \mathcal{F}$, by an obvious argument.

Secondly, $\phi\theta(y) = \{[e]_{\bigcup y} \mid e \in \bigcup y\}$, for $y \in \mathcal{C}(\mathrm{Pr}(\mathcal{F}))$. To show $rhs = y$ we use Lemma 2.7 repeatedly:

$$[e]_x \subseteq z \iff [e]_x = [e]_z\ ,$$

whenever $e \in x$ and z in $\mathcal{F}$.
From $e \in [e]_x \subseteq z$ we get $[e]_z \subseteq [e]_x$. Hence $e \in [e]_z \subseteq x$ ensuring the converse inclusion $[e]_x \subseteq [e]_z$, so $[e]_x = [e]_z$.
"$y \subseteq rhs$": $[e]_x \in y \Rightarrow [e]_x \subseteq \bigcup y \Rightarrow [e]_x = [e]_{\bigcup y} \in rhs$.
"$rhs \subseteq y$: Assume $p \in rhs$. Then $p = [e]_{\bigcup y}$ with $e \in \bigcup y$. We have $e \in [e']_x \in y$ for some e', x with $e' \in x$. So $[e]_x \subseteq [e']_x \in y$ ensuring $[e]_x \in y$. As $[e]_x \subseteq \bigcup y$ we obtain $p = [e]_{\bigcup y} = [e]_x$, so $p \in y$. □

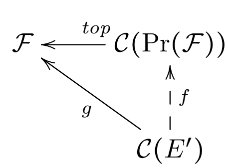
Remark. The above proposition ensures that the partial orders comprising stable families ordered by inclusion and the orders of configurations of event structures are the same to within isomorphism; both coincide with the orders of finite elements of "prime algebraic domains" in which every finite, or isolated, element dominates only finitely many elements.

The operation Pr is right adjoint to the "inclusion" functor, taking an event structure E to the stable family $\mathcal{C}(E)$. The unit of the adjunction at an event structure E is a map $E \to \mathrm{Pr}(\mathcal{C}(E))$ which takes an event e to the prime configuration $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$. The counit at a stable family $\mathcal{F}$ is a map $top_{\mathcal{F}} : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ which takes a prime configuration $[e]_x$ to e; this is well-defined as a function by coincidence-freeness (see the proof of Theorem 2.9.

Theorem 2.9. *There is a map $top_{\mathcal{F}} : \mathrm{Pr}(\mathcal{F}) \to \mathcal{F}$ given by $top_{\mathcal{F}}([e]_x) = e$ for $e \in x \in \mathcal{F}$. In fact, $\mathrm{Pr}(\mathcal{F})$, $top_{\mathcal{F}}$ is cofree over $\mathcal{F}$ i.e. for any map $g : \mathcal{C}(E') \to \mathcal{F}$ of stable families with E' a prime event structure, there is a unique map $f : E' \to \mathrm{Pr}(\mathcal{F})$ such that $g = top_{\mathcal{F}} f$.*

Proof. By Proposition 2.8, $\mathrm{Pr}(\mathcal{F})$ is a prime event structure. We require that $top_{\mathcal{F}} : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ above is a map. Firstly we need top is well-defined as a function $top : P \to E$ where $P = \{[e]_x \mid e \in x \in \mathcal{F}\}$. Suppose $[e]_x = [e']_y$ for $e \in x$ and $x \in \mathcal{F}$ and $e' \in y$ and $y \in \mathcal{F}$. Then by the coincidence-freeness of $\mathcal{F}$ we have $e = e'$, giving top well-defined as a (total) function. From the definition, if z is a configuration of $\mathrm{Pr}(\mathcal{F})$ then $z = \{[e]_x \mid e \in x\}$ for some $x \in \mathcal{F}$; thus $top(z) = \bigcup z = x \in \mathcal{F}$. Let z be a configuration of $\mathrm{Pr}(\mathcal{F})$ so $p, p' \in z$ and $top(p) = top(p') = e$ say. Then $p = p' = [e]_{\bigcup z}$. Thus top is a map of stable families.

We show $\mathrm{Pr}(\mathcal{F})$, $top_{\mathcal{F}}$ is cofree over $\mathcal{F}$. Let $g : \mathcal{C}(E') \to \mathcal{F}$ be a map of stable families where E' is a prime event structure $E' = (E', \mathrm{Con}', \leq')$. We require a unique map $f : E' \to \mathrm{Pr}(\mathcal{F})$ s.t. the following diagram commutes:



Define $f : E' \to P$ by

$$f(e') = \begin{cases} [g(e')]_{g[e']} & \text{if } g(e') \text{ is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Above $[e']$ is the downwards closure of e' in E'. Let $x \in \mathcal{C}(E')$. Then

$$\begin{aligned} fx &= \{[g(e')]_{g[e']} \mid e' \in x \ \&\ g(e') \text{ is defined}\} \\ &= \{[e]_{gx} \mid e \in gx\} \end{aligned}$$

where we have observed that $[g(e')]_{g[e']} \subseteq gx$ when $e' \in x$, so $[g(e')]_{g[e']} = [g(e')]_{gx}$. Hence fx is a configuration of $\mathrm{Pr}(F)$. If $e, e' \in x$ and $f(e) = f(e')$ (both defined) then $g(e) = g(e')$ (both defined) so $e = e'$, as g is a map. Thus f is a map. Clearly $topf = g$ so f makes the diagram commute.

Let $f' : E' \to \mathrm{Pr}(\mathcal{F})$ be a map such that the diagram commutes *i.e.* $topf = g$. We require $f' = f$. Let $e' \in E'$. Firstly note if $g(e')$ is defined then because *top* is a total function we must have $f'(e)$ defined which agrees with f. So suppose that $g(e)$ defined. Then $f'(e)$ is a prime configuration of F s.t. $top(f'(e)) = g(e)$. Now *top* is just union so using the assumed commutation we get

$$f'(e) \subseteq \bigcup f'[e] = top f'[e] = g[e]$$

As $f'(e)$ is a prime configuration in $g[e]$ and $top(f'(e)) = g(e)$ we have $f'(e) = [g(e)]_{g[e]}$, *i.e.* $f'(e) = f(e)$.

Consequently f is the unique map making the diagram commute. □

Theorem 2.9 gives a bijection between maps maps $g : \mathcal{C}(E) \to \mathcal{F}$ of stable families and maps $f : E \to \mathrm{Pr}(\mathcal{F})$ of event structures where E is an event structure and $\mathcal{F}$ is a stable family. The bijection is natural in E. As is well-known there is a unique extension of Pr to a functor so that the bijection is also natural in $\mathcal{F}$. Once extended in this way we obtain the natural bijection of an adjunction.

Corollary 2.10. *The functor $\mathcal{C}(\_)$ from the category of event structures to the category of stable families has a right adjoint the functor which acts as* Pr *on stable families and as follows on a map $f : \mathcal{A} \to \mathcal{B}$ of stable families: the map $\mathrm{Pr}(f) : \mathrm{Pr}(\mathcal{A}) \to \mathrm{Pr}(\mathcal{B})$ takes $[a]_x$, an event of $\mathrm{Pr}(\mathcal{A})$, where $a \in x \in \mathcal{A}$, to the event $[f(a)]_{fx}$ of $\mathrm{Pr}(\mathcal{B})$ if $f(a)$ is defined, and to undefined otherwise.*

The unit of the adjunction at an event structure E is the isomorphism $E \cong \mathrm{Pr}(\mathcal{C}(E))$ taking e to $[e]$. The counit at a stable family $\mathcal{F}$ is given by $top_{\mathcal{F}} : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$.

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be a map of stable families. We must first be sure that $\mathrm{Pr}(f)$ is well-defined as a partial function. Suppose $[a]_x = [a']_y$ for $a \in x \in \mathcal{A}$ and $b \in y \in \mathcal{B}$. We require $\mathrm{Pr}(f)([a]_x) = \mathrm{Pr}(f)([a']_y)$ when either is defined. Firstly, $a = a'$ by the coincidence-freeness of $\mathcal{A}$. Suppose $f(a)$ is defined. Then,

$$[f(a)]_{fx} \subseteq f[a]_x = f[a]_y \subseteq fy\,.$$

Hence by Lemma 2.7, $[f(a)]_{fx} = [f(a)]_{fy}$, *i.e.* $\mathrm{Pr}(f)([a]_x) = \mathrm{Pr}(f)([a']_y)$.

We should check that $\mathrm{Pr}(f)$ is a map of event structures. By Proposition 2.8, a configuration y of $\mathrm{Pr}(\mathcal{A})$ has the form $\{[a]_x \mid a \in x\}$ for some $x \in \mathcal{A}$. Under $\mathrm{Pr}(f)$ it is sent to

$$\{[f(a)]_{fx} \mid a \in x \ \&\ f(a) \text{ is defined}\} = \{[b]_{fx} \mid b \in fx\}\,,$$

a configuration of $\mathrm{Pr}(\mathcal{B})$. Moreover, if $[a]_x, [a']_{x'} \in y$ and $\mathrm{Pr}(f)([a]_x) = \mathrm{Pr}(f)([a']_{x'})$, then $[f(a)]_{fx} = [f(a')]_{fx'}$. But now $f(a) = f(a')$ as $\mathcal{B}$ is coincidence-free and $a, a' \in \bigcup y \in \mathcal{A}$ which implies $a = a'$. As $[a]_x, [a]_{x'} \subseteq \bigcup y$ from Lemma 2.7 we deduce $[a]_x = [a]_{\bigcup y} = [a]_{x'}$, as required.

The map $\mathrm{Pr}(f)$ clearly makes the diagram

$$\begin{array}{ccc} \mathcal{B} & \xleftarrow{top_{\mathcal{B}}} & \mathcal{C}(\mathrm{Pr}(\mathcal{B})) \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \mathrm{Pr}(f)} \\ \mathcal{A} & \xleftarrow{top_{\mathcal{A}}} & \mathcal{C}(\mathrm{Pr}(\mathcal{A})) \end{array}$$

commute Hence, $\mathrm{Pr}(f)$ gives the unique extension of Pr to a functor which makes the bijection (between maps $g : \mathcal{C}(E) \to \mathcal{F}$ of stable families and maps $f : E \to \mathrm{Pr}(\mathcal{F})$ of event structures) given by the cofreeness property of Theorem 2.9 natural, so forming an adjunction.

It is easily checked that the putative unit and counit maps do indeed correspond to the identities on $\mathcal{C}(E)$ and $\mathrm{Pr}(\mathcal{F})$, respectively, as required for their to be unit and counit. □

Remark. The fact that the unit is an isomorphism and the fact that the left adjoint is full and faithful are in fact equivalent and say that the adjunction is in a *coreflection*. Later it will play a role in obtaining products of event structures from those of stable families.

2.3 Process constructions

2.3.1 Products

Let $\mathcal{A}$ and $\mathcal{B}$ be stable families with events A and B, respectively. Their product, the stable family $\mathcal{A} \times \mathcal{B}$, has events comprising pairs in $A \times_* B =_{\mathrm{def}} \{(a, *) \mid a \in A\} \cup \{(a, b) \mid a \in A \ \&\ b \in B\} \cup \{(*, b) \mid b \in B\}$, the product of sets with partial functions, with (partial) projections π_1 and π_2—treating $*$ as 'undefined'—with configurations

$x \in \mathcal{A} \times \mathcal{B}$ iff

x is a finite subset of $A \times_* B$ such that

(a) $\pi_1 x \in \mathcal{A} \ \&\ \pi_2 x \in \mathcal{B}$,

(b) $\forall e, e' \in x.\ \pi_1(e) = \pi_1(e')$ or $\pi_2(e) = \pi_2(e') \Rightarrow e = e'$, &

(c) $\forall e, e' \in x.\ e \neq e' \Rightarrow \exists y \subseteq x.\ \pi_1 y \in \mathcal{A} \ \&\ \pi_2 y \in \mathcal{B} \ \&\ (e \in y \iff e' \notin y)$.

Note how (a) and (b) express that the projections are maps while (c) says the structure $\mathcal{A} \times \mathcal{B}$ is coincidence-free.

In checking that $\mathcal{A} \times \mathcal{B}$, π_1, π_2 is a product in the category of stable families we shall use the following lemma showing that the direct image under a partial function preserves intersections when the function is locally injective.

Lemma 2.11. *Let $\theta : E_0 \rightharpoonup E_1$ be a partial function between sets E_0 and E_1. Let $X \subseteq \mathcal{P}(E_0)$. Then if*

$$\forall e, e' \in \bigcup X \;.\; \theta(e) = \theta(e') \implies e = e'$$

then $\theta \bigcap X = \bigcap \theta X$.

Proof. Suppose $\theta(e) = \theta(e')$ (both defined) implies $e = e'$ for every $e, e' \in \bigcup x$. Clearly θ is monotonic w.r.t. $\subseteq$ so $\theta \bigcap X \subseteq \bigcap \theta X$. Take $e \in \bigcap \theta X$ and $x \in X$. For some $e' \in x$ we have $\theta(e') = e$. Take $y \in X$. Then for some $e_y \in y$ we have $\theta(e_y) = e$. However $e_y, e \in \bigcup X$ and $\theta(e_y) = \theta(e')$. Thus by hypothesis $e_y = e'$. Therefore $e' \in \bigcap X$ so $e \in \theta \bigcap X$. This establishes the converse inclusion; so $\theta \bigcap X = \bigcap \theta X$, as required. □

Theorem 2.12. *For stable families $\mathcal{A}$ and $\mathcal{B}$ the construction $\mathcal{A} \times \mathcal{B}$ with projections π_1 and π_2 described above is the product in the category of stable families.*

Proof. Suppose $x \subseteq \mathcal{A} \times \mathcal{B}$ and $e, e' \in x$. We shall say "y is a separating set for e, e' in x" when $y \subseteq x$ and $\pi_1(y) \in \mathcal{A}$ and $\pi_2(y) \in \mathcal{B}$ and $e \in y \iff e' \notin y$.

We first check $\mathcal{F} =_{\text{def}} \mathcal{A} \times \mathcal{B}$ is a stable family.

Complete. Suppose $X \subseteq \mathcal{F}$ and $X \uparrow$. We require $\bigcup X$ satisfies (a)-(c) in the definition of product.

(a) Clearly $\pi_i \bigcup X = \bigcup \pi_i X$. As X is compatible in F so are $\pi_1 X$ in $\mathcal{A}$ and $\pi_2 X \in \mathcal{B}$. Thus $\pi_1(\bigcup X) \in \mathcal{A}$ and $\pi_2(\bigcup X) \in \mathcal{B}$.

(b) By the compatibility of X, if $e, e' \in \bigcup X$ and $\pi_i(e) = \pi_i(e')$, both being defined, for $i = 1$ or 2, then $e = e'$.

(c) Suppose $e, e' \in \bigcup X$ and $e \neq e'$. Then $\exists x, y \in X \;.\; e \in x \;\&\; e' \in y$. If either $e \notin y$ or $e' \notin x$ we have respectively either y or x is a separating set for e, e' in $\bigcup X$. Otherwise $e, e' \in x$ or $e, e' \in y$. Then as both x and y satisfy (c) we obtain the required separating set.

Stable. Suppose $\varnothing \neq X \subseteq \mathcal{F}$ and $X \uparrow$. We require X satisfies (a)-(c).

(a) By lemma 2.11, $\pi_i \bigcap X = \bigcap \pi_i X$. But $\bigcap \pi_1 X \in \mathcal{A}$, as $\pi_1 X$ is a compatible set in $\mathcal{A}$, and similarly $\bigcap \pi_2 X \in \mathcal{B}$, so we have $\pi_1(\bigcap X) \in \mathcal{A}$ and $\pi_2(\bigcap X) \in \mathcal{B}$.

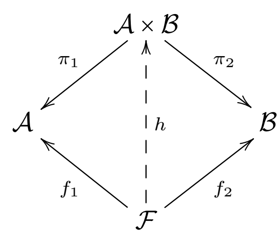
(b) As any $x \in X$ satisfies (b) and $\bigcap X \subseteq x$ certainly $\bigcap X$ satisfies (b).

(c) Suppose $e, e' \in \bigcap X$ and $e \neq e'$. Choose $x \in X$. Because $x \in \mathcal{F}$ there is a separating set y for e, e' in x. Take $v = y \cap \bigcap X$. Clearly $y, \bigcap X \subseteq x$ so because $\mathcal{A}$ and $\mathcal{B}$ are stable, by lemma 2.11*** $\pi_1 v = \pi_1 y \cap \pi_1 \bigcap X) \in \mathcal{A}$ and $\pi_2 v = \pi_2 y \cap \pi_2 \bigcap X \in \mathcal{B}$. This makes v a separating set for e, e' in $\bigcap X$.

Coincidence-free. Suppose $e, e' \in x \in F$ and $e \neq e'$. As x satisfies (c) there is a separating set y for e, e' in x. We further require $y \in F$. Clearly y satisfies (a), (b). To Show y satisfies (c), assume $\epsilon, \epsilon' \in y$ and $\epsilon \neq \epsilon'$. Take a separating set v

for ϵ, ϵ' in x. Take $u = v \cap y$. Then, just as in the proof of stability, part (c), we get u is a separating set for ϵ, ϵ' in x.

Thus we have shown $\mathcal{A} \times \mathcal{B}$ is a stable family. It remains to show that with projections π_1, π_2 it forms the product in the category of stable families. First note π_1 and π_2 are maps by (a), (b) in the construction of the product . Suppose there are maps $f_1 : \mathcal{F} \to \mathcal{A}$ and $f_2 : \mathcal{F} \to \mathcal{B}$ are maps of stable families. We require a unique map h such that the following diagram commutes:



Take h so that

$$h(e) = \begin{cases} (f_1(e), f_2(e)) & \text{if } f_1(e) \text{ is defined or } f_2(e) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

In a pair $(f_1(e), f_2(e))$ we shall identify undefined with $*$.
Obviously $\pi_i \circ h = f_i$ in the category of sets with partial functiosn, for $i = 1, 2$ so provided h is a map of stable families it is unique so the diagram commutes. To show h is a map we need:

$$\forall x \in \mathcal{F} \,.\, hx \in \mathcal{F} \tag{I}$$

$$\forall x \in \mathcal{F} \forall e, e' \in x \,.\, h(e) = h(e') \implies e = e' \tag{II}$$

We prove (II) first:

Suppose $e, e' \in x \in \mathcal{F}$. Then if $h(e) = h(e')$ then $f_i(e) = f_i(e')$, both being defined, for either $i = 1$ or $i = 2$. As each f_i is a map $e = e'$, as required to prove (II).

Now we prove (I). Let $x \in \mathcal{F}$. We need hx satisfies (a)-(c) in the construction of the product. Both (a) and (b) follow from the commutations $\pi_i \circ h = f_i$ using the map properties of f_1 and fa_2. To prove (c), suppose $e, e' \in hx$ and $e \neq e'$. Then $e = h(\epsilon)$ and $e' = h(\epsilon')$ for some $\epsilon, \epsilon' \in x$. We must have $\epsilon \neq \epsilon'$. Thus as $\mathcal{F}$ is coincidence-free we have some $y \in \mathcal{F}$ such that $y \subseteq x$ and $\epsilon \in y \iff \epsilon' \notin y$. As we know h satisfies (II) above it follows that one and only one of e, e' is in hy. The commutations $\pi_i \circ h = f_i$ give $\pi_1 hy \in \mathcal{A}$ and $\pi_2 hy \in \mathcal{B}$. Thus hy separates e, e' in x.

Thus finally we have shown $\mathcal{A} \times \mathcal{B}$ with projections π_1, π_2 is a product in the category of stable families. □

Proposition 2.13. *Let $x \in \mathcal{A} \times \mathcal{B}$, a product of stable families with projections π_1 and π_2. Then, for all $y \subseteq x$,*

$$y \in \mathcal{A} \times \mathcal{B} \iff \pi_1 y \in \mathcal{A} \;\&\; \pi_2 y \in \mathcal{B}\,.$$

Proof. Straightforwardly from the definition of $\mathcal{A} \times \mathcal{B}$. □

Right adjoints preserve products. Hence if $\mathcal{A}\times\mathcal{B}$, π_1, π_2 is a product of stable families then $\mathrm{Pr}(\mathcal{A}) \times \mathrm{Pr}(\mathcal{B})$, $\mathrm{Pr}(\pi_1)$, $\mathrm{Pr}(\pi_2)$ is a product of event structures. Consequently we obtain a product of event structures A and B by first regarding them as stable families $\mathcal{C}(A)$ and $\mathcal{C}(B)$, forming their product

$$\mathcal{C}(A) \times \mathcal{C}(B), \pi_1, \pi_2$$

and then constructing the event structure

$$A \times B =_{\mathrm{def}} \mathrm{Pr}(\mathcal{C}(A) \times \mathcal{C}(B))$$

with projections the composite maps

$$\Pi_1 : A \times B \xrightarrow{\mathrm{Pr}(\pi_1)} \mathrm{Pr}(\mathcal{C}(A)) \cong A \quad \text{and} \quad \Pi_2 : A \times B \xrightarrow{\mathrm{Pr}(\pi_2)} \mathrm{Pr}(\mathcal{C}(B)) \cong B$$

—the isomorphisms are inverses to those of the unit of the adjunction. The projections can be simplified:

Proposition 2.14. *Let A and B be event structures.*

$$A \times B =_{\mathrm{def}} \mathrm{Pr}(\mathcal{C}(A) \times \mathcal{C}(B))$$

and its projections as $\Pi_1 =_{\mathrm{def}} \pi_1 top : A \times B \to A$ and $\Pi_2 =_{\mathrm{def}} \pi_2 top : A \times B \to B$.

Proof. For example,

$$\Pi_1 : A \times B \xrightarrow{\mathrm{Pr}(\pi_1)} \mathrm{Pr}(\mathcal{C}(A)) \cong A$$

takes an event $[e]_x \in A \times B$ via $\mathrm{Pr}(\pi_1)$ to $[\pi_1(e)]_{\pi_1 x}$ if $\pi_1(e)$ is defined, by Corollary 2.10, whence to $\pi_1(e)$ under the isomorphism, *i.e.* to $\pi_1 \circ top([e]_x)$. □

2.3.2 Restriction

The *restriction* of $\mathcal{F}$ to a subset of events R is the stable family $\mathcal{F} \upharpoonright R =_{\mathrm{def}} \{x \in \mathcal{F} \mid x \subseteq R\}$. Defining $E \upharpoonright R$, the restriction of an event structure E to a subset of events R, to have events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency induced by E, we obtain $\mathcal{C}(E \upharpoonright R) = \mathcal{C}(E) \upharpoonright R$.

Proposition 2.15. *Let $\mathcal{F}$ be a stable family and R a subset of its events. Then,* $\mathrm{Pr}(\mathcal{F} \upharpoonright R) = \mathrm{Pr}(\mathcal{F}) \upharpoonright top^{-1} R$.

We remark that we can regard restriction as arising as an equaliser. *E.g.* for an event structure E and a subset R of events, the inclusion map $E \upharpoonright R \hookrightarrow E$ is the equaliser of the two maps id_E, the identity map on E, and $r : E \to E$, which acts as identity on events with down-closure in R and is undefined elsewhere.

2.3.3 Synchronized compositions

Synchronized parallel compositions are obtained as restrictions of products to those events which are allowed to synchronize or occur asynchronously. For example, the synchronized composition of Milner's CCS on stable families $\mathcal{A}$ and $\mathcal{B}$ (with labelled events) is defined as $\mathcal{A} \times \mathcal{B} \upharpoonright R$ where R comprises events which are pairs $(a, *)$, $(*, b)$ and (a, b), where in the latter case the events a of $\mathcal{A}$ and b of $\mathcal{B}$ carry complementary labels. Similarly, synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. By Proposition 2.15, we can equivalently form a synchronized composition of event structures by forming the synchronized composition of their stable families of configurations, and then obtaining the resulting event structure—this has the advantage of eliminating superfluous events earlier.

Products of stable families within the subcategory of total maps can be obtained by restricting the product (w.r.t. partial maps). Construct

$$\mathcal{A} \times_t \mathcal{B} = \mathcal{A} \times \mathcal{B} \upharpoonright A \times B$$

where we restrict to the cartesian product of the sets of events of $\mathcal{A}$ and $\mathcal{B}$, called A and B respectively; projection maps are obtained from the projection functions from the cartesian product. Explicitly, assume $\mathcal{A}$ and $\mathcal{B}$ have underlying sets A and B and that their product $A \times B$ has projections π_1 and π_2 to the left and right components. Then, define the family of configurations of the *product* of stable families with total maps to consist of

$z \in \mathcal{A} \times_t \mathcal{B}$ iff
z is a finite subset of $A \times B$ such that $\pi_1 z \in \mathcal{A}$ & $\pi_2 z \in \mathcal{B}$,
$\forall e, e' \in z.\ \pi_1(e) = \pi_1(e')$ or $\pi_2(e) = \pi_2(e') \Rightarrow e = e'$, &
$\forall e, e' \in z.\ e \neq e' \Rightarrow \exists z' \subseteq z.\ \pi_1 z' \in \mathcal{A}\ \&\ \pi_2 z' \in \mathcal{B}\ \&\ (e \in z' \iff e' \notin z')$.

Proposition 2.16. *$\mathcal{A} \times_t \mathcal{B}$ with projections π_1 and π_2 is a product in the category of stable families with total maps.*

Products of stable families within the subcategory of total maps have a particularly simple characterisation:

Proposition 2.17. *Finite configurations of a product $\mathcal{A} \times_t \mathcal{B}$ of stable families with total maps are* secured bijections *$\theta : x \cong y$ between configurations $x \in \mathcal{A}$ and $y \in \mathcal{B}$, such that the transitive relation generated on θ by taking $(a, b) \le (a', b')$ if $a \le_x a'$ or $b \le_y b'$ is a partial order.*

Proof. Assume θ is a secured bijection between $x \in \mathcal{A}$ and $y \in \mathcal{B}$ with generated partial order $\le$. As such θ is a subset of ordered pairs $A \times B$. It satisfies $\pi_1 \theta = x \in \mathcal{A}$ and $\pi_2 \theta = y \in \mathcal{B}$. Suppose $e, e' \in \theta$ are distinct pairs. As $\le$ is a partial order, the down-closure $[e]$ does not contain e', or *vice versa*. W.l.o.g. assume $[e]$ does not contain e'. Note, from the definition of $\le$, that $\pi_1[e]$ is a $\le_x$-down-closed subset of x so $\pi_1[e] \in \mathcal{A}$; similarly, $\pi_2[e] \in \mathcal{B}$. Taking $z' =_{\text{def}} [e]$ we fulfil the final condition required of z in order for it to be in $\mathcal{A} \times_t \mathcal{B}$.

Conversely, let $z \in \mathcal{A} \times_t \mathcal{B}$. Then $x =_{\text{def}} \pi_1 z \in \mathcal{A}$ and $y =_{\text{def}} \pi_2 z \in \mathcal{B}$. Moreover, because both π_1 and π_2 are total and locally injective maps from $z \subseteq A \times B$ they induce a bijection $z : x \cong y$. For $(a,b), (a',b') \in z$, write

$$(a,b) \leq_1 (a',b') \text{ iff } a \leq_x a' \text{ or } b \leq_y b' .$$

For z to be a secured bijection we require that $\leq = \leq_1^*$ is a partial order.

We first show that if $(a,b) \leq (a',b')$ then

$$\forall z' \subseteq z.\ \pi_1 z' \in \mathcal{A} \ \&\ \pi_2 z' \in \mathcal{B} \ \&\ (a',b') \in z' \implies (a,b) \in z' .$$

It is clearly sufficient to show this for $(a,b) \leq_1 (a',b')$, *i.e.* when $a \leq_x a'$ or $b \leq_y b'$. Suppose $a \leq_x a'$ (the other case is similar) and that $z' \subseteq z$ with $\pi_1 z' \in \mathcal{A}$, $\pi_2 z' \in \mathcal{B}$ and $(a',b') \in z'$. As $a' in \pi_1 z' \subset x$ and $a \leq_x a'$ we must also have $a \in \pi_1 z'$. As z is a bijection, with $z' \subseteq z$ this is only possible if $(a,b) \in z'$, as required.

Suppose $(a,b) \leq (a',b')$ and $(a',b') \leq (a,b)$. Then,

$$\forall z' \subseteq z.\ \pi_1 z' \in \mathcal{A} \ \&\ \pi_2 z' \in \mathcal{B} \ \&\ ((a,b) \in z' \iff (a',b') \in z') .$$

For $z \in \mathcal{A} \times_t \mathcal{B}$ this entails $(a,b) = (a',b')$. We have shown $\leq$ to be anti-symmetric so a partial order. A configuration z of the product is a secured bijection, as required. □

2.3.4 Pullbacks

The construction of pullbacks can be viewed as a special case of synchronized composition. Once we have products of event structures pullbacks are obtained by restricting products to the appropriate equalizing set. Pullbacks of event structures can also be constructed via pullbacks of stable families, in a similar manner to the way we have constructed products of event structures. We obtain pullbacks of stable families as restrictions of products. Suppose $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ are maps of stable families. Let A, B and C be the sets of events of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. The set $P =_{\text{def}} \{(a,b) \mid f(a) = g(b)\}$ with projections π_1, π_2 to the left and right, forms the pullback, in the category of sets, of the functions $f : A \to C$, $g : B \to C$. We obtain the pullback in stable families of f, g as the stable family $\mathcal{P}$, consisting of those subsets of P which are also configurations of the product $\mathcal{A} \times \mathcal{B}$—its associated maps are the projections π_1, π_2 from the events of $\mathcal{P}$. When f and g are total maps we obtain the pullback in the subcategory of stable families with total maps.

As a corollary of Proposition 2.17 we obtain a simple characterization of pullbacks of total maps within stable families:

Lemma 2.18. *Let $\mathcal{P}, \pi_1, \pi_2$ form a pullback of total maps $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ in the category of stable families. Configurations of $\mathcal{P}$ are precisely those composite bijections $\theta : x \cong fx = gy \cong y$ between configurations $x \in \mathcal{A}$ and $y \in \mathcal{B}$ s.t. $fx = gy$ for which the transitive relation generated on θ by taking $(a,b) \leq (a',b')$ if $a \leq_x a'$ or $b \leq_y b'$ is a partial order.*

For future reference we give the stand-alone construction of pullbacks of total maps in stable families. Let $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ be total maps of stable families. Assume $\mathcal{A}$ and $\mathcal{B}$ have underlying sets A and B. Define $D =_{\rm def} \{(a, b) \in A \times B \mid f(a) = g(b)\}$ with projections π_1 and π_2 to the left and right components. Define a family of configurations of the *pullback* to consist of

$z \in \mathcal{D}$ iff

z is a finite subset of D such that $\pi_1 z \in \mathcal{A}$ & $\pi_2 z \in \mathcal{B}$,

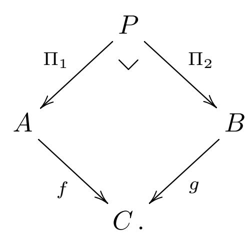
$\forall e, e' \in z.\ e \neq e' \Rightarrow \exists z' \subseteq z.\ \pi_1 z' \in \mathcal{A}\ \&\ \pi_2 z' \in \mathcal{B}\ \&\ (e \in z' \iff e' \notin z')$.

The extra local injectivity property we needed in the definition of product is not necessary here; it follows from the definition of D and that f and g are locally injective.

Just as for products, we obtain the pullback of event structures by first forming the pullback in stable families of their families of configurations and then applying Pr.

As a corollary of Lemma 2.18 we obtain a useful way to understand configurations of the pullback of total maps on event structures.

Proposition 2.19. *When $f : A \to C$ and $g : B \to C$ are total, maps of event structures, in their pullback P, Π_1, Π_2*



the finite configurations of P correspond to composite bijections

$$\theta : x \cong fx = gy \cong y$$

between finite configurations x of A and y of B such that $fx = gy$, for which the transitive relation generated on θ by $(a, b) \leq (a', b')$ if $a \leq_A a'$ or $b \leq_B b'$ forms a partial order.

Chapter 3

Games and strategies

Very general nondeterministic concurrent games and strategies are presented. The intention is to formalize distributed games in which both Player (or a team of players) and Opponent (or a team of opponents) can interact in highly distributed fashion, without, for instance, enforcing that their moves alternate. Strategies, those nondeterministic plays which compose well with copy-cat strategies, are characterized.

3.1 Event structures with polarities

We shall represent both a game and a strategy in a game as an event structure with polarity, comprising an event structure together with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ or $-$ to its events E. The events correspond to (occurrences of) moves. The two polarities $+/-$ express the dichotomy: Player/Opponent; Process/Environment; Prover/Disprover; or Ally/Enemy. Maps of event structures with polarity are maps of event structures which preserve polarity.

3.2 Operations

3.2.1 Dual

The *dual*, $E^\perp$, of an event structure with polarity E comprises a copy of the event structure E but with a reversal of polarities. It obviously extends to a functor.

3.2.2 Simple parallel composition

This operation simply juxtaposes two event structures with polarity. Let $(A, \leq_A, \text{Con}_A, pol_A)$ and $(B, \leq_B, \text{Con}_B, pol_B)$ be event structures with polarity. The events of $A \| B$ are $(\{1\} \times A) \cup (\{2\} \times B)$, their polarities unchanged, with: the only relations of causal dependency given by $(1,a) \le (1,a')$ iff $a \le_A a'$ and $(2,b) \le (2,b')$ iff $b \le_B b'$; a subset of events C is consistent in $A\|B$ iff $\{a \mid (1,a) \in C\} \in \mathrm{Con}_A$ and $\{b \mid (2,b) \in C\} \in \mathrm{Con}_B$. The operation extends to a functor—put the two maps in parallel. The empty event structure with polarity $\varnothing$ is the unit w.r.t. $\|$.

3.3 Pre-strategies

Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A *pre-strategy* in A is a total map $\sigma : S \to A$ from an event structure with polarity S. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept can be refined to that of *strategy* (and *winning strategy*.

A map from a pre-strategy $\sigma : S \to A$ to a pre-strategy $\sigma' : S' \to A$ is a map $f : S \to S'$ such that

$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ & \searrow^{\sigma} & \downarrow{\scriptstyle \sigma'} \\ & & A \end{array}$$

commutes. Accordingly, we regard two pre-strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ as essentially the same when they are isomorphic, and write $\sigma \cong \sigma'$, *i.e.* when there is an isomorphism of event structures $\theta : S \cong S'$ such that

$$\begin{array}{ccc} S & \overset{\theta}{\cong} & S' \\ & \searrow^{\sigma} & \downarrow{\scriptstyle \sigma'} \\ & & A \end{array}$$

commutes.

Let A and B be event structures with polarity. Following Joyal [?], a pre-strategy from A to B is a pre-strategy in $A^\perp \| B$, so a total map $\sigma : S \to A^\perp \| B$. It thus determines a span

$$\begin{array}{ccccc} & & S & & \\ & \swarrow^{\sigma_1} & & \searrow^{\sigma_2} & \\ A^\perp & & & & B\,, \end{array}$$

of event structures with polarity where σ_1, σ_2 are *partial* maps. In fact, a pre-strategy from A to B corresponds to such spans where for all $s \in S$ either, but not both, $\sigma_1(s)$ or $\sigma_2(s)$ is defined. Two pre-strategies σ and τ from A to B are isomorphic, $\sigma \cong \tau$, when their spans are isomorphic.

We write $\sigma : A \dashrightarrow B$ to express that σ is a pre-strategy from A to B. Note a pre-strategy in a game A coincides with a pre-strategy from the empty game $\sigma : \emptyset \dashrightarrow A$.

3.3.1 Concurrent copy-cat

Identities on games are given by copy-cat strategies—strategies for Player based on copying the latest moves made by Opponent.

Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}_A \to A^\perp \| A$. It describes a concurrent, or distributed, strategy based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity.

For $c \in A^\perp \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component, *i.e.*

$$\overline{(1,a)} = (2,a) \text{ and } \overline{(2,a)} = (1,a)\,.$$

Proposition 3.1. *Let A be an event structure with polarity. There is an event structure with polarity $\mathbb{C}_A$ having the same events and polarity as $A^\perp \| A$ but with causal dependency $\le_{\mathbb{C}_A}$ given as the transitive closure of the relation*

$$\le_{A^\perp \| A} \cup \{(\overline{c}, c) \mid c \in A^\perp \| A \;\&\; pol_{A^\perp \| A}(c) = +\}\,.$$

and finite subsets of $\mathbb{C}_A$ consistent if their down-closure w.r.t. $\le_{\mathbb{C}_A}$ are consistent in $A^\perp \| A$. Moreover,
(i) $c \rightarrowtriangle c'$ in $\mathbb{C}_A$ iff

$$c \rightarrowtriangle c' \text{ in } A^\perp \| A \text{ or } pol_{A^\perp \| A}(c') = + \;\&\; \overline{c} = c'\,;$$

(ii) $x \in \mathcal{C}(\mathbb{C}_A)$ iff

$$x \in \mathcal{C}(A^\perp \| A) \;\&\; \forall c \in x.\ pol_{A^\perp \| A}(c) = + \implies \overline{c} \in x\,.$$

Proof. It can first be checked that defining

$$\begin{aligned} c \le_{\mathbb{C}_A} c' \text{ iff } & (i)\ c \le_{A^\perp \| A} c' \text{ or} \\ & (ii)\ \exists c_0 \in A^\perp \| A.\ pol_{A^\perp \| A}(c_0) = + \;\& \\ & \qquad c \le_{A^\perp \| A} \overline{c_0} \;\&\; c_0 \le_{A^\perp \| A} c'\,, \end{aligned}$$

yields a partial order. Note that

$$c \le_{A^\perp \| A} d \text{ iff } \overline{c} \le_{A^\perp \| A} \overline{d}\,,$$

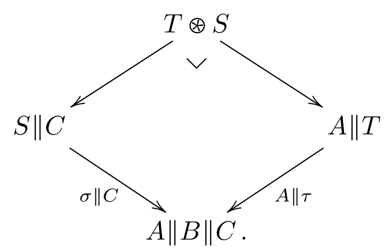
used in verifying transitivity and antisymmetry. The relation $\le_{\mathbb{C}_A}$ is clearly the transitive closure of $\le_{A^\perp \| A}$ together with all extra causal dependencies $(\overline{c}, c)$ where $pol_{A^\perp \| A}(c) = +$. The remaining properties required for $\mathbb{C}_A$ to be an event structure follow routinely.

(i) From the above characterization of $\leq_{\mathbb{CC}_A}$.
(ii) From $\mathbb{CC}_A$ and $A^\perp \| A$ sharing the same consistency relation and the extra causal dependency adjoined to $\mathbb{CC}_A$. □

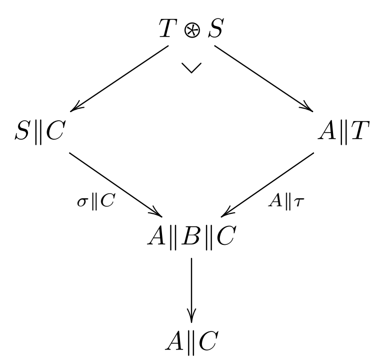
Based on Proposition 3.1, define the *copy-cat* pre-strategy from A to A to be the pre-strategy $\gamma_A : \mathbb{CC}_A \to A^\perp \| A$ where $\mathbb{CC}_A$ comprises the event structure with polarity $A^\perp \| A$ together with extra causal dependencies $\overline{c} \leq_{\mathbb{CC}_A} c$ for all events c with $pol_{A^\perp \| A}(c) = +$, and γ_A is the identity on the set of events common to both $\mathbb{CC}_A$ and $A^\perp \| A$.

3.3.2 Composition

We present the composition of pre-strategies via pullbacks. Given two pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$.



There is an obvious partial map of event structures $A \| B \| C \to A \| C$ undefined on B and acting as identity on A and C. The partial map from $T \circledast S$ to $A \| C$ given by following the diagram (either way round the pullback square)



factors through the projection of $T \circledast S$ to V, those events at which the partial map is defined:

$$T \circledast S \to P \downarrow V \to A \| C\,.$$

The resulting total map $T \circledast S \downarrow V \to A \| C$ gives us the composition $\tau \odot \sigma : T \odot S \to A^\perp \| C$ once we reinstate polarities.

Deconstructing composition

It is helpful to deconstruct the definition of composition of pre-strategies. It is based on pullbacks of event structures and these in turn on pullbacks of stable families.

Given pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ form the pullback in stable families:

$$\begin{array}{ccccc} & & \mathcal{C}(T) \circledast \mathcal{C}(S) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathcal{C}(S\|C) & & & & \mathcal{C}(A\|T) \\ & \searrow_{\sigma\|C} & & \swarrow_{A\|\tau} & \\ & & \mathcal{C}(A\|B\|C)\,. & & \end{array}$$

The pullback $\mathcal{C}(T) \circledast \mathcal{C}(S)$ is composition without hiding given as a stable family. Composition without hiding as an event structure is given as the pullback $\mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S))$; its projections take a prime to the the appropriate component of its top element. Composition (with hiding) is given as the projection

$$\mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S)) \downarrow V$$

to the set of 'visible' events V, those with image in the game A or the game C.

3.3.3 Duality

A pre-strategy $\sigma : A \longrightarrow\!\!\!\!\!+\, B$ corresponds to a dual pre-strategy $\sigma^\perp : B^\perp \longrightarrow\!\!\!\!\!+\, A^\perp$. This duality arises from the correspondence

$$\begin{array}{ccc} \begin{array}{ccc} & S & \\ {}^{\sigma_1}\swarrow & & \searrow^{\sigma_2} \\ A^\perp & & B \end{array} & \longleftrightarrow & \begin{array}{ccc} & S & \\ {}^{\sigma_2}\swarrow & & \searrow^{\sigma_1} \\ (B^\perp)^\perp & & A^\perp\,. \end{array} \end{array}$$

It is easy to check that the dual of copy-cat, $\gamma_A^\perp$, is isomorphic, as a span, to the copy-cat of the dual, $\gamma_{A^\perp}$, for A an event structure with polarity. It is also straightforward, though more involved, to show that the dual of a composition of pre-strategies $(\tau \odot \sigma)^\perp$ is isomorphic as a span to the composition $\sigma^\perp \odot \tau^\perp$.

3.4 Strategies

Two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient in order for copy-cat to behave as identity w.r.t. the composition of pre-strategies. It becomes compelling to define a *(nondeterministic) concurrent strategy*, in general, as a pre-strategy which is receptive and innocent.

Receptivity. Say a pre-strategy $\sigma : S \to A$ is *receptive* when $\sigma x \xrightarrow{a}\subset$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \xrightarrow{s}\subset$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$. Receptivity ensures that no Opponent move which is possible is disallowed.
Innocence. Say a pre-strategy σ is *innocent* when it is both +-innocent and −-innocent:
+-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.
−-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

The definition of a pre-strategy $\sigma : S \to A$ ensures that the moves of Player and Opponent respect the causal constraints of the game A. Innocence restricts Player further. Locally, within a configuration, Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$. Thus innocence gives Player the freedom to await Opponent moves before making their move, but prevents Player having any influence on the moves of Opponent beyond those stipulated in the game A; more surprisingly, innocence also disallows any immediate causality of the form $\oplus \rightarrowtriangle \oplus$, purely between Player moves, not already stipulated in the game A.

Theorem 3.2. *Let $\sigma : S \to A$ be a pre-strategy in a game A. Then, $\gamma_A \odot \sigma \cong \sigma$ iff σ is receptive and innocent.*

We omit the proof which is quite technical.

3.5 Concurrent strategies

Define a *strategy* to be a pre-strategy which is receptive and innocent. We obtain a bicategory, **Strat**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \leftrightarrow B$, with identities the copy-cat strategies, and the 2-cells are maps of pre-strategies. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback and partial-total factorisation). The bicategory satisfies the conditions expected of a *compact-closed* bicategory. (Though, with the addition of extra structure such as winning conditions or payoff, compact-closure weakens to $\ast$-autonomy; the isomorphism $(A\|B)^\perp \cong A^\perp\|B^\perp$ of compact-closure disappears.)

3.5.1 Alternative characterizations

Via saturation conditions

An alternative description of concurrent strategies exhibits the correspondence between innocence and earlier "saturation conditions," *reflecting* specific independence, in the work of Laird, and Ghica and Murawski.

Proposition 3.3. *A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that* *(i)* $\sigma x \stackrel{a}{-\!\!\!\subset}\ \&\ pol_A(a) = - \Rightarrow \exists ! s \in S.\ x \stackrel{s}{-\!\!\!\subset}\ \&\ \sigma(s) = a$, *for all* $x \in \mathcal{C}(S)$, $a \in A$;

(ii)(+) If $x \stackrel{e}{-\!\!\!\subset} x_1 \stackrel{e'}{-\!\!\!\subset}\ \&\ pol_S(e) = +$ *in* $\mathcal{C}(S)$ *and* $\sigma x \stackrel{\sigma(e')}{-\!\!\!\subset}$ *in* $\mathcal{C}(A)$, *then* $x \stackrel{e'}{-\!\!\!\subset}$ *in* $\mathcal{C}(S)$; *and*

(ii)(−) If $x \stackrel{e}{-\!\!\!\subset} x_1 \stackrel{e'}{-\!\!\!\subset}\ \&\ pol_S(e') = -$ *in* $\mathcal{C}(S)$ *and* $\sigma x \stackrel{\sigma(e')}{-\!\!\!\subset}$ *in* $\mathcal{C}(A)$, *then* $x \stackrel{e'}{-\!\!\!\subset}$ *in* $\mathcal{C}(S)$.

Proof. Note that if $x \stackrel{e}{-\!\!\!\subset} x_1 \stackrel{e'}{-\!\!\!\subset}$ then either $e\ co\ e'$ or $e \rightarrowtriangle e'$. Condition (ii) is a contrapositive reformulation of innocence. □

Via lifting conditions

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^+ x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{+\}$, *i.e.* the configuration x' extends the configuration x solely by events of +ve polarity. With this notation in place we can give an attractive characterization of concurrent strategies:

Lemma 3.4. *A strategy in a game* A *comprises a total map of event structures with polarity* $\sigma : S \to A$ *such that*
(i) whenever $y \subseteq^+ \sigma x$ *in* $\mathcal{C}(A)$ *there is a (necessarily unique)* $x' \in \mathcal{C}(S)$ *so that* $x' \subseteq x\ \&\ \sigma x' = y$, *i.e.*

$$\begin{array}{ccc} x' & \subseteq & x \\ \sigma \downarrow & & \downarrow \sigma \\ y & \subseteq^+ & \sigma x\,, \end{array}$$

and
(ii) whenever $\sigma x \subseteq^- y$ *in* $\mathcal{C}(A)$ *there is a unique* $x' \in \mathcal{C}(S)$ *so that* $x \subseteq x'\ \&\ \sigma x' = y$, *i.e.*

$$\begin{array}{ccc} x & \subseteq & x' \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma x & \subseteq^- & y\,. \end{array}$$

Proof. Let $\sigma : S \to A$ be a total map of event structures with polarity. We show σ is a strategy iff (i) and (ii).
"Only if": (i) It suffices to show the seemingly weaker property (i)′ that

$$y \stackrel{a}{-\!\!\!\subset} \sigma x\ \&\ pol(a) = + \implies \exists x' \in \mathcal{C}(S).\ x' -\!\!\!\subset x\ \&\ \sigma x' = y$$

for $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$. Then (i), with $y \subseteq^+ \sigma x$, follows by considering a covering chain $y -\!\!\!\subset \cdots -\!\!\!\subset \sigma x$. (The uniqueness of x is a direct consequence of σ being a total map of event structures.) To show (i)′, suppose $y \stackrel{a}{-\!\!\!\subset} \sigma x$ with a +ve. Then $\sigma(s) = a$ for some unique $s \in x$ with s +ve. Supposing s were not $\leq$-maximal in x, then $s \rightarrowtriangle s'$ for some $s' \in x$. By +-innocence $a = \sigma(s) \rightarrowtriangle \sigma(s') \in \sigma x$ implying a is not $\leq$-maximal in σx. This contradicts $y \stackrel{a}{-\!\!\!\subset} \sigma x$. Hence s is $\leq$-maximal and $x' =_{\text{def}} x \setminus \{s\} \in \mathcal{C}(S)$ with $x' -\!\!\!\subset x$ and $\sigma x' = y$.

(ii) Assuming $\sigma x \subseteq^- y$ we can form a covering chain

$$\sigma x \xrightarrow{a_1}\!\!\subset y_1 \cdots \xrightarrow{a_n}\!\!\subset y_n = y\,.$$

By repeated use of receptivity we obtain the existence of x' where $x \subseteq x'$ and $\sigma x' = y$. To show the uniqueness of x' suppose $x \subseteq z, z'$ and $\sigma z = \sigma z' = y$. Suppose that $z \neq z'$. Then, without loss of generality, suppose there is a $\leq_S$-minimal $s' \in z'$ with $s' \notin z$. Then $[s') \subseteq z$, with s of $-$ve polarity. Now $\sigma(s') \in y$ so there is $s \in z$ for which $\sigma(s) = \sigma(s')$. We have $[s), [s') \subseteq z$ so $[s) \uparrow [s')$. We show $[s) = [s')$. Suppose $s_1 \rightarrowtriangle s$. Then by $-$-innocence, $\sigma(s_1) \rightarrowtriangle \sigma(s)$. As $\sigma(s') = \sigma(s)$ and σ is a map of event structures there is $s_2 < s'$ such that $\sigma(s_2) = \sigma(s_1)$. But s_1, s_2 both belong to the configuration $[s) \cup [s')$ so $s_1 = s_2$, as σ is a map, and $s_1 < s'$. Symmetrically, if $s_1 \rightarrowtriangle s'$ then $s_1 < s$. It follows that $[s) = [s')$. Now both $[s) \xrightarrow{s}\!\!\subset$ and $[s) \xrightarrow{s'}\!\!\subset$ with $\sigma(s) = \sigma(s')$ where both s, s' have $-$ve polarity. As σ is receptive, $s = s'$. This implies $s' \in z$, a contradiction. Hence, $z = z'$ and we have established uniqueness of x'.

"If": Assume σ satisfies (i) and (ii). Clearly σ is receptive by (ii). We establish innocence via an observation, that in any event structure E,

$$(\exists x, x_1 \in \mathcal{C}(E).\ x \xrightarrow{s}\!\!\subset x_1 \xrightarrow{s'}\!\!\subset\,) \iff s \rightarrowtriangle s' \text{ or } s\ co\ s'\,.$$

Suppose $s \rightarrowtriangle_S s'$ and $pol(s) = +$. Then $x \xrightarrow{s}\!\!\subset x_1 \xrightarrow{s'}\!\!\subset x'$ for some $x, x_1, x' \in \mathcal{C}(S)$. Hence $\sigma x \xrightarrow{s}\!\!\subset \sigma x_1 \xrightarrow{s'}\!\!\subset \sigma x'$. Either, as required, $\sigma(s) \rightarrowtriangle_S \sigma(s')$ or $\sigma(s)$ *co* $\sigma(s')$. Assume the latter. Then $\sigma x \xrightarrow{\sigma(s')}\!\!\subset y_2 \xrightarrow{\sigma(s)}\!\!\subset \sigma x'$ where $y_2 = x \cup \{\sigma(s')\}$, with $pol(\sigma(s)) = +$. From (i) we obtain a unique $x_2 \in \mathcal{C}(S)$ such that $x_2 \subseteq x'$ and $\sigma x_2 = y_2$. As σ is a total map of event structures, we obtain $x_2 \xrightarrow{s}\!\!\subset x'$ and subsequently $x \xrightarrow{s'}\!\!\subset x_2$, contradicting $s \rightarrowtriangle_S s'$.

Suppose $s \rightarrowtriangle_S s'$ and $pol(s') = -$. The case where $pol(s) = +$ is covered by the previous argument. Suppose $pol(s) = -$. Then $x \xrightarrow{s}\!\!\subset x_1 \xrightarrow{s'}\!\!\subset x'$ for some $x, x_1, x' \in \mathcal{C}(S)$. Again, $\sigma x \xrightarrow{s}\!\!\subset \sigma x_1 \xrightarrow{s'}\!\!\subset \sigma x'$. Assume, to obtain a contradiction, that $\sigma(s)$ *co* $\sigma(s')$. Then $\sigma x \xrightarrow{\sigma(s')}\!\!\subset y_2 \xrightarrow{\sigma(s)}\!\!\subset \sigma x'$, where $y_2 = x \cup \{\sigma(s')\}$. As σ is already known to be receptive, we obtain

$$x \xrightarrow{e'}\!\!\subset x_2 \xrightarrow{e}\!\!\subset x'' \ \&\ \sigma x_2 = y_2 \ \&\ \sigma x'' = \sigma x'\,.$$

From the uniqueness part of (ii) we deduce $x'' = x'$. As σ is a total map of event structures, $e = s$ and $e' = s'$. Thus $x \xrightarrow{s'}\!\!\subset$, which contradicts $s \rightarrowtriangle_S s'$. Via the observation we conclude that $\sigma(s) \rightarrowtriangle_S \sigma(s')$. $\square$

Chapter 4

Deterministic strategies

An important special case is that of *deterministic* concurrent strategies. They coincide with Melliès and Mimram's *receptive ingenuous strategies*.

We say an event structure with polarity S is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \text{Con}_S \implies X \in \text{Con}_S\,,$$

where $Neg[X] =_{\text{def}} \{s' \in S \mid pol(s') = - \ \&\ \exists s \in X.\ s' \leq s\}$. In other words, S is deterministic iff any finite set of moves is consistent when it causally depends only on a consistent set of opponent moves. Say a strategy $\sigma : S \to A$ is *deterministic* if S is deterministic.

Lemma 4.1. *An event structure with polarity S is deterministic iff*

$$\forall s, s' \in S, x \in \mathcal{C}(S).\ \ x \overset{s}{\longrightarrow\!\!\subset}\ \&\ x \overset{s'}{\longrightarrow\!\!\subset}\ \&\ pol(s) = + \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. *"Only if"*: Assume S is deterministic, $x \overset{s}{\longrightarrow\!\!\subset}$, $x \overset{s'}{\longrightarrow\!\!\subset}$ and $pol(s) = +$. Take $X =_{\text{def}} x \cup \{s, s'\}$. Then $Neg[X] \subseteq x \cup \{s\}$ so $Neg[X] \in \text{Con}_S$. As S is deterministic, $X \in \text{Con}_S$ and being down-closed $X = x \cup \{s, s'\} \in \mathcal{C}(S)$.
"If": Assume S satisfies the property stated above in the proposition. Let $X \subseteq_{\text{fin}} S$ with $Neg[X] \in \text{Con}_S$. Then the down-closure $[Neg[X]] \in \mathcal{C}(S)$. Clearly $[Neg[X]] \subseteq [X]$ where all events in $[X] \setminus [Neg[X]]$ are necessarily +ve. Suppose, to obtain a contradiction, that $X \notin \text{Con}_S$. Then there is a maximal $z \in \mathcal{C}(S)$ such that

$$[Neg[X]] \subseteq z \subseteq [X]$$

and some $e \in [X] \setminus z$, necessarily +ve, for which $[e) \subseteq z$. Take a covering chain

$$[e) \overset{s_1}{\longrightarrow\!\!\subset} z_1 \overset{s_2}{\longrightarrow\!\!\subset} \cdots \overset{s_k}{\longrightarrow\!\!\subset} z_k = z\,.$$

As $[e) \overset{e}{\longrightarrow\!\!\subset} [e]$ with e +ve, by repeated use of the property of the lemma—illustrated below—we obtain $z \overset{e}{\longrightarrow\!\!\subset} z'$ in $\mathcal{C}(S)$ with $[Neg[X]] \subseteq z' \subseteq [X]$, which contradicts the maximality of z.

$$
\begin{array}{ccccccccccc}
[e] & \overset{s_1}{-\!\!\subset} & z'_1 & \overset{s_2}{-\!\!\subset} & \cdots & \overset{s_k}{-\!\!\subset} & z'_k & = & z' \\
{\scriptstyle e}\Big\uparrow & & {\scriptstyle e}\Big\uparrow & & \cdots & & {\scriptstyle e}\Big\uparrow & & \\
[e) & \overset{s_1}{-\!\!\subset} & z_1 & \overset{s_2}{-\!\!\subset} & \cdots & \overset{s_k}{-\!\!\subset} & z_k & = & z
\end{array}
$$

□

So, above, an event structure with polarity can fail to be deterministic in two ways, either with $pol(s) = pol(s') = +$ or with $pol(s) = +$ & $pol(s') = -$. In particular, the copy-cat strategy need not be deterministic.

4.1 The bicategory of deterministic strategies

In general for an event structure with polarity A the copy-cat strategy can fail to be deterministic, illustrated in the examples below.

Example 4.2. (i) Take A to consist of two +ve events and one −ve event, with any two but not all three events consistent. The construction of $\mathbb{C}\mathbb{C}_A$ is pictured:

$$
\begin{array}{cccc}
 & \ominus \rightarrowtriangle \oplus & \\
A^\perp & \ominus \rightarrowtriangle \oplus & A \\
 & \oplus \leftarrowtriangle \ominus &
\end{array}
$$

Here γ_A is not deterministic: take x to be the set of all three −ve events in $\mathbb{C}\mathbb{C}_A$ and s, s' to be the two +ve events in the A component.
(ii) Take A to consist of two events, one +ve and one −ve event, inconsistent with each other. The construction $\mathbb{C}\mathbb{C}_A$:

$$
\begin{array}{cccc}
A^\perp & \ominus \rightarrowtriangle \oplus & A \\
 & \oplus \leftarrowtriangle \ominus &
\end{array}
$$

To see $\mathbb{C}\mathbb{C}_A$ is not deterministic, take x to be the singleton set consisting *e.g.* of the −ve event on the left and s, s' to be the +ve and −ve events on the right.

Fortunately copy-cat is deterministic iff the underlying game is free of races between Player and Opponent.

Lemma 4.3. *Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff A satisfies*

$$\forall x \in \mathcal{C}(A).\ x \overset{a}{-\!\!\subset}\ \&\ x \overset{a'}{-\!\!\subset}\ \&\ pol(a) = +\ \&\ pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(race-free)}$$

Proof. *"Only if":* Suppose $x \in \mathcal{C}(A)$ with $x \overset{a}{-\!\!\subset}$ and $x \overset{a'}{-\!\!\subset}$ where $pol(a) = +$ and $pol(a') = -$. Construct $y =_{\text{def}} \{(1, \overline{b}) \mid b \in x\} \cup \{(1, a)\} \cup \{(2, b) \mid b \in x\}$. Then $y \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ with $y \overset{(2,a)}{-\!\!\subset}$ and $y \overset{(2,a')}{-\!\!\subset}$, by Proposition 3.1(ii). Assuming $\mathbb{C}\mathbb{C}_A$ is deterministic, we obtain $y \cup \{(2, a), (2, a')\} \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$, so $y \cup \{(2, a), (2, a')\} \in \mathcal{C}(A^\perp \| A)$. This entails $x \cup \{a, a'\} \in \mathcal{C}(A)$, as required to show **(race-free)**.

"If": Assume A satisfies (**race-free**). It suffices to show for $X \subseteq_{\text{fin}} \mathbb{CC}_A$, with X down-closed, that $Neg[X] \in \text{Con}_{\mathbb{CC}_A}$ implies $X \in \text{Con}_{\mathbb{CC}_A}$. Recall for Z down-closed, $Z \in \text{Con}_{\mathbb{CC}_A}$ iff $Z \in \text{Con}_{A^\perp \| A}$.

Let $X \subseteq_{\text{fin}} \mathbb{CC}_A$ with X down-closed. Assume $Neg[X] \in \text{Con}_{\mathbb{CC}_A}$. Observe

(i) $\{c \mid c \in X \ \&\ pol(c) = -\} \subseteq Neg[X]$ and

(ii) $\{\overline{c} \mid c \in X \ \&\ pol(c) = +\} \subseteq Neg[X]$ as by Proposition 3.1, X being down-closed must contain $\overline{c}$ if it contains c with $pol(c) = +$.

Consider $X_2 =_{\text{def}} \{a \mid (2, a) \in X\}$. Then X_2 is a finite down-closed subset of A. From (i),

$$X_2^- =_{\text{def}} \{a \in X_2 \mid pol(a) = -\} \in \text{Con}_A \,.$$

From (ii),

$$X_2^+ =_{\text{def}} \{a \in X_2 \mid pol(a) = +\} \in \text{Con}_A \,.$$

We show (**race-free**) implies $X_2 \in \text{Con}_A$.

Define $z^- =_{\text{def}} [X_2^-]$ and $z^+ =_{\text{def}} [X_2^+]$. Being down-closures of consistent sets, $z^-, z^+ \in \mathcal{C}(A)$. We show $z^- \uparrow z^+$ in $\mathcal{C}(A)$. First note $z^- \cap z^+ \in \mathcal{C}(A)$. If $a \in z^- \setminus z^- \cap z^+$ then $pol(a) = -$; otherwise, if $pol(a) = +$ then $a \in z^+$ a well as $a \in z^-$ making $a \in z^- \cap z^+$, a contradiction. Similarly, if $a \in z^+ \setminus z^- \cap z^+$ then $pol(a) = +$. We can form covering chains

$$z^- \cap z^+ \stackrel{p_1}{-\!\!\!\subset} x_1 \stackrel{p_2}{-\!\!\!\subset} \cdots \stackrel{p_k}{-\!\!\!\subset} x_k = z^- \quad \text{and} \quad z^- \cap z^+ \stackrel{n_1}{-\!\!\!\subset} y_1 \stackrel{n_2}{-\!\!\!\subset} \cdots \stackrel{n_l}{-\!\!\!\subset} y_l = z^+$$

where each p_i is +ve and each n_j is −ve.

Consequently, by repeated use of (**race-free**), we obtain $x_k \cup y_l \in \mathcal{C}(A)$, *i.e.* $z^+ \cup z^- \in \mathcal{C}(A)$, as is illustrated below. But $X_2 \subseteq z^+ \cup z^-$, so $X_2 \in \text{Con}_A$. A similar argument shows $X_1 =_{\text{def}} \{a \in A^\perp \mid (1, a) \in X\} \in \text{Con}_{A^\perp}$. It follows that $X \in \text{Con}_{A^\perp \| A}$, so $X \in \text{Con}_{\mathbb{CC}_A}$ as required.

$$\begin{array}{ccccccccc}
y_l & \stackrel{p_1}{-\!\!\!\subset} & x_1 \cup y_l & \stackrel{p_2}{-\!\!\!\subset} & x_2 \cup y_l & \stackrel{p_3}{-\!\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\!\subset} & x_k \cup y_l \\
{\scriptstyle n_l}\uparrow & & {\scriptstyle n_l}\uparrow & & {\scriptstyle n_l}\uparrow & & & & {\scriptstyle n_l}\uparrow \\
\vdots & & \vdots & & \vdots & \cdots & \cdots & \cdots & \vdots \\
{\scriptstyle n_2}\uparrow & & {\scriptstyle n_2}\uparrow & & {\scriptstyle n_2}\uparrow & & & & {\scriptstyle n_2}\uparrow \\
y_1 & \stackrel{p_1}{-\!\!\!\subset} & x_1 \cup y_1 & \stackrel{p_2}{-\!\!\!\subset} & x_2 \cup y_1 & \stackrel{p_3}{-\!\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\!\subset} & x_k \cup y_1 \\
{\scriptstyle n_1}\uparrow & & {\scriptstyle n_1}\uparrow & & {\scriptstyle n_1}\uparrow & & & & {\scriptstyle n_1}\uparrow \\
z^- \cap z^+ & \stackrel{p_1}{-\!\!\!\subset} & x_1 & \stackrel{p_2}{-\!\!\!\subset} & x_2 & \stackrel{p_3}{-\!\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\!\subset} & x_k
\end{array}$$

□

Via the next lemma, when games satisfy (**race-free**) we can simplify the condition for a strategy to be deterministic.

Lemma 4.4. *Let $\sigma : S \to A$ be a strategy. Suppose $x \xrightarrow{s}\subset y$ & $x \xrightarrow{s'}\subset y'$ & $pol_S(s) = -$. Then, $\sigma y \uparrow \sigma y'$ in $\mathcal{C}(A) \implies y \uparrow y'$ in $\mathcal{C}(S)$. A fortiori, if A satisfies* (**race-free**) *then so does S.*

Proof. Assume $\sigma y \uparrow \sigma y'$ in $\mathcal{C}(A)$, so $\sigma y' \xrightarrow{\sigma(s)}\subset \sigma y \cup \sigma y'$ in $\mathcal{C}(A)$. As $\sigma(s)$ is $-$ve, by receptivity, there is a unique $s'' \in S$, necessarily $-$ve, such that $\sigma(s'') = \sigma(s)$ and $y' \xrightarrow{s''}\subset x \cup \{s', s''\}$ in $\mathcal{C}(S)$. In particular, $x \cup \{s', s''\} \in \mathcal{C}(S)$. By $-$-innocence, we cannot have $s' \twoheadrightarrow s''$, so $x \cup \{s''\} \in \mathcal{C}(S)$. But now $x \xrightarrow{s}\subset$ and $x \xrightarrow{s''}\subset$ with $\sigma(s) = \sigma(s'')$ and both s, s'' $-$ve and hence $s'' = s$ by the uniqueness part of receptivity. We conclude that $x \cup \{s', s\} \in \mathcal{C}(S)$ so $y \uparrow y'$. □

Corollary 4.5. *Assume A satisfies* (**race-free**) *of Lemma 4.3. A strategy $\sigma : S \to A$ is deterministic iff it is* weakly-deterministic*, i.e. for all +ve events $s, s' \in S$ and configurations $x \in \mathcal{C}(S)$,*

$$x \xrightarrow{s}\subset \ \& \ x \xrightarrow{s'}\subset \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. *"Only if"*: clear. *"If"*: Let $x \xrightarrow{s}\subset$ and $x \xrightarrow{s'}\subset$ where $pol_S(s) = +$. For S to be deterministic we require $x \cup \{s, s'\} \in \mathcal{C}(S)$. The above assumption ensures this when $pol_S(s') = +$. Otherwise $pol_S(s') = -$ with $\sigma x \xrightarrow{\sigma(s)}\subset$ and $\sigma x \xrightarrow{\sigma(s')}\subset$. As A satisfies (**race-free**), $\sigma x \cup \sigma(s), \sigma(s') \in \mathcal{C}(A)$. Now by Lemma 4.4, $x \cup \{s, s'\} \in \mathcal{C}(S)$. □

Lemma 4.6. *The composition $\tau \odot \sigma$ of deterministic strategies σ and τ is deterministic.*

Proof. (Sketch) Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be deterministic strategies. The composition $T \odot S$ is constructed as the projection of $T \circledast S$ to those moves visible in $A^\perp \| C$. The proof proceeds by first showing that $T \circledast S$ is *deterministic* in the following sense:

$$\forall e, e' \in S, x \in \mathcal{C}(T \circledast S).\ \ x \xrightarrow{e}\subset \ \& \ x \xrightarrow{e'}\subset \ \& \ pol(e) \neq - \implies x \cup \{e, e'\} \in \mathcal{C}(T \circledast S)\,.$$

Recall $T \circledast S$ generally has synchronisation events, regarded as being of neutral polarity. So its being deterministic forbids immediate conflict between a +ve or neutral event and any other. That $T \circledast S$ is deterministic can be shown by case analysis on the form of the events e, e' using the definition of pullback. From $T \circledast S$ being deterministic it follows that $T \odot S$ is deterministic. □

We thus obtain a sub-bicategory of concurrent games and strategies: its objects are (**race-free**) (*cf.* Lemma 4.3) and its maps are deterministic strategies.

4.2 A category of deterministic strategies

In fact, the bicategory of deterministic strategies is equivalent to a category via the following lemma.

Lemma 4.7. *Let $\sigma : S \to A$ be a deterministic strategy. Then,*

$$\sigma y \subseteq \sigma x \implies y \subseteq x$$

for all $x, y \in \mathcal{C}(S)$. In particular, a deterministic strategy σ is injective on configurations, i.e., $\sigma x = \sigma y$ implies $x = y$, for all $x, y \in \mathcal{C}(S)$ (so is mono as a map of event structures).

Proof. Let $\sigma : S \to A$ be a deterministic strategy. We show

$$x \supseteq z \longrightarrow\!\!\!\!\subset y \ \&\ \sigma y \subseteq \sigma x \implies y \subseteq x\,,$$

for $x, y, z \in \mathcal{C}(S)$, by induction on $|x \setminus z|$.

Suppose $x \supseteq z \overset{e}{\longrightarrow\!\!\!\!\subset} y$ and $\sigma y \subseteq \sigma x$. There are x_1 and event $e_1 \in S$ such that $z \overset{e_1}{\longrightarrow\!\!\!\!\subset} x_1 \subseteq x$. If $\sigma(e_1) = \sigma(e)$ then e_1, e have the same polarity; if $-$ve $e_1 = e$, by receptivity; if +ve $e_1 = e$, by determinacy with the local injectivity of σ. Either way $y \subseteq x$. Suppose $\sigma(e_1) \neq \sigma(e)$. We show in all cases $y \cup \{e_1\} \subseteq x$, so $y \subseteq x$.
Case $pol(e_1) = +$ or $pol(e) = +$: As σ is deterministic, e_1 and e are concurrent giving $x_1 \overset{e}{\longrightarrow\!\!\!\!\subset} y \cup \{e_1\}$. By induction we obtain $y \cup \{e_1\} \subseteq x$.
Case $pol(e_1) = pol(e) = -$: From Lemma 4.4, we deduce that e_1 and e are concurrent yielding $x_1 \overset{e}{\longrightarrow\!\!\!\!\subset} y \cup \{e_1\}$, and by induction $y \cup \{e_1\} \subseteq x$.

Another, simpler induction on $|y \setminus z|$ now yields

$$x \supseteq z \subseteq y \ \&\ \sigma y \subseteq \sigma x \implies y \subseteq x\,,$$

for $x, y, z \in \mathcal{C}(S)$, from which the result follows. □

Via the above lemma we can provide an alternative description of deterministic strategies in a game A as certain subfamilies of $\mathcal{C}(A)$—see the course slides. This description shows deterministic strategies to coincide with the receptive ingenuous strategies of Melliès and Mimram. For deterministic strategies we can consider composition up to identical families; we obtain a category in place of a bicategory.

Chapter 5

Special cases of concurrent games and strategies

Stable spans, profunctors and stable functions

The sub-bicategory of **Strat** where the events of games are purely +ve is equivalent to the bicategory of stable spans. In this case, strategies correspond to *stable spans*:

$$\begin{array}{ccccccc} & S & & \longleftrightarrow & & S^{+} & \\ {\scriptstyle \sigma_1}\swarrow & & \searrow{\scriptstyle \sigma_2} & & {\scriptstyle \sigma_1^-}\swarrow & & \searrow{\scriptstyle \sigma_2^+} \\ A^{\perp} & & B & & A & & B\,, \end{array}$$

where S^+ is the projection of S to its +ve events; σ_2^+ is the restriction of σ_2 to S^+, necessarily a rigid map by innocence; σ_2^- is a *demand map* taking $x \in \mathcal{C}(S^+)$ to $\sigma_1^-(x) = \sigma_1[x]$; here $[x]$ is the down-closure of x in S.

Stable functions

If we further restrict strategies to be deterministic (and, strictly, event structures to be countable) we obtain a bicategory equivalent to Berry's *dI-domains and stable functions*.

Ingenuous strategies

Deterministic concurrent strategies coincide with the *receptive ingenuous strategies* of Melliès and Mimram.

Closure operators

Deterministic strategies have been presented as closure operators by Abramsky and Melliès. A deterministic strategy $\sigma : S \to A$ determines a closure operator

φ on possibly infinite configurations $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \& \ Neg[\{s\}] \subseteq x\}\,.$$

Clearly φ preserves intersections of configurations and is continuous. The closure operator φ on $\mathcal{C}^\infty(S)$ induces a *partial* closure operator φ_p on $\mathcal{C}^\infty(A)$. This in turn determines a closure operator $\varphi_p^\top$ on $\mathcal{C}^\infty(A)^\top$, where configurations are extended with a top $\top$, take $y \in \mathcal{C}^\infty(A)^\top$ to the least, fixed point of φ_p above y, if such exists, and $\top$ otherwise.

Simple games

"Simple games" of game semantics arise when we restrict **Strat** to objects and deterministic strategies which are 'tree-like'—alternating polarities, with conflicting branches, beginning with opponent moves. *Conway games* tree-like, but where only strategies need alternate and begin with opponent moves.