

Parallel full abstraction for PCF in concurrent Hyland-Ong games

Simon Castellan, Pierre Clairambault, Glynn Winskel

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Abstract

In this technical report, we build on the category Cho of concurrent Hyland-Ong games introduced in [CCW14] and give an intensionally fully abstract model for PCF, where the strategies interpreting terms behave in a parallel fashion.

This requires us to generalize to the concurrent case the notions of visibility, innocence and well-bracketing familiar from sequential Hyland-Ong games. We give an interpretation of terms of PCF as strategies in Cho satisfying these conditions (plus determinism). Finally we prove that our model has the finite definability property, and deduce that its extensional collapse is fully abstract.

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1 Introduction

The structure of the document is as follows. We build on top of the \sim -bicategory Cho of [CCW14], that we consider already constructed.

In Section 2, we introduce the various conditions on strategies necessary to prove definability. There are four of them: visibility, innocence, well-bracketing and determinism. For each of those conditions we show: (1) that it is satisfied by copycat, and by lifted maps in general – hence ensuring its satisfaction by structural morphisms of Cho, (2) that it is stable under weak equivalence – so that it naturally makes sense in the quotient category, (3) that it is preserved by the limit construction – so that it is satisfied by the fixpoint operator, and last but not least (4) that it is preserved by composition. Amongst all our conditions, visibility plays a particular role: it structures strategies so that one can reason on them using the notion of *grounded causal chains*, which somehow resemble sequential plays. Innocence and well-bracketing depend on visibility, in the sense that their proof of stability under composition exploits it. In the presence of all four conditions, we show that strategies can be represented more compactly by their *reduced forms*, which plays the same role with respect to full strategies as view functions play with respect to extended presentations of innocent strategies in sequential HO games.

In Section 3, we use this development to give the parallel intensionally fully abstract interpretation of PCF. First we detail the interpretation of PCF in Cho, and show soundness and adequacy (using standard methods). Then, we show that strategies on PCF types that satisfy our conditions and whose reduced form is finite are definable in PCF up to observational equivalence. From there, it is quite direct to prove that our model is fully abstract.

2 Conditions on strategies

In this section, the conditions we investigate deal with causality and consistency and are completely orthogonal with symmetry. We will occasionally refer to \sim -strategies just by strategies.

2.1 Visible strategies

We start by introducing the condition of visibility. In the usual HO game semantics, visibility says that the justifier of any move should appear in its view. In the following we claim that the right generalization of the notion of view in our setting is that of *grounded causal chains*:

Definition 2.1 (Grounded causal chain). *If S is an event structure, a gcc of S is a finite sequence (s_0, \dots, s_n) of events of S where s_0 is minimal and we have for*

$0 \leq i < n, s_i \rightarrow s_{i+1}$ where $s \rightarrow s'$ denotes immediate causal dependency: $s < s'$ and no events in between.

Whereas our strategies are concurrent, gccs are always sequential and alternating. In fact, by courtesy one can see that a gcc is a P-view with an explicit choice of copy indices.

Note however that although in sequential HO-games, gccs (P-views) are always causally down-closed, here a Player move could require the conjunction of several gccs to be played. Grounded causal chains represent running threads and are useful to observe *forking* (creation of new threads) and *joining* (waiting for several threads to terminate and merging them in a single thread).

Notations on gccs We will use the following notations for gccs:

- $\text{gcc}(S)$: the set of gccs of S
- $\text{gcc}(s)$ for $s \in S$: the set of gccs ending in s
- q_ω where $q \in \text{gcc}(S)$ is the last event of q
- $q_{i \leq \dots \leq j}$ where $q \in \text{gcc}(S)$ is the sub-sequence of q consisting of q_i, \dots, q_j . It is a gcc only if $i = 0$. We write $q_{\leq i}$ for $q_{0 \leq \dots \leq i}$.
- $s \in q$ for $s \in S$ and $q \in \text{gcc}(S)$ means that $s = q_i$ for some i .
- $|q|$ for $q \in \text{gcc}(S)$ is the length of the sequence

To define visibility, we will need the notion of justifiers:

Definition 2.2 (Justifiers). Let $\sigma : S \rightarrow A$ be a map of event structures where A is a forest. Let $s \in S$ be an event of the strategy that is not minimal. Because A is a forest, σs has a unique predecessor a_0 . Since $\sigma[s]$ is a down-closed set of $!A$, $a_0 \in \sigma[s]$ and by local injectivity we can define the justifier of s (written $\text{just}(s)$) as the unique event $s_0 < s$ mapped to a_0 .

The link $\text{just } s \rightarrow s$ is called the *pointer* of S (terminology used in for instance “follow a pointer”)

Because of courtesy, Opponent always points to the last move which is unique:

Lemma 2.3. If $\sigma : S \rightarrow A$ is a courteous map of event structures to an arena, then for any non-minimal $s \in S$ such that σs is negative, $\text{just}(s)$ is the unique event $s' \in S$ such that $s' \rightarrow s$.

Proof. Take a $s' \in S$ such that $s' \rightarrow s$ (it exists because s is non-minimal), then by courtesy we have $\sigma s' \rightarrow \sigma s$ and thus $s' = \text{just } s$ by definition of the justifier. \square

In our setting, being visible means that any gcc of a non-minimal event must feature its justifier.

Definition 2.4 (Visible map). A map of event structures $\sigma : S \rightarrow A$ (where A is a forest) is visible when for all $s \in S$ and $q \in \text{gcc}(s)$, $\text{just}(s) \in q$.

Iterating the definition, we see that the image of any gcc by σ is actually a down-closed subset of A (ie. a configuration because A is conflict-free).

Lemma 2.5. A map $\sigma : S \rightarrow A$ is visible iff for all $q \in \text{gcc}(S)$, σq is a configuration of A .

Proof. (\Leftarrow) Let $s \in S$ be a non-minimal event of S and q a gcc of s . We know that σq is a configuration of A , thus it must contain the projection of the justifier of s . Thus there exists $s_0 \in q$ such that $\sigma s_0 = \sigma(\text{just } s)$. Because $s_0 \in q$ we know that $s_0 \leq s$ and by definition we must have that s_0 is the justifier of s .

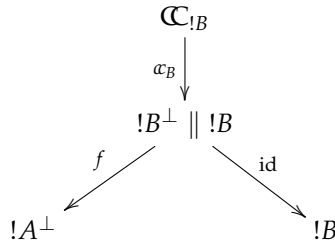
(\Rightarrow) Let s be an event. We prove the result on induction on the cardinal of $[s]$.

- If $[s] = 1$ then s is minimal and for all gcc $q \in \text{gcc}(s)$ we have $q = [s]$ thus $\sigma q = \sigma[s]$ is a configuration of A .
- Otherwise we write s_0 the justifier of s . Let q be a gcc of s and write s_n for the event in q before s . Because $s_n < q$ we know that $\sigma q_{0 \leq \cdot < \omega}$ is down-closed. Because σ is visible, the justifier of s appears in $q_{0 \leq \cdot < \omega}$. Thus $\sigma q_{0 \leq \cdot < \omega}$ contains $[\sigma s_0]$ because it is down-closed. Then we conclude by remarking that.

$$\sigma q = \{\sigma s\} \cup \sigma q_{0 \leq \cdot < \omega} = \{\sigma s\} \cup [\sigma s_0] \cup \sigma q_{0 \leq \cdot < \omega} = [\sigma s] \cup \sigma q_{0 \leq \cdot < \omega}$$

and the last set is a union of down-closed sets. □

Visibility of lifted maps We recall that given a map $f : !B^\perp \rightarrow !A^\perp$ satisfying strong-receptivity and courtesy, we can build a strategy written $\bar{f} \in \text{Cho}(A, B)$ as follows (see section 3.6 of [CCW14] for more details):



Lemma 2.6. Given a arena A , the event structure \mathbb{C}_A is an arena.

Proof. All the immediate causalities of \mathbb{C}_A are inherited from A or of the form $(1, a) \rightarrow (2, a)$ when a is positive (so negative in A^\perp and $(1, a) \rightarrow (2, a)$ when a is negative. So all immediate causalities are alternating thus \mathbb{C}_A is alternating.

So we are left to prove that \mathbb{C}_A is a forest. Assume we have $\alpha, \beta, \gamma \in \mathbb{C}_A$ distinct with $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$. If γ is negative in $A^\perp \parallel B$ then by courtesy of α_A , we would have $\alpha\alpha \rightarrow \alpha\gamma$ and $\alpha\beta \rightarrow \alpha\gamma$ contradicting the fact that $A^\perp \parallel A$ is a forest.

So γ must be positive, assume for instance $\gamma = (2, c)$ and $c \in A$. Then we have three cases depending on the nature of the immediate causalities:

- $\alpha = (1, c)$ and $\beta = (1, c)$ so $\alpha = \beta$ absurd
- $\alpha = (1, c)$ and $\beta = (2, b)$ with $b \rightarrow_A c$. That cannot be because we have $\beta = (2, b) \rightarrow (1, b) \rightarrow (1, c) \rightarrow (2, c)$ (causal chain in \mathbb{C}_A) that contradicts $\beta \rightarrow \gamma$
- $\alpha = (2, a)$ and $\beta = (2, b)$ with $a \rightarrow_A c$ and $b \rightarrow_A c$ which is absurd because A is a forest.

All cases being absurd, \mathbb{C}_A is a forest and thus an arena. \square

From Lemma 2.6, it is easy to derive that lifted maps (so copycat in particular) are visible.

Lemma 2.7. *Given a $f : !B^\perp \rightarrow !A^\perp$ strong-receptive and courteous, the strategy \bar{f} is visible.*

Proof. Let q be a gcc of \mathbb{C}_B . By Lemma 2.6, we know that \mathbb{C}_B is a forest and thus q is down-closed in \mathbb{C}_B . q is then a configuration and is mapped to a configuration in $!A^\perp \parallel !B$ by \bar{f} as desired. \square

Stability under weak equivalence Visibility is stable under weak equivalence

Lemma 2.8. *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !A$ be \sim -strategies such that $\sigma \simeq \tau$ and σ is visible. Then τ is visible.*

Proof. Call $\varphi : S \rightarrow T$ and $\psi : T \rightarrow S$ the two maps arising from the weak equivalence. They are both rigid. If $q \in \text{gcc}(T)$, then ψq is a gcc of S , thus $\sigma(\psi q)$ is a configuration of $!A$ by visibility of σ . $[q]$ being a configuration of T implies that we have the following isomorphism in the symmetry in $!A$:

$$\sigma(\psi[q]) \stackrel{\theta}{\cong} \tau[q]$$

By using restriction on $\sigma(\psi([q]))$ it follows that $\tau q \in \mathcal{C}(!A)$. \square

Parallel composition of visible strategies We now prove that visibility is stable under parallel composition.

Lemma 2.9. *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !B$ be visible \sim -strategies. Then $\sigma \parallel \tau$ is visible.*

Proof. Let q be a gcc of $S \parallel T$. By definition of $S \parallel T$, either all the elements of q are in S or all in T . We can then apply the visibility of σ or τ and conclude. \square

Limits of visible strategies We now prove that a limit of a chain of visible strategies is visible. (See Section 4.5 of [CCW14] for more details on limit construction in Cho)

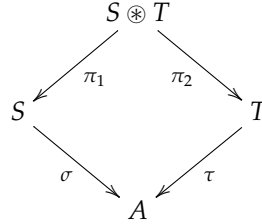
Lemma 2.10. *Assume $\sigma_0 \trianglelefteq \sigma_1 \trianglelefteq \dots$ is a chain of visible \sim -strategies. Then $\sigma = \bigsqcup_{i \in \mathbb{N}} \sigma_i$ is a visible \sim -strategy.*

Proof. Let ρ be a gcc of $\bigcup_{i \in \mathbb{N}} S_i$ where S_i is the event structures underlying σ_i . ρ being finite there exists a $j \in \mathbb{N}$ such that ρ is a gcc of S_j . σ_j being visible we have that $\sigma \rho = \sigma_j \rho$ is a configuration of $!A$ and σ is visible. \square

2.2 Visibility in the interaction

We now need to check that visibility is stable under composition. It turns out that visibility gives fruitful consequences on the structure of the interaction of two strategies, which are going to be essential. This all follows from the fact that when calculating the interaction of two visible strategies, a gcc of the interaction induces two gccs of the two strategies that represent their view of the interaction. (Related to the notions of P-views and O-views in sequential games)

In all this section we assume two visible \sim -strategies $\sigma : S \rightarrow !A^\perp$ and $\tau : T \rightarrow !A$, and we consider $\sigma \circledast \tau$ their interaction defined as their pullback, forgetting the polarities:



Lemma 2.11. *Let p be an event of $S \circledast T$. Then $\pi_i(\text{just } p) = \text{just}(\pi_i p)$.*

Proof. Because $\sigma \circledast \tau(p) = \sigma(\pi_1 p)$, p and $\pi_1 p$ have the same justifier image. We have that $\text{just}(\pi_1 p) \leq \pi_1 p \in \pi_1[p]$ thus there exists $p' \leq p$ such that $\pi_1 p' = \text{just } \pi_1 p$. Moreover,

$$\sigma \circledast \tau(p') = \sigma(\pi_1 p') = \sigma(\text{just } \pi_1(p)) = \sigma(\text{just } \pi_1 p) = \sigma(\text{just } p).$$

Thus by definition of the justifier, $p' = \text{just } p$ and the desired equation holds. \square

Views in the interaction We first show how to extract from a gcc of $S \circledast T$ a gcc of S and a gcc of T following the usual notion of views in HO games.

Proposition 2.12. *Let q be a gcc in $S \otimes T$ ending at p . There exists two sub-sequences of q , written q^σ and q^τ , both ending at p such that $\pi_1 q^\sigma$ is a gcc of S and $\pi_2 q^\tau$ is a gcc of T .*

Moreover, if $\pi_2 p$ is negative there exists $i \in \mathbb{N}$ such that $\pi_2 q_i = \text{just}(\pi_2 q)$.

It is to be noted that q^σ and q^τ are **not** gccs in $S \otimes T$.

Proof. We proceed by induction on the length of q . If q is of length one then we let both $q^\sigma = q^\tau = q$.

If not, if q has length $n + 1$, write s_i for $\pi_1 q_i$ and t_i for $\pi_2 q_i$. One of s_{n+1} or t_{n+1} must be positive, because σ and τ play on dual games. For instance, assume s_{n+1} is positive. Then by Lemma 2.11 of [CCW14], we get $s_n \rightarrow_S s_{n+1}$. We let $q^\sigma = q_{\leq n}^\sigma \cdot p_{n+1}$. By the induction hypothesis, we have indeed that $\pi_1 q^\sigma$ is a gcc of S .

If t_{n+1} is minimal, then we can let $q^\tau = p_{n+1}$. Otherwise, we need to look at the justifier of t_{n+1} . By visibility of σ , we know that $\sigma(\pi_1 q^\sigma) \in \mathcal{C}(A)$. Because $p_n \in q^\sigma$, we have that $\tau t_{n+1} \in \sigma(\pi_1 q^\sigma)$. This set is down-closed, so it means that $\tau(\text{just } t) \in \sigma(\pi_1 q^\sigma)$, ie. there exists $q_i \in q$ such that $\sigma s_i = \tau(\text{just } t)$. Then $\tau t_i = \tau(\text{just } t)$ and t_i and $\text{just } t$ are consistent (they are in $\pi_2[q]$ which is a consistent set of $S \otimes T$), so $t_i = \text{just } t$. Then we let

$$q^\tau = q_{\leq i}^\tau \cdot p_{n+1}$$

By induction and by Lemma 2.3 $\pi_2 q^\tau$ is a gcc of T . We also proved the last property of the proposition. \square

In the following, we will use the following inductive properties of q^σ and q^τ :

- $(q \cdot p)^\sigma = q^\sigma \cdot p$ (if $\pi_1 p$ is positive)
- $(q \cdot p)^\sigma = q_{\leq j}^\sigma \cdot p$ where j is the index of the justifier of $\pi_1 p$, which exists (otherwise)

Corollary 2.13. *The map $\sigma \otimes \tau$ is visible.*

Proof. Let $p \in S \otimes T$ and q a gcc of p . Assume for instance $\pi_1 p$ is negative. Then, we know by Proposition 2.12 that there exists $i \in \mathbb{N}$ such that $\pi_1(\text{just } p) = \pi_1 q_i$. Since q_i and $\text{just } p$ are consistent (both below p) we must have $\text{just } p = q_i$ by local injectivity which concludes the proof. \square

The forking lemma Visibility has also very interesting consequences on the structure of the gccs that are going to be very useful later on. Grounded causal chains are a very good tool to study when the strategies spawns new threads (forking) and when it waits for threads to terminate (joining).

Definition 2.14 (Sibling ground causal chains). *Let S be an event structure and q, q' two consistent gccs on S such that $q_1 = q'_1$. Then q and q' are called sibling gccs, and the top event $q_i = q'_i$ of $q \cap q'$ is called their divergence. Moreover, if $q_\omega \notin q'$*

and $q'_\omega \notin q$, they are said to be forking. The (necessarily disjoint and non-empty) segments $q_{i<-}$ and $q'_{j<-}$ are called the branches of the forking gccs.

The divergence of two sibling gccs is the latest point in time where they met. They can have forked and joined several times before reaching that point. But after the divergence, if they continue, they are disjoint. We say q and q' diverge (or fork) at $q_i = q_j$.

Lemma 2.15. *Let q be a gcc of $S \otimes T$, and $i \leq |q|$. We have $\pi_1 q_{\leq i}^\sigma$ and $\pi_1 q^\sigma$ are sibling gccs. If they don't diverge at $\pi_1 q_i$, then they diverge at some positive event.*

Proof. We proceed by induction on $|q|$. If $i = |q|$ they are trivially diverging at $\pi_1 q_i$.

Otherwise, we have $q_i < q_\omega$ and we look at the last event q_j in q^σ (before q_ω).

- If $q_i < q_j$ then we can apply the induction hypothesis to $q_{\leq i}$ and $q_{\leq j}$, and conclude because $\pi_1 q^\sigma$ extends $\pi_1 q_{\leq j}^\sigma$ and $\pi_1 q_\omega \notin \pi_1 q_{\leq i}^\sigma$.
- If $q_j \leq q_i$, since $i \neq q_i$ we must have followed the justifier of $\pi_1 q_\omega$ which is negative. So $\pi_1 q_j$ is positive. We can apply the induction hypothesis and get that the gccs $\pi_1 q_{\leq j}^\sigma$ and $\pi_1 q_{\leq i}^\sigma$ are sibling gccs. Either they meet on $\pi_1 q_{\leq j}^\sigma$ and then $\pi_1 q_{\leq i}^\sigma$ and $\pi_1 q^\sigma$ diverge at $\pi_1 q_j$ which is positive so it is fine. Or they diverge at some positive events and so do $\pi_1 q_{\leq i}^\sigma$ and $\pi_1 q^\sigma$.

□

The next lemma is a key lemma to understanding the forks in the interaction. If there is a fork in the interaction, then both players see a fork in their respective view, and either they see this fork, or they see an Opponent fork (the divergence is a Player move).

Lemma 2.16 (Forking Lemma). *Assume we have two gccs q, q' of $S \otimes T$ forking at p . Then $\pi_1 q^\sigma$ and $\pi_1 q'^\sigma$ are gccs of S forking either at $\pi_1 p$ or at a positive event of S , and the same for $\pi_2 q^\tau$ and $\pi_2 q'^\tau$.*

Proof. We proceed by induction on the length of q . Let r be the event in q^σ before q_ω . We write χ for $q_{\leq r}$.

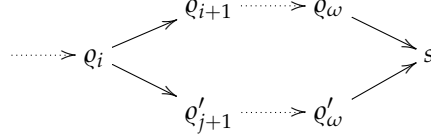
There are two cases:

- If $r < q'_\omega$ then we can apply Lemma 2.15 to get that $\pi_1 \chi^\sigma$ and $\pi_1 q'^\sigma$ are sibling gccs. Since $r < q_\omega$ and $q_\omega \notin q'$, $\pi_1 q^\sigma$ and $\pi_1 q'^\sigma$ are forking. Their divergence is either positive (which is fine) or $\pi_1 r$. If it is $\pi_1 p$ but not $\pi_1 q$, it means that $r < q < q_\omega$, ie. we followed a pointer to get to r which means that $\pi_1 r$ is positive.
- Otherwise, χ and q' are forking gccs, forking at p so we can apply the induction hypothesis to get that $\pi_1 \chi^\sigma$ and $\pi_1 q'^\sigma$ are forking gccs, forking at either $\pi_1 q$ or a positive event. Thus so are $\pi_1 q^\sigma$ and $\pi_1 q'^\sigma$ because $q_\omega \notin q'$. □

Having dealt with the forking behaviour, we can also say things about joins.

Definition 2.17 (Wait event). *Let $\sigma : S \rightarrow A$ be a strategy and q, q' two forking gccs of S . An event $s \in S$ is a wait event for q and q' if $q_\omega \rightarrow s$ and $q'_\omega \rightarrow s$.*

The corresponding diagram is the following:



We consider an interaction between visible strategies $\sigma : S \rightarrow A^\perp$ and $\tau : T \rightarrow A$.

Lemma 2.18. *Let q, q' be gccs of $S \otimes T$ with a wait event p such that $\pi_1 p$ is positive. Then $\pi_1 p$ is a wait event for $\pi_1 q^\sigma$ and $\pi_1 q'^\sigma$, and similarly if $\pi_2 p$ is positive instead.*

Proof. By Lemma 2.16, we know that $\pi_1 q^\sigma$ and $\pi_1 q'^\sigma$ are forking gccs. Since $\pi_1 p$ is positive, we know that both $q_\omega \rightarrow p$ and $q'_\omega \rightarrow p$ which is what we wanted. \square

Deadlock-free interaction Visible strategies have deadlock-free interaction meaning that every bijection of the form $x \cong y$ induced by $\sigma x = \tau y$ is secured.

Lemma 2.19. *Let $s : S \rightarrow !A$ and $\tau : T \rightarrow !A^\perp$ be visible \sim -strategies. Let $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$ and θ such that $\sigma x \cong \tau y$. Then the bijection $\varphi : x \cong y$ induced by θ is secured, meaning that the transitive closure of " $s \leq_S s'$ or $\varphi s \leq_T \varphi s'$ " defines an order on x .*

Proof. It is enough to show that if we cannot have $s <_S s'$ and $\varphi s' <_T \varphi s$ at the same time. Assume it is the case and pick a minimal $(s, s') \in S^2$ (for the product order) which satisfies the property above. We look at s_0 the justifier of s' . By visibility, it is comparable with s . Moreover, we have $\theta(\sigma(s_0)) = \tau(\text{just}(\varphi s'))$. It means that $\varphi s_0 = \text{just}(\varphi s')$.

- Assume $s = s_0$. Then we have $\sigma s \leq \sigma s'$ and thus $\tau(\varphi s) \leq \tau(\varphi s')$ and thus $\varphi s \leq \varphi s'$ which is absurd.
- Assume $s < s_0$. So we have $s < s_0$ and $\varphi s_0 < \varphi s' < \varphi s$. So (s, s_0) is a smaller pair satisfying the property which contradicts the minimality of (s, s') .
- Assume $s_0 < s$. We can look at the justifier t_0 of φs , and if $\varphi s' \leq t_0$ then we can conclude by the same argument as above, so we can assume $t_0 < \varphi s'$. It means that t_0 and s_0 must be negative, and s' and φs must be positive. This means that s and $\varphi s'$ are negative and their justifiers (s_1 and t_1) happens right before them. Then $(s_1, \varphi^{-1} t_1)$ is a smaller pair than (s, s') .

□

Lemma 2.20. *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !A$ be visible \sim -strategies. Let $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$ such that $\sigma x \stackrel{\theta}{\cong} \tau y$. Then there exists a configuration $z \in \mathcal{C}(S \otimes T)$ such that $\pi_1 z \cong x$ and $\pi_2 z \cong y$ are symmetries in S and T . Moreover, if θ is reflexivity, then so are those equalities.*

Proof. We use the bi-pullback property (Proposition 3.16 of [CCW14]). By the previous lemma, we can build an event structure z which has the same events as x and y and all the causalities from x and y . So there are maps $z \rightarrow x$ and $z \rightarrow y$, and we get two maps $z \rightarrow S$ and $z \rightarrow T$ that commutes up to symmetry in A . So by the bi-pullback property we have a configuration of $S \otimes T$ that satisfies the property we want.

Moreover if θ is reflexivity, then we can use the pullback property instead to get what we want. □

Composition of visible strategies We finish this section by proving that visible strategies are stable under composition.

Lemma 2.21. *Let $\sigma : S \rightarrow !A^\perp \parallel !B$ and $\tau : T \rightarrow !B^\perp \parallel !C$ be visible \sim -strategies, then so is $\tau \odot \sigma$.*

Proof. Since the identity strategy ($id_A : A \rightarrow A \in \text{Cho}(A)$) is clearly visible and visible strategies are stable under parallel composition (Lemma 2.9) we have that the interaction $(\sigma \parallel C) \otimes (A \parallel \tau) : (S \parallel C) \otimes (A \parallel T) \rightarrow A \parallel B \parallel C$ is visible (Lemma 2.13).

Let ϱ be a gcc of $S \odot T$. By definition, ϱ can be extended to a gcc $\bar{\varrho}$ of the interaction such that ϱ is a sub-sequence of $\bar{\varrho}$. Then by visibility of the interaction we know that $\sigma \bar{\varrho} \in \mathcal{C}(A \parallel B \parallel C)$. By definition of $\bar{\varrho}$, we have that

$$(\tau \odot \sigma)\varrho = (\sigma \parallel C) \odot (A \parallel \tau)(\bar{\varrho}) \cap (!A \parallel !C) \in \mathcal{C}(!A \parallel !C)$$

□

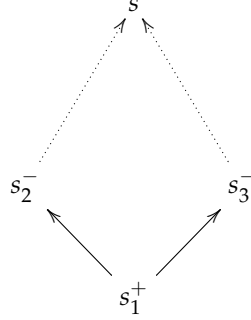
2.3 Innocence

Innocence in our setting is the inability to join forks started by Opponent.

Definition 2.22 (Normal configurations). *A configuration x of a strategy $\sigma : S \rightarrow A$ is normal when Opponent points at most once in x to any Player move, i.e. x has a unique minimal event, and for all $s^+, s_1^-, s_2^- \in x$ such that $s \rightarrow s_1$ and $s \rightarrow s_2$ we have $s_1 = s_2$.*

Definition 2.23 (Innocence). *A strategy $\sigma : S \rightarrow A$ is innocent whenever for each $s \in S$, each prime configuration $[s]$ of S is normal.*

In particular the following picture is not possible:



Notice that innocence is independent of the labelling function σ .

We proceed to prove the basic properties of this notion:

Lemma 2.24 (Innocence of lifted maps). *Let $f : !B^\perp \rightarrow !A^\perp$ be strong-receptive and courteous, \bar{f} is an innocent strategy.*

Proof. It amounts to checking that \mathbb{C}_A is innocent for any arena A . This is the case because CC_A is a forest (Lemma 2.6). \square

Lemma 2.25 (Stability by weak equivalence). *Innocence is stable by weak equivalence.*

Proof. Since weak equivalence maps are rigid and preserve polarities, they preserve normal configurations and prime configurations. \square

Lemma 2.26 (Stability by parallel composition). *If $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !B$ are innocent \sim -strategies, then so is $\sigma \parallel \tau : S \parallel T \rightarrow !A \parallel !B$.*

Proof. Prime configurations correspond to primes of S and primes of T and the result follows from that fact. \square

Lemma 2.27. *Assume $\sigma_0 \leq \sigma_1 \leq \dots$ is a chain of strategies in $\text{Cho}(A)$ that are all innocent. Then $\sigma = \bigsqcup_{i \in \mathbb{N}} \sigma_i$ is an innocent strategy.*

Proof. Let $s \in \bigsqcup S$. By definition there exists a $i \in \omega$ such that $s \in S_i$ and thus $[s]$ is normal in S_i . Because $S_i \leq S$ $[s]$ is also normal in S . \square

Thanks to the Forking Lemma, proving the stability by composition of innocence is not too hard. However, this requires visibility; indeed innocence is not stable without visibility.

Lemma 2.28 (Stability by composition). *Let $\sigma : S \rightarrow !A^\perp \parallel !B$ and $\tau : T \rightarrow !B^\perp \parallel !C$ be visible and innocent \sim -strategies. Then $\tau \odot \sigma$ is innocent.*

Proof. Take a prime configuration $[p] \in S \odot T$. Assume it is not normal, then it means that there are two gccs ϱ and ϱ' forking at a positive event and which have a wait event p . We can complete the gccs to have two gccs of the pullback $\bar{\varrho}$ and $\bar{\varrho}'$ with a wait event \bar{p} . Since ϱ and ϱ' fork at a positive visible event, and positive visible events cannot enable invisible events, $\bar{\varrho}$ and $\bar{\varrho}'$ also fork at this positive event. Assume for instance $\pi_1 \bar{p}$ is positive. Then by Lemma 2.18, we have that $\pi_1 \varrho^\sigma$ and $\pi_1 \varrho'^\sigma$ fork at a positive event and have $\pi_1 \bar{p}$ as a wait event, which contradicts the innocence of $\sigma \parallel C$. □

2.4 Well-bracketing

2.4.1 Definition

In usual HO game semantics, well-bracketing is used to ensure that the strategies respect the natural control flow. In particular it rules out the interpretation of control operators that act on the context of evaluation (such as `call/cc`). The control flow of the execution is usually modelled by Question/Answers in HO game semantics and we will use the same ideas here. To do so, we introduce a new kind of arenas: *arenas with question-answer labelling*. They are pairs $(A, \lambda : A \rightarrow \{Q, A\})$ of an arena A and a function λ assigning to each move of the arena a Question/Answer labelling such that initial moves of A have label Q .

To capture exactly the types of PCF, one can add the requirement that answers are maximal events in arenas, but this is not necessary for the stability under composition of the condition.

We define a new category $\text{Cho}^?$ whose objects are arenas with question-answer labelling, the set of morphisms from (A, λ_A) to (B, λ_B) is just $\text{Cho}(A, B)$. As Cho , $\text{Cho}^?$ satisfies the laws of CCC up to weak equivalence. From now on, we only consider arenas that are equipped with a Question/Answer labelling.

As for polarities if $\sigma : S \rightarrow !A$ plays on an arena A with question-answer labelling, S inherits a Question/Answer labelling so it makes sense to speak of a question or an answer in S as an event of S whose projection in the game has label Q or A . We start by giving some standard vocabulary related to questions and answers:

Definition 2.29. Let $\sigma : S \rightarrow !A$ be a strategy, and a consistent set of event $X \subseteq S$.

- An answer to $q \in S$ is an answer $a \in S$ such that $\sigma q \rightarrow \sigma a$, in particular just $a = q$ thus every answer of S is the answer of its justifier.
- q is answered in X if there is an answer to q in X
- X is complete if all questions of X are answered in X .
- A question q is pending in X if it is not answered in X

- If X is totally ordered by \leq_S , the last pending question of X , if it exists, is the maximum of the set

$$\{q \in X \mid q \text{ has no answer in } X\}.$$

X is then well-bracketed if for every every answer a of X answering to a question $q \in X$, q is the last pending question in $X \cap [a)$.

The traditional condition of well-bracketing in HO games would then read “every gcc of S is well-bracketed as a set of events”. But this is not enough and not stable under composition in itself, by nature of this concurrent world. This is because here we have a form of a very generic concurrency which shares similarity with the UNIX syscall `fork`. Because there is no conflict between different answers to the same question in the arenas, a strategy can give several answers to the same question (as for instance the identity map $\text{id}_A : !A \rightarrow !A$ viewed as a strategy on $!A$). An innocent Opponent by receptivity will then handle all the answers concurrently in separate threads.

Using such a strategy along with communication between threads it is possible to derive a callcc-like strategy, thus we also need to require the answering discipline to be affine. Note that this condition will not be not be stable under composition without the first one as fork-like behaviours and callcc-like behaviours are inter-definable in presence of state and concurrency.

Moreover, we will need to add another condition to forbid joining of threads with pending questions. Indeed, we want to allow joining of threads only when they are completed, ie. all questions have been answered. This gives the definition of well-bracketed strategy:

Definition 2.30 (Well-bracketed strategy). *Let $\sigma : S \rightarrow !A$ be a map of event structures to an arena A with a Question-Answer labelling. It is well-bracketed when it satisfies the following conditions:*

- Every gcc of S is well-bracketed (as a set of events)
- If q and q' are two distinct sibling gccs ending at two positive answers of the same question, then they are forking and their divergence is positive.
- If we have two forking gccs q and q' of S , forking at $q_i = q'_j$ such that there exists a wait event for q and q' , then the sets $q_{i<}$ and $q'_{j<}$ are complete.

The condition (b) does not strictly coincide with the condition in the LICS submission, but is equivalent for the PCF arenas where answers do not justify questions.

Lemma 2.31. *Let $\sigma : S \rightarrow !A$ be a visible strategy satisfying well-bracketing.(b) and q be a gcc of S . Then q does not have two answers with the same justifier.*

Proof. Assume we have two answers q_i and q_j with the same justifier q_k (consequence of visibility). Then they have the same polarity. If they are negative, then by courtesy we have that $q_k = q_{i-1} = q_{j-1}$ implying $i = j$ which is absurd.

Otherwise, assume for instance $i < j$. Then $q_{\leq i}$ and $q_{\leq j}$ are two distinct gccs that are siblings and both end with an answer to the same question. So they must be forking which cannot be. \square

Lemma 2.32. *Let $\sigma : S \rightarrow !A$ be a visible strategy satisfying condition well-bracketing.(b). Then the following are equivalent:*

- σ satisfies well-bracketing.(a)
- condition (a'): For every gcc q of S ending in an answer whose justifier is q_i , then the segment $q_{i < \omega}$ is complete.

Proof. • If σ satisfies well-bracketing (a) and let q be such a gcc. Then q_i is the last pending question at $q_{< \omega}$ meaning no questions in $q_{i < \omega}$ are pending and thus they are all answered

- Conversely, if $q_{i < \omega}$ is complete, then by the previous lemma, q_i must be pending at $q_{< \omega}$ and it must be the last one by completeness. \square

Lemma 2.33 (Stability by weak equivalence). *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !A$ be strategies that are weakly equivalent. If σ is well-bracketed, then so is τ .*

Proof. Let $(f, g) : S \rightarrow T$ be a weak equivalence between σ and τ . Because g preserves labels, it also preserves the Question/Answer labelling of S and T induced by σ and τ . g also preserves justifiers so it preserves the relationship of “being an answer to”: t is answer to t' if and only if gt is an answer to gt' . Thus it implies that g both reflects and preserves the notion of well-bracketing and the notion of completeness: X is complete in T iff gX is complete in S and similarly for well-bracketing.

Adding the fact that g is rigid and thus preserves gccs, it entails that τ is well-bracketed. \square

Lemma 2.34. *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !B$ be well-bracketed strategies. Then $\sigma \parallel \tau$ is well-bracketed.*

Proof. We check the properties.

- (a) Let q be a gcc of $S \parallel T$. By definition of $S \parallel T$, q lives in S or T . For instance, assume q lives in S . By well-bracketing of σ q is well-bracketed
- (b) Let q and q' be gccs of $S \parallel T$ ending with answers to the same question, then they both live in S or both in T and we can conclude by well-bracketing of σ or τ .
- (c) Because of the wait event, the forking gccs both live in either S and T and we can conclude by well-bracketing of σ or τ . \square

Lemma 2.35. *Let $f : !B^\perp \rightarrow !A$ be a courteous and strong-receptive map. Then $\bar{f} \in \text{Cho}^?(A, B)$ is well-bracketed.*

Proof. We check the properties. We recall that $\bar{f} = \mathbb{C}_{!B} \xrightarrow{c_{!B}} !B^\perp \parallel !B \xrightarrow{f \parallel !B} !A^\perp \parallel !B$.

- (a) Let ϱ be a gcc of $\mathbb{C}_{!B}$. For instance, let $((1, q), (1, a))$ be a pair of question/answer in ϱ . Write $X = \{e \in \mathbb{C}_{!B} \mid (1, q) < e < (1, a)\}$. If q is positive, then $X = \emptyset$ and thus complete. If q is negative, $X = \{(0, q), (0, a)\}$ which is also complete.
- (b) Let ϱ and ϱ' be gccs of $\mathbb{C}_{!B}$ ending at positive answers to the same question. Then they fork at the dual of the question, which is positive.
- (c) There is no joins in \mathbb{C}_B .

□

Lemma 2.36. *Assume $\sigma_0 \trianglelefteq \sigma_1 \trianglelefteq \dots$ is a chain of strategies that are all well-bracketed. Then $\sigma = \bigsqcup_{i \in \mathbb{N}} \sigma_i$ is a well-bracketed strategy.*

Proof. All conditions only involve finite configurations or gccs which are finite, so it is clear. □

2.4.2 Stability under composition

As noticed before, the identity strategy on an arena A is not always well-bracketed (fails condition (b)), so we consider directly the pullback of the composition. We will need visibility to relate the gccs of the interaction with the gccs of the strategies. Let $\sigma : S \rightarrow !A^\perp \parallel !B$ and $\tau : T \rightarrow !B^\perp \parallel !C$ be visible and well-bracketed strategies and consider the following pullback:

$$\begin{array}{ccc}
 & (S \parallel C) \otimes (A \parallel T) & \\
 \swarrow & & \searrow \\
 S \parallel C & & A \parallel T \\
 \searrow \sigma \parallel C & & \swarrow A \parallel \tau \\
 & A \parallel B \parallel C &
 \end{array}$$

We write $\sigma \otimes \tau$ for the pullback map $(S \parallel C) \otimes (A \parallel T) \rightarrow A \parallel B \parallel C$.

It does not make sense to say that $\sigma \otimes \tau$ is well-bracketed because condition (b) mentions polarities and there is no clear notion of polarity on $A \parallel B \parallel C$. Conditions (a) and (c) still make sense.

We recall than an event of the interaction is **visible** if it is projected to $A \parallel C$ or **invisible** otherwise

Lemma 2.37. *Let q be a gcc of the interaction. If $\pi_1 q_\omega$ is negative and if q_i is a question in the view for σ of q such that $\chi = q_{>i} \cap q^\sigma$ is complete, then $q_{>i}$ is complete.*

Proof. By induction on the length of $q_{>i}$. First, since χ is complete, q_ω cannot be a question, it has to be an answer. There are two cases:

- The justifier of q_ω is after q_i : call q_j the justifier. By induction, we have that $q_{>j}$ is complete. Moreover, since $\pi_1 q_j$ is negative and we have that $q_{i<.<j} \cap q^\tau$ is complete, we can apply the induction hypothesis to get that $q_{i<.<j}$ is complete and thus $q_{>i}$ is complete.
- The justifier of q_ω is equal to q_i (it cannot be before by completeness of $q_{>i} \cap q^\sigma$). Again by completeness, the event q_j right before q_ω has to be either q_i (which then is fine since we have $q_{>i} = \{q_\omega\}$ which is complete), or an answer whose first projection is positive. We can look at its justifier. Because $\pi_2 q^\tau$ is well-bracketed, and $q_i \in q^\tau$ (by visibility, as it is the justifier of $q_\omega \in q^\tau$), this justifier q_k must appear after q_i . There are three cases:
 - $q_i = q_k$: then $q_{>i} = q_{i<.<\omega} \cdot q_\omega$ so we can apply the induction hypothesis to $q_{i<.<\omega}$ to conclude
 - $i = k - 1$, then $q_{>i} = q_{i<.<\omega} \cdot q_\omega$ so can also conclude by hypothesis
 - otherwise, q_{k-1} is an answer (if it were a question, it could not be answered before q_ω which would contradict well-bracketing of τ). And then we can apply the induction hypothesis to $q_{j<.<\omega}$ and $q_{i<.<j}$ and conclude.

□

Lemma 2.38. $\sigma \otimes \tau$ satisfies condition (a') and (c) of well-bracketing.

Proof. • Condition (a'): Let q be a gcc of $(S \parallel C) \otimes (A \parallel T)$ and (q_i, a) be pair Question/Answer in q . Wlog, we can assume q ends at a and $\pi_1 a$ is negative. Since $q_{>i} \cap q^\sigma = \{q_\omega\}$ is complete by Lemma 2.37, we have that $q_{>i}$ is complete as desired.

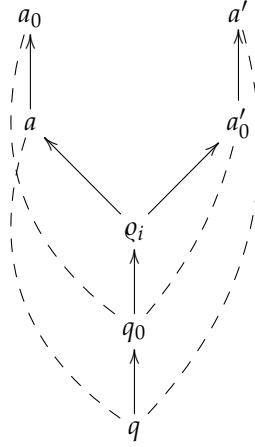
- Condition (c). Let q and q' be forking gccs at $q_i = q'_j$ of the interaction and p a wait event for q and q' with for instance $\pi_1 p$ positive. By visibility we know by Lemma 2.18. Then $\pi_1 q^\sigma$ and $\pi_1 q'^\sigma$ are forking with wait event $\pi_1 p$. Because $\pi_1 p$ is positive, we know that $\pi_1 q_{>i}^\sigma$ and $\pi_1 q'_{>j}^\sigma$ are complete, thus $q_{>i}$ and $q'_{>j}$ are complete by Lemma 2.37.

□

Lemma 2.39. $\sigma \odot \tau$ is well-bracketed.

Proof. We show it satisfies conditions (a'), (b) and (c).

- (a') Let ϱ be a gcc of $S \odot T$ ending at an answer. We can complete it into a gcc of the interaction and apply Lemma 2.38 to conclude as justifiers of visible events are visible.
- (c) Same reasoning as for (a')
- (b) Assume we have two gccs ϱ and ϱ' of the pullback ending by two visible answers to the same question. We write $\varrho_i = \varrho'_i$ for their divergence. We need to show it is visible and positive.
- (a) Claim: there is a question a' in ϱ' that answers the same question as a and happens strictly after ϱ'_j . Indeed, because $\varrho'_{k < \omega}$ is complete, the question answered by a is answered in ϱ' by some a' . Clearly a' cannot belong to ϱ as otherwise, we would have $a < a'$ by minimality of a question which is absurd because a' is in ϱ so happens before the fork and thus before a .
- (b) Claim: Between ϱ'_j and a' no answer points before ϱ'_j . Indeed, call q the question of a and a' and assume there is a'_0 in ϱ' answering q_0 appearing before a' . Then by a similar argument as before there is a a_0 in $\varrho_{j < \omega}$ answering q_0 . Because a is minimal, we have $a \leq a_0$. Since q, q' are both in ϱ they are comparable. Assume eg. $q \leq q_0$. Then at a , q_0 is pending because its answer (a_0) did not occur yet contradicting the fact that q must be the last pending question. See on the picture:



- (c) If $\pi_1 a$ is visible in S and negative, then by courtesy of σ and minimality, the fork must also be the justifier of a and thus positive which is fine. A similar argument can be made if $\pi_2 a$ is visible in T and negative. We derive an absurdity from the fact that $\pi_1 a$ is invisible and we assume wlog that $\pi_1 a$ is positive.

- (d) We call q_a the prefix of q ending at a . We have $q_a^\sigma = q_{\leq i}^- \cdot q_n^+ \cdot a_n^- \cdot \dots \cdot q_1^+ \cdot a_1^- \cdot a^+$ (polarities are for S). We prove that by induction. The first element in q_a before a has to be a question by well-bracketing of q^σ , that will be our a_1 . Since a is positive for S , we have that this answer is negative. Thus before that in q_a^σ there is the justifier to this answer, ie. a question q_1 happening after the fork by (b). This question is negative, and has to be preceded (for S) by either a positive answer a_2 in q_a^σ or the fork q_i , indeed a question distinct from q_i would be pending at a which is absurd because a points before q_i . In the first case we can continue inductively and in the second case we have a sequence of desired shape.
- (e) By (d), we know that $q_i \in q_a^\sigma$ and $q'_j \in q_{a'}^{\sigma}$. Thus we know that q_a^σ and $q_{a'}^\sigma$ are forking at q_i and thus q_i must be positive thus contradicting (d).

□

2.5 Determinism

Since our arenas are conflict-free, it is easy to define the notion of deterministic strategy:

Definition 2.40. *Let $\sigma : S \rightarrow !A$ be a \sim -strategy. It is deterministic when every finite subset of S is consistent. (S is said to be conflict-free)*

Lemma 2.41. *Determinism is stable under parallel composition and lifted maps are deterministic.*

Proof. It is easy to see that if S and T are conflict-free, then so is $S \parallel T$, and by construction \mathbb{C}_A is conflict-free for any arena A . □

Lemma 2.42. *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !A$ be \sim -strategies that are weakly equivalent. Then if σ is deterministic, so is τ .*

Proof. Write $f : S \rightarrow T$ and $g : T \rightarrow S$ for the maps involved in the weak equivalence. Assume we have a configuration x of T that can be extended by two events t and t' (necessarily positive by receptivity) such that $x \cup \{t, t'\}$ is not a configuration of t . There are two cases:

- $gt = gt'$: then we have that $x \cup \{t\} \cong fg(x \cup \{t\}) \cong x \cup \{t'\}$: this contradicts thin because t and t' are positive.
- Otherwise $gt \neq gt'$. Then $f(gx \cup \{gt, gt'\})$ is a configuration of T , that extends $fgx \cup \{fgt\} \cong x \cup \{t\}$. Thus $x \cup \{t\} \xrightarrow{t'_0} \subset$. Likewise $x \cup \{t'\} \xrightarrow{t_0} \subset$ with $x \cup \{t, t'_0\} \cong x \cup \{t', t_0\}$. By restriction, we have $x \cup \{t\} \cong x \cup \{t_0\}$ thus contradicting again thin.

□

Lemma 2.43. *Let $\sigma : S \rightarrow !A^\perp \parallel !B$ and $\tau : T \rightarrow !B^\perp \parallel !C$ be deterministic \sim -strategies. Then $\tau \odot \sigma$ is a deterministic strategy.*

Proof. It follows from the fact that the pullback of two conflict-free event structures is conflict-free and that conflict in the restriction is inherited from the conflict of the interaction. \square

2.6 Reduced form

In this subsection, we write $!^+A$ for the sub-arena of $!A$ consisting in the index functions $\alpha : [a] \rightarrow \omega$ that map negative events to zero.

In all this subsection, let $\sigma : S \rightarrow !A$ be a PCF_{\parallel} -strategy.

Definition 2.44. *First, we define the **reduced form** of S to be the event structure having as set of events:*

$$S_{\text{rf}} = \{s \in S \mid \forall s' \leq s, \text{pol}(s') = - \implies \text{ind}(\sigma s) = 0\}$$

and causality, consistency inherited from S .

There is no need to equip S_{rf} with a symmetry, by the following lemma:

Lemma 2.45. *If $x \stackrel{\theta}{\cong}_S y$ with $x, y \in \mathcal{C}(S_{\text{rf}})$, then $\theta = \text{id}_x$.*

Proof. We prove this by induction on a covering chain:

$$\theta_0 \text{---} \text{C} \theta_1 \text{---} \text{C} \dots \text{---} \text{C} \theta_n$$

for θ , writing $x_i \stackrel{\theta_i}{\cong}_S y_i$. There are two cases, depending on the polarity of the extension in the symmetry.

- If $\theta_i \text{---} \text{C}^{(s_1, s_2)} \theta_{i+1}$ with $\text{pol}(s_1) = \text{pol}(s_2) = -$, then since $s_1, s_2 \in S_{\text{rf}}$ we have $\text{ind}(\sigma s_1) = \text{ind}(\sigma s_2) = 0$. By induction hypothesis $\theta_i = \text{id}_{x_i}$. Therefore, we have:

$$\text{id}_{\sigma x_i} \text{---} \text{C}^{(\sigma s_1, \sigma s_2)} \sigma \theta_{i+1}$$

so by definition of the symmetry on $!A$, $\text{lbl}(\sigma s_1) = \text{lbl}(\sigma s_2)$. But we have already seen that they have the same index as well, and they have the same dependency in σx_1 since symmetries in $!A$ are order-isomorphisms. Therefore, $\sigma s_1 = \sigma s_2$. Since σ is strong-receptive, $s_1 = s_2$.

- If $\theta_i \text{---} \text{C}^{(s_1, s_2)} \theta_{i+1}$ with $\text{pol}(s_1) = \text{pol}(s_2) = +$, then we also have by induction hypothesis that $\theta_i = \text{id}_{x_i}$. So id_{x_i} has two extensions:

$$\text{id}_{x_i} \text{---} \text{C}^{(s_1, s_1)} \quad \text{id}_{x_i} \text{---} \text{C}^{(s_1, s_2)}$$

Since they coincide on the left component and σ is thin, we must have $s_1 = s_2$ and $\theta_{i+1} = \text{id}_{x_{i+1}}$.

□

By definition if $s \in S_{\text{rf}}$ then $\sigma s \in !^+ A$ so by restriction, σ induces a reduced PCF_{\parallel} -strategy $\sigma_{\text{rf}} : S_{\text{rf}} \rightarrow !^+ A$.

Given a reduced form one can recover a PCF_{\parallel} -strategy.

Lemma 2.46. *Let $\sigma : S \rightarrow !^+ A$ be a reduced PCF_{\parallel} -strategy. Then there exists $\bar{\sigma} : \bar{S} \rightarrow !A$ such that $\bar{\sigma}_{\text{rf}} \simeq \sigma$*

Proof. We need to close σ by strong-receptivity. So we build an event structure \bar{S} with events, causality as $!^- S$ and no conflict.

Symmetry on \bar{S} is given as in $!^- A$: a bijection between configuration of \bar{S} is in the symmetry whenever it is an order isomorphism and preserves the label in S .

The map $\bar{\sigma}$ will send $(\alpha^- : [s^-] \rightarrow \omega) \in \bar{S}$ to the event of $!A$ having σs as label and $\text{ind} \alpha$ as copy index. However for $\alpha^+ : [s^+] \rightarrow \omega$, $\bar{\sigma} \alpha^+$ is set to have the same label as σs , but copy index an injective function of the index of $\sigma(s)$ and the copy indices of negative dependencies of α that lie strictly between just α and α .

$\bar{\sigma}$ is locally injective: indeed if there are minimal α, α' consistent in \bar{S} such that $\bar{\sigma}(\alpha) = \bar{\sigma}(\alpha')$, then we have $\text{lbl} \alpha = \text{lbl} \alpha'$ by local injectivity of σ . Moreover, if α and α' are negative, then it means that we have $\text{ind} \alpha = \text{ind} \alpha'$, hence by minimality $\alpha = \alpha'$. If α and α' are positive, then we know also that $\text{ind} \alpha = \text{ind} \alpha'$. By construction, it follows that α and α' have the same negative dependencies between their justifier to themselves, but they have the same justifier since $\bar{\sigma}(\alpha) = \bar{\sigma}(\alpha')$, therefore $\alpha = \alpha'$. The other axioms of maps of event structures are easily verified, and symmetry is preserved.

Courtesy is easily checked because $\alpha \rightarrow \alpha'$ induces $\text{lbl} \alpha \rightarrow \text{lbl} \alpha'$. Strong-receptivity holds by definition: if we have $\alpha \stackrel{\theta}{\cong} \alpha'$ such that $\bar{\sigma} \alpha \stackrel{\sigma \theta}{\cong} \bar{\sigma} \alpha'$ can be extended by (β, β') , then θ induces a configuration of $x \in \mathcal{C}(S)$ such that σx can be extended by a symmetric event $a \in !^+ A$. By receptivity of σ we get an event s with which we can extend α and α' the copy indices of β and β' . Visibility, innocence and well-bracketing follow directly from those for σ and determinism holds by construction. □

As a consequence of this lemma, symmetry is unique up to weak-equivalence: two PCF_{\parallel} -strategies that share the same underlying event structure and map to the game are weakly equivalent. This is not the case without innocence.

Lemma 2.47. *Let $\sigma : S \rightarrow !A$ be a PCF_{\parallel} -strategy. Then for any $x \in \mathcal{C}(S)$ normal, there exists a unique $x \stackrel{\theta}{\cong}_S y$ with $y \in \mathcal{C}(S_{\text{rf}})$.*

Proof. By induction on x . Assume $x \cup \{s\}$ is normal, and that there is a unique $x \stackrel{\theta}{\cong}_S y$ with $y \in \mathcal{C}(S_{\text{rf}})$.

If s is negative minimal, then $x = \emptyset$ – otherwise, $x \cup \{s\}$ would have two minimal events, contradicting normality. Then, θ is extended with (s, s') where

s' is the unique move in S mapping to the minimal event of $!A$ having the same label as σs and copy index 0.

If s is negative non-minimal, then take $s_0 \rightarrow s$ its justifier, and $\alpha_0 = \sigma s_0$. Necessarily we have $(\alpha_0, \alpha'_0) \in \sigma\theta$. Let α' be the unique move of $!A$ with label the same as α , with immediate predecessor α'_0 and copy index 0. Necessarily $\alpha' \notin y$ – otherwise there would be $(\alpha'', \alpha') \in \sigma\theta$ and $s'' \in x$ such that $\sigma s'' = \alpha''$, but then s, s'' would be two Opponent moves with the same justifier, which is forbidden by normality. We extend θ to $\theta \cup \{(s, s')\}$, which satisfies the requirements. By construction, this is the only extension of θ satisfying the requirements.

If s is positive, then we use the extension property to get $\theta \cup \{(s, s')\}$ which satisfies the required properties. For uniqueness, assume there is another extension (s, s'') of θ satisfying the requirement. By determinism, $y \cup \{s', s''\}$ is consistent. Therefore, since σ is thin, $\theta \cup \{(s, s'), (s, s'')\}$ is a valid isomorphism – but this is obviously not the case since it is not injective. \square

Lemma 2.48. *Let $\sigma : S \rightarrow !A$ be a PCF_{\parallel} -strategy. There exists a function $f : S \rightarrow S_{\text{rf}}$ satisfying the following properties:*

- (i) f preserves and reflects the order
- (ii) If $s \in S_{\text{rf}}$, $f(s) = s$
- (iii) For all normal configuration $x \in \mathcal{C}(S)$, $x \cong fx$.

Proof. Take $s \in S$. By Lemma 2.47, there exists a unique $[s] \stackrel{\theta}{\cong}_S y$ with $y \in \mathcal{C}(S_{\text{rf}})$. As a member of the isomorphism family θ preserves and reflects the order, therefore $y = [s']$ for some $s' \in S_{\text{rf}}$, we set $f(s) = s'$. So defined, f preserves and reflects the order by construction. Condition (ii) is satisfied by Lemma 2.45. Finally, (iii) is satisfied by construction. \square

Lemma 2.49. *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !A$ be PCF_{\parallel} -strategies and let $f : S \rightarrow T$ be a rigid map of event structures such that $f; \tau \sim \sigma$. Then f preserves the symmetry on normal configurations.*

Proof. Assume we have $x \stackrel{\theta}{\cong} y$ with $x, y \in \mathcal{C}(S)$ normal configurations. We prove that by induction on θ . For empty θ the result is clear, assume $\theta = \theta_0 \xrightarrow{(s, s')} \text{---}$, and write $x_0 = \pi_1 \theta_0$ and $y_0 = \pi_2 \theta_0$. We have by induction hypothesis $f x_0 \stackrel{f \theta_0}{\cong} f y_0$. Write $f_S : T \rightarrow T_{\text{rf}}$ for the map of Lemma 2.48. Because $f x_0$ and $f y_0$ are normal, we have $f_T(f x_0) \cong f_T(f y_0)$ and hence $f_T(f x_0) = f_T(f y_0)$ by Lemma 2.45. But then $f_T(f x_0)$ can be extended by two events $f_T f s$ and $f_T f s'$. We know that those events have same label as $\text{lbl}(f_T f s) = \text{lbl}(s)$, and same justifier as $f_T f$ is rigid and thus preserves justifier which is in $f_T f x_0 = f_T f y_0$ so it has to be the same. If s and s' are negative then we can conclude by receptivity. Otherwise, by thin we know that we cannot have $[s] = [s']$ (unless $s = s'$) because those events are symmetric. But by induction we have

$f[s] \cong f[s']$ which in turn means $f_T f[s] = f_T f[s']$ and this implies $[s] = [s']$ thus $s = s'$. \square

Lemma 2.50. *Let $\sigma : S \rightarrow !A$ be a PCF_{\parallel} -strategy. There is a weak equivalence between $\overline{\sigma}_{\text{rf}}$ and σ .*

Proof. Define $\varphi : S \rightarrow !^-S$ by $\varphi(s) = (\alpha : [f_S s] \rightarrow \omega)$ (with f_S the function of Lemma 2.48) where $\alpha(s_0^-) = \text{ind}(\sigma_{s_0})$. This map preserves and reflects the order, and $\varphi; \overline{\sigma}_{\text{rf}} \sim \sigma$.

Conversely, we build ψ by induction. Let $\alpha : [s] \rightarrow \omega \in !^-A$. We can consider $\psi[\alpha] \in \mathcal{C}(S)$

- If s is negative, then by receptivity there is a unique s' such that $\psi[\alpha] \xrightarrow{s'} \text{---} \subset$ and $\sigma\psi[\alpha] \cong \sigma[s']$. We have $\varphi s' = \alpha$ by definition and we let $\psi\alpha = s'$
- If s is positive we have $\psi[\alpha] \cong [s]$ so this isomorphism can be extended by a pair (s', s) . By thin such an s' is unique, and by definition we have $f_S s' = s$, thus letting $\psi\alpha = s'$ works.

\square

Lemma 2.51. *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !A$ be PCF_{\parallel} -strategies. The following are equivalent:*

- $\sigma \simeq \tau$
- There exists two maps of event structures $f : S \rightarrow T$ and $g : T \rightarrow S$ such that $f; g \sim \text{id}_S$, $g; f \sim \text{id}_T$, $f; \tau \sim g$ and $g; \sigma \sim \tau$.
- There exists an isomorphism $f : S_{\text{rf}} \rightarrow T_{\text{rf}}$ such that $f; \tau \sim \sigma$

Proof. • We have (i) \Rightarrow (ii)

- (ii) \Rightarrow (iii) Write $f : S \rightarrow T$ and $g : T \rightarrow S$ for the equivalence. We let φ be the map

$$\varphi : S_{\text{rf}} \xrightarrow{\text{is}} S \xrightarrow{f} T \xrightarrow{f_T} T_{\text{rf}}$$

(where f_T is given by Lemma 2.48). φ^{-1} is given by the obvious corresponding map. Let $x \in \mathcal{C}(S_{\text{rf}})$. We have the following calculation:

$$f_S(g(f_T(fx))) \cong f_S(g(f(x))) \cong f_S x \cong x$$

This calculation is valid because x is normal implying that all configurations in the calculations are, so all maps preserve symmetry by Lemma 2.49. Thus φ is an isomorphism and it commutes with the projection on the game up to symmetry.

- (iii) \Rightarrow (i): Assume we have an isomorphism $\varphi : S_{\text{rf}} \simeq T_{\text{rf}}$ commuting with the projection on the game. It is easy to see how φ extends to $\overline{\varphi} : \overline{\sigma}_{\text{rf}} \simeq \overline{\tau}_{\text{rf}}$ by letting $\overline{\varphi}(\alpha : [s] \rightarrow \omega) = [\varphi s] \cong \varphi[s] \cong [s] \xrightarrow{\alpha} \omega$ by rigidity of φ . By transitivity we can then conclude by Lemma 2.50.

\square

$$\begin{array}{c}
\text{NAT} \quad \frac{}{n \Downarrow n} \quad \text{ISZZ} \quad \frac{M \Downarrow 0}{\text{iszero } M \Downarrow \text{true}} \quad \text{ISZS} \quad \frac{M \Downarrow n+1}{\text{iszero } M \Downarrow \text{false}} \quad \text{SUCC} \quad \frac{M \Downarrow n}{\text{succ } M \Downarrow n+1} \\
\\
\text{PREDS} \quad \frac{M \Downarrow n+1}{\text{pred } M \Downarrow n} \quad \text{PREDZ} \quad \frac{M \Downarrow 0}{\text{pred } M \Downarrow 0} \quad \text{TRUE} \quad \frac{}{\text{true} \Downarrow \text{true}} \quad \text{FALSE} \quad \frac{}{\text{false} \Downarrow \text{false}} \\
\\
\text{IFT} \quad \frac{M \Downarrow \text{true} \quad N_1 \Downarrow V}{\text{if } M N_1 N_2 \Downarrow V} \quad \text{IFF} \quad \frac{M \Downarrow \text{false} \quad N_2 \Downarrow V}{\text{if } M N_1 N_2 \Downarrow V} \\
\\
\text{APP} \quad \frac{M \Downarrow (\lambda x.M') \quad M'[N/x] \Downarrow V}{M N \Downarrow V} \quad \text{LAMBDA} \quad \frac{}{\lambda x.M \Downarrow \lambda x.M} \quad \text{FIX} \quad \frac{M (Y M) \Downarrow V}{Y M \Downarrow V}
\end{array}$$

Figure 1: Big-Step operational semantics of PCF

3 Interpretation of PCF and full abstraction

In this section, we detail the interpretation of terms of PCF.

Reminder on PCF We first recall the definition of terms and types of PCF.

Types. $A, B ::= \mathbb{B} \mid \mathbb{N} \mid A \rightarrow B$

Terms. $M, N ::= x \mid \lambda x. M \mid M N \mid Y \mid$
 $\text{tt} \mid \text{ff} \mid \text{if } M_1 M_2 M_3 \mid$
 $n \mid \text{succ } M \mid \text{pred } M \mid \text{iszero } M$

The typing rules are standard, with the exception of the rule for if:

$$\frac{\Gamma \vdash M : \mathbb{B} \quad \Gamma \vdash N_1 : \mathbb{X} \quad \Gamma \vdash N_2 : \mathbb{X}}{\Gamma \vdash \text{if } M N_1 N_2 : \mathbb{X}}$$

This might seem to be quite restrictive, but the full conditional can be recovered by setting if on higher-order type $A_1 \rightarrow \dots \rightarrow A_n \rightarrow \mathbb{X}$ as syntactic sugar for

$$\lambda x^{A_1} \dots \lambda x^{A_n}. \text{if } M (N_1 x_1 \dots x_n) (N_2 x_1 \dots x_n)$$

The (standard) big-step operational semantics is given for reference in Figure 3.

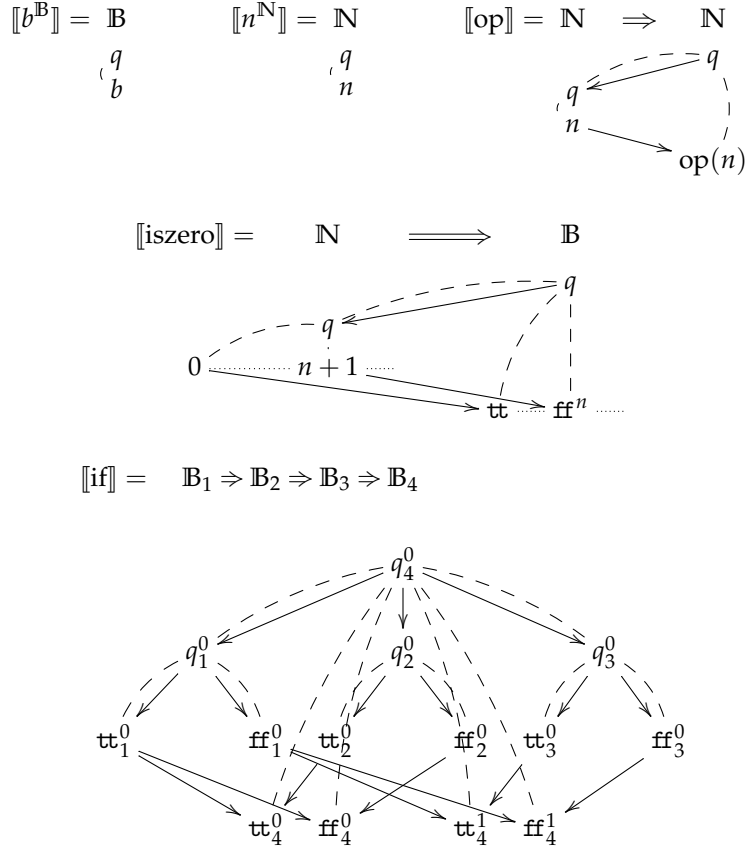


Figure 2: Interpretation of the basic combinators of PCF

3.1 Interpretation, soundness and adequacy

The interpretation follows the standard lines of the interpretation of the simply-typed λ -calculus in a CCC. The additional combinators of PCF are generated from the reduced forms of Figure 2.

3.1.1 Soundness

Soundness will follow from the CCC structure up to weak equivalence, the definition of fixpoints and an analysis of compositions with strategies for the basic combinators of PCF, generated from their reduced forms.

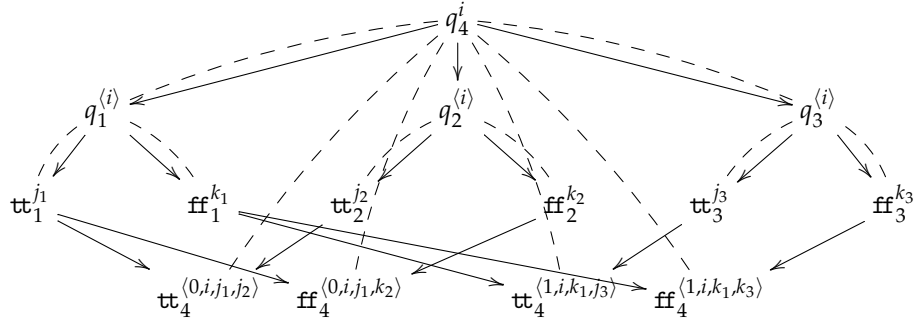
For the ease of writing, in this section we will often write $\sigma : A$ for a PCF type to mean that σ is a Cho-strategy $\sigma : S \rightarrow !A$. We will also often omit the semantic brackets when it is not ambiguous, and write for instance expressions

like $\text{if} \odot \langle \sigma_1, \sigma_2, \sigma_3 \rangle$.

Lemma 3.1. *For any PCF_{\parallel} -strategies $\sigma_1 : \mathbb{B}, \sigma_2 : \mathbb{X}, \sigma_3 : \mathbb{X}$, we have:*

$$\begin{aligned} \text{if} \odot \langle \llbracket \text{tt} \rrbracket, \llbracket v_1 \rrbracket, \llbracket v_2 \rrbracket \rangle &\simeq \llbracket v_1 \rrbracket \\ \text{if} \odot \langle \llbracket \text{ff} \rrbracket, \llbracket v_1 \rrbracket, \llbracket v_2 \rrbracket \rangle &\simeq \llbracket v_2 \rrbracket \\ \text{if} \odot \langle \perp, \llbracket v_1 \rrbracket, \llbracket v_2 \rrbracket \rangle &\simeq \perp \end{aligned}$$

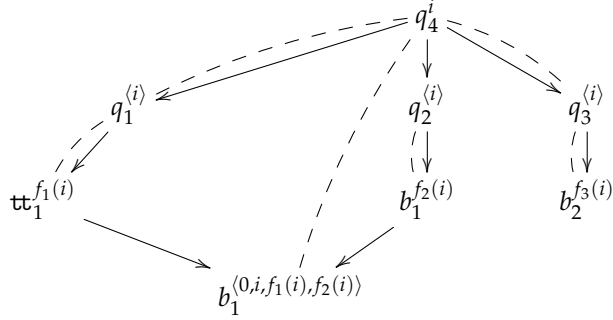
Proof. We treat the case $\mathbb{X} = \mathbb{B}$. By definition of the extended PCF_{\parallel} -strategy corresponding to a reduced form, the full event structure for if can be represented by (for some fixed injections $\langle - \rangle : \omega \rightarrow \omega$ and $\langle -, -, -, - \rangle : \omega^4 \rightarrow \omega$):



Note that the dependency of the final answers to i is not required for this assignment of copy indices to avoid any collisions – we include it because the naive generation of a full strategy from a reduced form in the LICS submission produces it. Note however that it would not be present in the optimized version discussed in this technical report (See Lemma 2.46). Both versions are however weakly equivalent.

It is then direct to compute the interaction pullback, and deduce that the composition satisfies the required laws. We do it here for $\text{if} \odot \langle \llbracket \text{tt} \rrbracket, \llbracket b_1 \rrbracket, \llbracket b_2 \rrbracket \rangle$. Since all strategies involved are deterministic, events in the interaction pullback are entirely determined by compatible pairs of events (taking the primes does not change the event structure up to isomorphism). We display typical configurations of the pullback below, following our usual convention of representing events by their image in the game. The interaction pullback comprises,

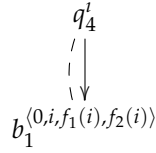
for each $i \in \omega$:



where f_1, f_2 and f_3 are the functions determining the choice of copy indices respectively for $\llbracket \text{tt} \rrbracket$, $\llbracket b_1 \rrbracket$ and $\llbracket b_2 \rrbracket$.

Then, hiding removes from the pullback all events hereditarily justified by q_1 and q_2 , leaving the strategy:

\mathbb{B}



which is as required weakly equivalent to $\llbracket b_1 \rrbracket$. □

Lemma 3.2. *We have the following equations:*

$$\begin{aligned}
 \text{succ} \odot \llbracket n \rrbracket &\simeq \llbracket n + 1 \rrbracket \\
 \text{succ} \odot \perp &\simeq \perp \\
 \text{pred} \odot \llbracket 0 \rrbracket &\simeq \llbracket 0 \rrbracket \\
 \text{pred} \odot \llbracket n + 1 \rrbracket &\simeq \llbracket n \rrbracket \\
 \text{pred} \odot \perp &\simeq \perp \\
 \text{iszero} \odot \llbracket 0 \rrbracket &\simeq \llbracket \text{tt} \rrbracket \\
 \text{iszero} \odot \llbracket n + 1 \rrbracket &\simeq \llbracket n \rrbracket \\
 \text{iszero} \odot \perp &\simeq \perp
 \end{aligned}$$

Proof. Similar verifications as for the previous lemma. □

Proposition 3.3. *For any $\vdash M : \mathbb{X}$, if $M \Downarrow v$, then $\llbracket M \rrbracket \simeq \llbracket v \rrbracket$.*

Proof. By induction on the operational semantics. □

3.1.2 Adequacy

We start by proving a few simple lemmas on PCF_{\parallel} -strategies on ground type.

Lemma 3.4. *If $\sigma : \mathbb{X}$ is a PCF_{\parallel} -strategy, then either $\sigma \simeq \llbracket v \rrbracket$ for some value $\vdash v : \mathbb{X}$, or σ is empty.*

Proof. Take $\sigma : \mathcal{S} \rightarrow !\mathbb{X}$, and assume that it is non-empty: it must have a positive move $s^+ \in S$. By courtesy, the immediate predecessors of s are Opponent moves $q^{i_1}, \dots, q^{i_n} \in S$, that necessarily map to distinct copies of the initial Opponent question. But by single-threadedness (or innocence) there can be only one of them, so $[s] = \{q^i, s\}$. Moreover, q^i enables exactly one positive move. If it enabled another one, it would be compatible since σ is deterministic, which would immediately contradict well-bracketing.

So for each q^i , there is exactly one answer v_i^j . But then the labels of each v_i^j s need to be the same label v . Indeed assume v_i^j and $v_{i'}^{j'}$ have distinct labels. We have $\{q^i\} \stackrel{\{(q^i, q^{i'})\}}{\cong_S} \{q^{i'}\}$ by receptivity, so by extension we must have

$$\{q^i, v_i^j\} \stackrel{\{(q^i, q^{i'}), (v_i^j, v_{i'}^{j'})\}}{\cong_S} \{q^{i'}, v_{i'}^{j'}\}$$

as well. But this maps to a valid isomorphism in $!\mathbb{X}$, so $v_i^j, v_{i'}^{j'}$ have the same label. From there it is immediate to write a weak equivalence between σ and $\llbracket v \rrbracket$. \square

Definition 3.5. *We define a relation \sim_A between closed terms of type A and PCF_{\parallel} -strategies on arena $\llbracket A \rrbracket$ by induction on types, with:*

- $M \sim_{\mathbb{X}} \sigma$ iff: if $\sigma \simeq \llbracket v \rrbracket$, then $M \Downarrow v$.
- $M \sim_{A \rightarrow B} \sigma$ iff for all $N \sim_A \tau$, $MN \sim_B ev \odot \langle \sigma, \tau \rangle$.

Before we prove the fundamental lemma of logical relations, we start with some useful lemmas.

Lemma 3.6. *We have the following properties:*

- (1) *If $M[N/x] N_1 \dots N_p \Downarrow v$, then $(\lambda x. M) N N_1 \dots N_p \Downarrow v$ as well,*
- (2) *If $M(Y M) N_1 \dots N_p \Downarrow v$, then $Y M N_1 \dots N_p \Downarrow v$ as well.*

Proof. (1) By induction on p . For $p = 0$ this is just by definition of the operational semantics. For $p + 1$, Necessarily we have $M[N/x] N_1 \dots N_p \Downarrow \lambda y. M'$ and $M'[N_{p+1}/y] \Downarrow v$. By induction hypothesis, we have $(\lambda x. M) N N_1 \dots N_p \Downarrow \lambda y. M'$ as well. So by definition of \Downarrow we have $(\lambda x. M) N N_1 \dots N_{p+1} \Downarrow v$ as required.

(2) By induction on p . As above, for 0 it is by definition and for $p + 1$ it follows directly by induction hypothesis. \square

Lemma 3.7. *We have the following properties:*

- (1) *If $M[N/x] \sim_A \sigma$, then $(\lambda x. M) N \sim_A \sigma$,*
- (2) *If $M(YM) \sim_A \sigma$, then $YM \sim_A \sigma$.*

Proof. (1) We prove by induction on A the slightly generalized statement that for all p and N_1, \dots, N_p , if $M[N/x] N_1 \dots N_p \sim_A \sigma$, then $(\lambda x. M) N_1 \dots N_p \sim_A \sigma$ as well. For A ground, then it follows directly from Lemma 3.6.

For $A \rightarrow B$, take $T \sim_A \tau$. By hypothesis, we have that $M[N/x] N_1 \dots N_n T \sim_B ev \odot \langle \sigma, \tau \rangle$. By induction hypothesis we have $(\lambda x. M) N N_1 \dots N_p T \sim_B ev \odot \langle \sigma, \tau \rangle$. By definition of logical relations this implies that $(\lambda x. M) N N_1 \dots N_p \sim_{A \rightarrow B} \sigma$ as required.

(2) As above, it is direct to prove by induction on A the generalized statement that for all p and N_1, \dots, N_p , we have that if $M(YM) N_1 \dots N_p \sim_A \sigma$, then $YM N_1 \dots N_p \sim_A \sigma$ as well. For ground types it follows from Lemma 3.6 and otherwise by induction hypothesis. \square

Lemma 3.8. *If $M \sim_A \sigma$ and $\sigma \simeq \tau$, we have $M \sim_A \tau$ as well.*

Proof. Straightforward by induction. \square

Lemma 3.9. *Assume $\sigma : !A$ is empty, then for any $\vdash M : A$, we have $M \sim_A \sigma$.*

Proof. By induction on A . For \mathbb{X} there is nothing to prove, and for $A \rightarrow B$ it immediately boils down to the induction hypothesis. \square

Lemma 3.10. *Assume that we have an ω -chain of PCF_{\parallel} -strategies on an arena A .*

$$\sigma_0 \sqsubseteq \sigma_1 \sqsubseteq \dots$$

Assume moreover that there is a term $\vdash M : A$ such that for any $i \in \omega$, $M \sim_A \sigma_i$, then we have:

$$M \sim_A \bigsqcup_{i \in \omega} \sigma_i$$

Proof. By induction on the type A . Assume first that A is a ground type \mathbb{X} . Assume that $\sqcup_i \sigma_i \simeq \llbracket v \rrbracket$. Then, there must be some $i_0 \in \omega$ such that $\sigma_{i_0} \simeq \llbracket v \rrbracket$ as well. By hypothesis, we can deduce that $M \Downarrow v$ as required, therefore $M \sim_{\mathbb{X}} \sqcup_i \sigma_i$ as well.

Assume now that for all $i \in \omega$, $M \sim_{A \rightarrow B} \sigma_i$. Take $N \sim_A \tau$, we need to prove that $MN \sim_B ev \odot \langle \sqcup_i \sigma_i, \tau \rangle$. But all operations on strategies are continuous for \sqsubseteq , therefore

$$ev \odot \langle \sqcup_i \sigma_i, \tau \rangle = \sqcup_i (ev \odot \langle \sigma_i, \tau \rangle)$$

Now by hypothesis and definition of $\sim_{A \rightarrow B}$, we know that for each i we have $MN \sim_B ev \odot \langle \sigma_i, \tau \rangle$. Therefore by induction hypothesis, $MN \sim_B \sqcup_i (ev \odot \langle \sigma_i, \tau \rangle)$, which concludes the proof. \square

Lemma 3.11. *For any term $x_1 : A_1, \dots, x_n : A_n \vdash M : A$, then for any $N_i \sim_{A_i} \sigma_i$ ($1 \leq i \leq n$), we have that $M[N_i/x_i] \sim_A \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$.*

Proof. By induction on the typing rules for PCF.

- For constants:

$$\Gamma \vdash v : \mathbb{X}$$

Obvious by definition of the interpretation.

- For variables:

$$\frac{}{x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i}$$

Direct from hypothesis, using the CCC laws.

- For application:

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

Direct from hypothesis, using the CCC laws.

- For abstraction:

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B}$$

Direct from hypothesis, using the CCC laws and Lemma 3.6.

- For fixpoints.

$$\frac{\Gamma \vdash M : A \rightarrow A}{\Gamma \vdash YM : A}$$

Assume $\Gamma = x_1 : A_1, \dots, x_n : A_n$, and take $N_i \sim_{A_i} \sigma_i$. Recall that $\llbracket YM \rrbracket$ is defined as $(\sqcup_i F^i(\perp)) \odot \llbracket M \rrbracket$, where:

$$F(\sigma) = \left((A \rightarrow A) \xrightarrow{\langle \text{id}_{A \rightarrow A}, \sigma \rangle} (A \rightarrow A) \times A \xrightarrow{ev_{A,A}} A \right)^{\dagger}$$

We need to check that

$$(YM)[N_i/x_i] \sim_A (\sqcup_i F^i(\perp)) \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$$

First, we prove by induction on i that for all $i \in \omega$,

$$(YM)[N_i/x_i] \sim_A (F^i(\perp)) \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$$

For $i = 0$ it follows from Lemma 3.9. Otherwise, we calculate:

$$\begin{aligned} & (F^{i+1}(\perp)) \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \\ \simeq & ev_{A,A} \odot \langle \text{id}_{A \rightarrow A}, F^i(\perp) \rangle \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \\ \simeq & ev_{A,A} \odot \langle \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle, F^i(\perp) \rangle \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \end{aligned}$$

But by hypothesis we know that $\llbracket M[N_i/x_i] \rrbracket \sim_{A \rightarrow A} \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$, and by induction hypothesis we know that $Y M[N_i/x_i] \sim_A F^i(\perp) \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$. By definition of $\sim_{A \rightarrow A}$, it follows that

$$(M[N_i/x_i]) (Y M[N_i/x_i]) \sim_A ev_{A,A} \odot \langle \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle, F^i(\perp) \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \rangle$$

By Lemmas 3.7 and 3.8, it follows that

$$Y M[N_i/x_i] \sim_A (F^{i+1}(\perp)) \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$$

so we conclude that it is true for any i . Therefore, from Lemma 3.10 we obtain:

$$Y M[N_i/x_i] \sim_A \llbracket Y M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$$

as was required.

- For conditionals.

$$\frac{\Gamma \vdash M_1 : \mathbb{B} \quad \Gamma \vdash M_2 : \mathbb{X} \quad \Gamma \vdash M_3 : \mathbb{X}}{\Gamma \vdash \text{if } M_1 M_2 M_3 : \mathbb{X}}$$

Write $\Gamma = x_1 : A_1, \dots, x_n : A_n$, and take $N_i \sim_{A_i} \sigma_i$. We need to show that $(\text{if } M_1 M_2 M_3)[N_i/x_i] \sim_{\mathbb{X}} \text{if } \odot \langle \llbracket M_1 \rrbracket, \llbracket M_2 \rrbracket, \llbracket M_3 \rrbracket \rangle \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$. By definition of substitution, CCC laws up to weak equivalence and Lemma 3.8, this amounts to:

$$\begin{aligned} (\text{if } (M_1[N_i/x_i]) (M_2[N_i/x_i]) (M_3[N_i/x_i])) \sim_{\mathbb{X}} & \text{if } \odot \langle \llbracket M_1 \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle, \\ & \llbracket M_2 \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \\ & \llbracket M_3 \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \rangle \end{aligned}$$

Assume that the right hand side is weakly equivalent to some $\llbracket v \rrbracket$. Since $\llbracket M_1 \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$ is a PCF_{\parallel} -strategy, by Lemma 3.4 it must be weakly equivalent to some $\llbracket b \rrbracket$, or empty. If it is empty, then so is the right-hand side by Lemma 3.1: absurd. Otherwise, two cases are possible. Wlog, assume that $\llbracket M_1 \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \simeq \llbracket \text{tt} \rrbracket$ and $\llbracket M_2 \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \simeq \llbracket v \rrbracket$. By induction hypothesis, we have that $M_1[N_i/x_i] \sim_{\mathbb{B}} \llbracket M_1 \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$. Therefore by definition of logical relations, it follows that $M_1[N_i/x_i] \Downarrow \text{tt}$. Likewise, we have $M_2[N_i/x_i] \sim_{\mathbb{X}} \llbracket M_2 \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$; so it follows by definition of logical relations that $M_2[N_i/x_i] \Downarrow v$. Therefore, $(\text{if } M_1 M_2 M_3)[N_i/x_i] \Downarrow v$ as required.

- For succ, pred, iszero: similar arguments. We detail it for iszero.

$$\frac{\Gamma \vdash M : \mathbb{N}}{\Gamma \vdash \text{iszero } M : \mathbb{B}}$$

Write $\Gamma = x_1 : A_1, \dots, x_n : A_n$, and take $N_i \sim_{A_i} \sigma_i$. Assume that $\llbracket \text{iszero } M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle \simeq \llbracket b \rrbracket$. We calculate:

$$\llbracket \text{iszero } M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle = \llbracket \text{iszero} \rrbracket \odot \llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$$

Since $\llbracket M \rrbracket \odot \langle \sigma_i \mid 1 \leq i \leq n \rangle$ is a PCF_{\parallel} -strategy, we know by Lemma 3.4 that it is either empty or $\llbracket n \rrbracket$ for some $n : \mathbb{N}$. If it is empty, then so is the right hand side above by Lemma 3.2 – absurd. Otherwise, by Lemma 3.2 again we either have $n = 0$ and $b = \text{tt}$ or $n > 0$ and $b = \text{ff}$. We detail the first case. By induction hypothesis, it follows that $M[N_i/x_i] \Downarrow 0$. By definition of the operational semantics it follows that $(\text{iszero } M)[N_i/x_i] = \text{iszero}(M[N_i/x_i]) \Downarrow \text{tt}$ as required. \square

From that, we can immediately deduce computational adequacy.

Proposition 3.12. *Let $\vdash M : \mathbb{X}$ be a closed term of ground type. Then if $\llbracket M \rrbracket \simeq \llbracket v \rrbracket$, we have $M \Downarrow v$.*

Proof. By Lemma 3.11, we have $M \sim_{\mathbb{X}} \llbracket M \rrbracket$. Computational adequacy follows by definition of logical relations. \square

3.2 Finite definability

Recall that two PCF terms M, N of the same type are **observationally equivalent** iff for all type-compatible context $C[-]$ such that $C[M]$ and $C[N]$ are closed terms of ground type, $C[M] \Downarrow$ iff $C[N] \Downarrow$. We write $M \simeq_{\text{obs}} N$. A similar observational quotient can be performed on PCF_{\parallel} -strategies. A **test** on arena A is any negative PCF_{\parallel} -strategy $\alpha : !A^{\perp} \parallel !\mathbb{X}$. We say that $\sigma, \sigma' : !A$ are **observationally equivalent** iff for all test α , $\alpha \odot \sigma \simeq \alpha \odot \sigma'$. Observational equivalence is generalized in the obvious way to $\sigma, \sigma' : A \xrightarrow{\text{PcfPar}} B$, by setting $\sigma \simeq_{\text{obs}} \sigma'$ iff $\Lambda(\sigma) \simeq_{\text{obs}} \Lambda(\sigma')$.

Although the definition considers arbitrary PCF_{\parallel} -strategies as test, we start as usual by remarking that it suffices to consider only finite ones.

Lemma 3.13. *Let $\sigma : S \rightarrow !A$ and $\tau : T \rightarrow !A$ be PCF_{\parallel} -strategies. Then $\sigma \simeq_{\text{obs}} \tau$ iff for any test $\alpha : A \xrightarrow{\text{PcfPar}} \mathbb{X}$ whose reduced form is finite, we have $\alpha \odot \sigma \simeq \alpha \odot \tau$.*

Proof. The direct implication is clear. Conversely we proceed by contraposition. Assume we have a test $\alpha : A \xrightarrow{\text{PcfPar}} \mathbb{X}$ such that $\alpha \odot \sigma \not\simeq \alpha \odot \tau$ does not hold. Write E for the underlying event structure of α . Assume for instance $\alpha \odot \sigma \simeq \llbracket v \rrbracket$ (by Lemma 3.4). Then there exists an event e of E such that $\text{lbl}(\alpha e) = v$ and a configuration x of S such that $(x, [e])$ is a configuration of the composition. Then consider just the strategy $[e] \xrightarrow{\alpha} !A^{\perp} \parallel !\mathbb{X}$ closed by receptivity (ie. add all negative events not played by $[e]$). This makes a PCF_{\parallel} -strategy α' whose reduced form is finite, and it is easy to see that $(x, [e])$ is still a configuration of $\alpha' \odot \sigma$ thus $\alpha' \odot \sigma \simeq \llbracket v \rrbracket$. But we can't have $\alpha' \odot \tau \simeq \llbracket v \rrbracket$ because this directly implies that $\alpha \odot \tau \simeq \llbracket v \rrbracket$ which is absurd. \square

In order to prove full abstraction, we start as usual by proving definability. We will start by doing it for first-order types (ie. of the form $\mathbb{X}_1 \rightarrow \dots \rightarrow \mathbb{X}_n \rightarrow \mathbb{X}$), then generalize to higher-order types using a decomposition procedure akin to the usual definability process of game semantics.

3.2.1 First-order case

Ground type. We assume familiarity with the standard domain semantics of PCF. For A a PCF type, we write $\llbracket A \rrbracket_{\mathcal{D}}$ for its domain interpretation.

We have seen in Lemma 3.4 that a PCF_{\parallel} -strategy $\sigma : !\mathbb{X}$ is either empty, or is weakly equivalent to $\llbracket v \rrbracket$ for some value $v : \mathbb{X}$. In other words, there is a bijection (up to weak equivalence) between PCF_{\parallel} -strategies $\sigma : !\mathbb{X}$ and $\llbracket \mathbb{X} \rrbracket_{\mathcal{D}}$. For $\sigma : !\mathbb{X}$ we write $\downarrow \sigma \in \llbracket \mathbb{X} \rrbracket_{\mathcal{D}}$, and reciprocally for $d \in \llbracket \mathbb{X} \rrbracket_{\mathcal{D}}$ we write $\uparrow d : !\mathbb{X}$ for the corresponding strategy. We generalize \uparrow to first-order by setting:

$$\begin{aligned} \downarrow \sigma & : \prod_{i \leq n} \llbracket \mathbb{X}_i \rrbracket_{\mathcal{D}} \rightarrow \llbracket \mathbb{X} \rrbracket_{\mathcal{D}} \\ \langle x_i \mid i \leq n \rangle & \mapsto \downarrow (\sigma \odot \langle \uparrow x_i \mid i \leq n \rangle) \end{aligned}$$

Our definability proof for PCF_{\parallel} -strategies on first-order types starts with the corresponding lemma, which is a variant of the standard “linear tests suffice” argument in game semantics.

To prove it, we will make use of the following technical lemma.

Lemma 3.14. *Let $\sigma : \mathcal{S} \rightarrow !A$ be a PCF_{\parallel} -strategy. Take $s^{\mathcal{Q},-}$ a question in \mathcal{S} , and two distinct answers a, a' to s . Assume moreover that for any two questions $q \in [a], q' \in [a']$ such that q, q' symmetric (i.e. $[q] \cong [q']$), answers to q and q' have the same label. Then, a and a' have the same label.*

Proof. We prove that for any two such distinct answers a, a' to the same question $q^{\mathcal{Q},-}$, for any $\varrho \in \text{gcc}(a), \varrho' \in \text{gcc}(a')$ such that symmetric questions in ϱ and ϱ' are answered with the same label, then a, a' have the same label. This is proved by induction on $|\varrho \cup \varrho'|$.

Note that ϱ and ϱ' cannot be included in one another, since their last event is a positive answer (not justifying anything). Necessarily, ϱ and ϱ' are forking at $\varrho_i = \varrho_j$ which is positive. If ϱ_{i+1} was a question, then ϱ_i would necessarily be a question as well (answers do not justify questions). So by well-bracketing it would have an answer in ϱ , which would need to be ϱ_{i+1} by courtesy, so ϱ_{i+1} and ϱ'_{i+1} are answers for $\varrho_i = \varrho'_j$ a question. By hypothesis, ϱ_{i+1} and ϱ'_{j+1} share the same label. By receptivity, it follows that:

$$[\varrho_i] \cup \{\varrho_{i+1}\} \stackrel{\text{id}_{[\varrho_i]} \cup \{(\varrho_{i+1}, \varrho_{j+1})\}}{\cong_{\mathcal{S}}} [\varrho'_j] \cup \{\varrho'_{j+1}\}$$

Let us write $\theta = \text{id}_{[\varrho_i]} \cup \{(\varrho_{i+1}, \varrho_{j+1})\}$. The left projection of θ extends to $[a]$, so θ extends to:

$$[a] \stackrel{\theta'}{\cong_{\mathcal{S}}} [a'']$$

Let us call q'' the image of q through θ' . Then q' and q'' terminate with two answers to q . But $|q'' \cup q'| < |q \cup q'|$, since they have the same length but at least one more event in common. Therefore by induction hypothesis, a'' and a' have the same label, hence a and a' have the same label.

To conclude the lemma, we apply this auxiliary property to any gccs $q \in \text{gcc}(a)$ and $q' \in \text{gcc}(a')$. \square

Lemma 3.15. For $\sigma, \tau : \Pi_{i \leq n} \mathbb{X}_i \xrightarrow{\text{PcfPar}} \mathbb{X}$, $\sigma \simeq_{\text{obs}} \tau$ iff $\downarrow \sigma = \downarrow \tau$.

Proof. \Rightarrow . Assume $\sigma \simeq_{\text{obs}} \tau$, and take $\langle x_i \mid 1 \leq i \leq n \rangle \in \Pi_{i \leq n} [\mathbb{X}_i]_{\mathcal{D}}$. We use the x_i s to define a test, using ccc syntax:

$$\begin{aligned} \alpha & : ((\Pi_{i \leq n} \mathbb{X}_i) \rightarrow \mathbb{X}) \xrightarrow{\text{PcfPar}} \mathbb{X} \\ & = ev \odot \langle \text{id}_{(\Pi_{i \leq n} \mathbb{X}_i) \rightarrow \mathbb{X}}, \langle (\uparrow x_i) \circ ! \mid 1 \leq i \leq n \rangle \rangle \end{aligned}$$

where $! : ((\Pi_{i \leq n} \mathbb{X}_i) \rightarrow \mathbb{X}) \xrightarrow{\text{PcfPar}} 1$ denotes the terminal projection (the empty strategy). Then, we calculate:

$$\begin{aligned} \alpha \odot \Lambda(\sigma) & = ev \odot \langle \text{id}_{(\Pi_{i \leq n} \mathbb{X}_i) \rightarrow \mathbb{X}}, \langle (\uparrow x_i) \circ ! \mid 1 \leq i \leq n \rangle \rangle \odot \Lambda(\sigma) \\ & \simeq ev \odot \langle \Lambda(\sigma), \langle \uparrow x_i \mid 1 \leq i \leq n \rangle \rangle \\ & \simeq \sigma \odot \langle \uparrow x_i \mid 1 \leq i \leq n \rangle \end{aligned}$$

The same reasoning holds for σ' , so from $\sigma \simeq_{\text{obs}} \sigma'$ we can deduce that $\sigma \odot \langle \uparrow x_i \mid 1 \leq i \leq n \rangle \simeq \sigma' \odot \langle \uparrow x_i \mid 1 \leq i \leq n \rangle$. Therefore, $\downarrow \sigma = \downarrow \sigma'$ by definition.

\Leftarrow . By Lemma 3.13, it suffices to consider finite tests. Take a finite test

$$\alpha : (\Pi_{1 \leq i \leq n} \mathbb{X}_i \rightarrow \mathbb{X}) \xrightarrow{\text{PcfPar}} \mathbb{X}'$$

Write \mathcal{T} for its internal event structure with symmetry.

For now on, fix a continuous function $f : \Pi_{1 \leq i \leq n} [\mathbb{X}_i]_{\mathcal{D}} \rightarrow [\mathbb{X}]_{\mathcal{D}}$. Given a question $t^Q \in T$, we say that t is **determined** iff:

- Either, there exists a value v such that for any $\sigma : \Pi_{1 \leq i \leq n} \mathbb{X}_i \xrightarrow{\text{PcfPar}} \mathbb{X}$ (on internal event structure S) such that $\downarrow \sigma = f$, for any interaction pullback $P = (S \parallel !\mathbb{X}') \otimes T$, for any occurrence $p \in P$ of t up to symmetry (ie. $[\pi_2 p] \stackrel{\theta}{\cong}_{\mathcal{T}} [t]$), p has an answer in P with label v ,
- Or, for any such interaction pullback with an occurrence of t up to symmetry, t has no answer.

We prove by induction on $>_{\mathcal{T}}$ that for all $t^Q \in T$, t is determined.

If t^Q is positive, it must be an occurrence of the initial question on \mathbb{X} . For each question t'^Q such that $\tau t \rightarrow \tau t'$, there is by induction a value $v_{t'}$ (possibly including divergence). There are n such questions up to symmetry, so we

have v_1, \dots, v_n . Then it is direct to check that σ has an answer v' to t in any interaction pullback iff $f(\langle v_i \mid 1 \leq i \leq n \rangle) = v'$.

If t^Q is negative, consider the set of answers $a \in T$ to t , such that in the segment $[t, a]$, questions are answered according to their determined value. By Lemma 3.14, any two such answers a, a' must share the same label v . We set t to be determined as v if there is a of label v , and \perp otherwise. Then if t appears in an interaction pullback with σ , all of $[t, a]$ must appear (up to symmetry) by receptivity of σ and determination of the questions in it. Therefore, a appears also in the pullback by receptivity of σ and t is determined.

We have proved that all questions in a finite test are determined. In particular, the initial question of α is determined. Therefore for any two $\sigma, \sigma' : \prod_{1 \leq i \leq n} \mathbb{X}_i \xrightarrow{\text{PcfPar}} \mathbb{X}$ such that $\downarrow \sigma = \downarrow \sigma'$, the initial question of α must have the same answer (if any, up to symmetry) against both σ and σ' , so $\alpha \odot \Lambda(\sigma) \simeq \alpha \odot \Lambda(\sigma')$. \square

Now, we define a **ground term** as a term of PCF extended with a constant $\perp_{\mathbb{X}} : \mathbb{X}$ for divergence, and no sub-terms of function type. Ground terms will suffice to define all functions on first-order types that are definable with finite strategies. We show the following lemma.

Lemma 3.16. *For any ground $\Gamma \vdash M : \mathbb{X}$, $\downarrow \llbracket M \rrbracket = \llbracket M \rrbracket_{\mathcal{D}}$.*

Proof. From the restriction on the type of sub-terms of M it contains only constants, variables, conditionals, first-order operations or divergence. Therefore, the lemma follows from an immediate induction on M , using Lemmas 3.1 and 3.2. \square

Now, we show that any first-order PCF $_{\parallel}$ -strategy is definable by a ground PCF term up to equality of the corresponding continuous functions.

Lemma 3.17. *If $\sigma : \prod_{i \leq n} \mathbb{X}_i \xrightarrow{\text{PcfPar}} \mathbb{X}$ is finite negative, there is a ground term $x_1 : \mathbb{X}_1, \dots, x_n : \mathbb{X}_n \vdash M_{\sigma} : \mathbb{X}$ s.t. $\llbracket M \rrbracket_{\mathcal{D}} = \downarrow \sigma$.*

Proof. By induction on n .

First of all, if σ has no answer to the initial question, then we set $M = \perp_{\mathbb{X}}$, which obviously satisfies the requirements.

Write $\sigma : \mathcal{S} \rightarrow !(\prod_{1 \leq i \leq n} \mathbb{X}_i^{\perp}) \parallel !\mathbb{X}$. If \mathcal{S} contains an answer a (with label v) to the initial question q , then consider first the case where the segment $[q, a]$ contains no positive question. It follows that $[q, a] = \{q, a\}$. Then, it is direct to check that $\downarrow \sigma$ is the constant function v , which is obviously definable with a ground term.

Otherwise, the initial question q has an answer a and the segment $[q, a]$ contains a positive question. Therefore, it contains a *minimal* positive question q' , that necessarily must immediately depend on q , and on no other event; it must be a minimal question in some \mathbb{X}_{i_0} . For each $d \in \llbracket \mathbb{X}_{i_0} \rrbracket_{\mathcal{D}}$, we define

$$\begin{aligned} \sigma_d & : \prod_{i \neq i_0} \mathbb{X}_i \xrightarrow{\text{PcfPar}} \mathbb{X} \\ & = \sigma \odot \langle \omega_1, \dots, \omega_{i_0-1}, \uparrow d, \omega_{i_0+1}, \dots, \omega_n \rangle \end{aligned}$$

But then, we remark that σ_{\perp} is empty. Indeed, if it was not then there would be an answer a' in S to q not depending on q' . Note that a and a' are consistent by determinism. But then gccs of answers a and a' must fork at q which is negative, impossible by well-bracketing.

We now distinguish cases depending on \mathbb{X}_{i_0} . If $\mathbb{X}_{i_0} = \mathbb{B}$, then by induction hypothesis we have M_{tt} and M_{ff} such that $\llbracket M_{\text{tt}} \rrbracket_{\mathcal{D}} = \downarrow \sigma_{\text{tt}}$ and $\llbracket M_{\text{ff}} \rrbracket_{\mathcal{D}} = \downarrow \sigma_{\text{ff}}$. We write:

$$M = \text{if } x_{i_0} M_{\text{tt}} M_{\text{ff}}$$

Then, we have that $\llbracket M \rrbracket_{\mathcal{D}} = \downarrow \sigma$. Indeed, take an input $\vec{d} = \langle d_i \mid 1 \leq i \leq n \rangle$. We reason by cases on d_{i_0} . If $d_{i_0} = \perp$, then $(\downarrow \sigma)(\vec{d}) = \perp$ by the remark above; accordingly $\llbracket M \rrbracket_{\mathcal{D}}(\vec{d}) = \perp$. For $d_{i_0} = \text{tt}$ or ff , then it follows that $\llbracket M \rrbracket_{\mathcal{D}}(\vec{d}) = (\downarrow \sigma)(\vec{d})$ by induction hypothesis.

Suppose now that $\mathbb{X}_i = \mathbb{N}$. Since we do not have a case construct in the language, we need to implement one. For $n \in \omega$, we define a ground term in context $x : \mathbb{N}$ by

$$\begin{aligned} \text{eq}_n(x) &: \mathbb{B} \\ &= \text{iszero}(\text{pred}(\dots(\text{pred } x))) \end{aligned}$$

with n occurrences of pred . Clearly, $\llbracket \text{eq}_n(x) \rrbracket_{\mathcal{D}} : \llbracket \mathbb{N} \rrbracket_{\mathcal{D}} \rightarrow \llbracket \mathbb{B} \rrbracket_{\mathcal{D}}$ is true on input n and false otherwise. Then we remark that because σ is finite, only finitely many answers of σ have other events depending on them. Write $K = \{k_1, \dots, k_p\} \subseteq \omega$ this finite set of answers. For each $k \in K$, by induction hypothesis we have M_k such that $\llbracket M_k \rrbracket_{\mathcal{D}} = \downarrow \sigma_k$.

We define a term in context $x_1 : \mathbb{X}_1, \dots, x_n : \mathbb{X}_n$ with:

$$M = \text{if } \text{eq}_{k_1}(x_{i_0}) M_{k_1} (\text{if } \text{eq}_{k_2}(x_{i_0}) M_{k_2} (\dots (\text{if } \text{eq}_{k_p}(x_{i_0}) M_{k_p} \perp_{\mathbb{X}}) \dots))$$

For the same reasons as for \mathbb{B} , we know that $\downarrow \sigma$ is i_0 -strict. Moreover, for any $\vec{d} \in \prod_{i \leq n} \llbracket \mathbb{X}_i \rrbracket_{\mathcal{D}}$ with $d_{i_0} \notin K$, we have $(\downarrow \sigma)(\vec{d}) = \perp$. Indeed, any answer in S to the initial opponent question must depend on q' by well-bracketing, but by construction no event depends on the answer d_{i_0} in S . Therefore, $\downarrow \sigma$ and $\llbracket M \rrbracket_{\mathcal{D}}$ agree on every input. \square

Putting everything together, we can finally deduce first-order definability up to observational equivalence.

Proposition 3.18. *If $\sigma : \prod_{i \leq n} \mathbb{X}_i \xrightarrow{\text{PcfPar}} \mathbb{X}$ is finite negative, there is $x_1 : \mathbb{X}_1, \dots, x_n : \mathbb{X}_n \vdash M_{\sigma} : \mathbb{X}$ s.t. $\llbracket M \rrbracket \simeq_{\text{obs}} \sigma$.*

Proof. By Lemma 3.17, there is a ground term M such that $\llbracket M \rrbracket_{\mathcal{D}} = \downarrow \sigma$. By Lemma 3.16, it follows that $\downarrow \llbracket M \rrbracket = \downarrow \sigma$. Replacing occurrences of $\perp_{\mathbb{X}}$ in M with $Y(\lambda x. x)$ yields a PCF term M' such that $\llbracket M \rrbracket \simeq \llbracket M' \rrbracket$. Therefore, $\downarrow \llbracket M' \rrbracket = \downarrow \sigma$. By Lemma 3.15, it follows that $\llbracket M' \rrbracket \simeq_{\text{obs}} \sigma$ as required. \square

3.2.2 Higher-order case

We have seen how to define syntactically the finite and first-order strategies. Building on this result, we show how to define in the syntax every finite strategy. In all this section we consider a strategy $\sigma : A_1 \times \dots \times A_n \xrightarrow{\text{PcfPar}} \mathbb{X}$. By the previous section, if A_i are all base types we know how to build a term M_σ such that $\llbracket M_\sigma \rrbracket \simeq_{\text{obs}} \sigma$: we assume that every A_i is an arrow type of the form $A_i = A_{1,i} \rightarrow \dots \rightarrow A_{p_i,i} \rightarrow \mathbb{X}_i$ where \mathbb{X}_i is a base type. We write S for the underlying event structure of σ .

Flow of a strategy We first extract the first-order part of σ , called the *flow* of σ , which is the sub-event structure of S where Opponent only plays answers (intuitively, in that part of the game, Player can call its arguments without having to provide their arguments). This is equivalent to taking all the events of S whose causal history in S is projected in the first-order part of the game $\prod_{1 \leq i \leq n} !\mathbb{X}_i^\perp \parallel !\mathbb{X}$, thus we define the flow σ in this way:

$$S_{\text{flow}} = \{s \in S \mid \sigma([s]) \subseteq !A^0\}$$

where A^0 is the (non-negative) arena $\prod_{1 \leq i \leq n} \mathbb{X}_i^\perp \parallel \mathbb{X}$.

Lemma 3.19. *Inheriting the causality of S , S_{flow} is an event structure which is closed under symmetry inside S thus is an event structure with symmetry. Moreover the inclusion function $S_{\text{flow}} \hookrightarrow S$ is a map of event structures (ie. S is down-closed.)*

Proof. Rigid inclusion. The fact that S_{flow} is an event structure is clear. We show that the inclusion map is a map of event structures, the only thing needed to be checked is that if we have $s \leq s' \in S_{\text{flow}}$ then $s \in S_{\text{flow}}$. This means $[s] \subseteq [s']$ and then $\sigma[s] \subseteq \sigma[s'] \subseteq !A^0$ which means that $s \in S_{\text{flow}}$.

Closed under symmetry. Let $x \cong_S^\theta y$ in the symmetry of S and $s \in x \cap S_{\text{flow}}$. Then we know that $\sigma[s] \subseteq A^0$. Because σ preserves the symmetry and θ is rigid, θ restricts to an isomorphism $\sigma[\theta s] \cong \sigma[s] \subseteq A^0$. The conclusion follows because A^0 is closed under symmetry in $!A$. \square

We call those events $q^{(+, \text{Qu})} \in S_{\text{flow}}$ that are in S_{rf} , the reduced form of S , the **primary questions** of S . Each such q maps to an initial question of some A_i . We write Q_i for the set of primary questions mapping to A_i .

Write $f_S : S \rightarrow S_{\text{rf}}$ for the function given by Lemma 2.48. We can define a function σ_{flow} from S_{flow} to $\prod_{1 \leq i \leq n, q \in Q_i} !\mathbb{X}_i^\perp \parallel !\mathbb{X}$ by case, for $s \in S_{\text{flow}}$:

- If $\sigma s = (2, a \in \mathbb{X})$, then $\sigma_{\text{flow}} s = (2, a) = \sigma s$
- If $\sigma s = (1, (i, q \in \mathbb{X}_i))$ is positive (and thus a question), we set $\sigma_{\text{flow}}(s) = (1, i, f_S q, q)$
- If $\sigma s = (1, (i, a \in \mathbb{X}_i))$ is negative (and thus an answer), we set $\sigma_{\text{flow}}(s) = (1, i, f_S(\text{just } a), a)$.

Lemma 3.20. σ_{flow} is a PCF_{\parallel} -strategy.

Proof. Most results follow from the inclusion $S_{\text{flow}} \hookrightarrow S$ being rigid.

- *Map of event structures.* Let $s \in S_{\text{flow}}$ and $a \in !\prod_{1 \leq i \leq n, q \in Q_i} !\mathbb{X}_i^{\perp} \parallel \mathbb{X}$ such that $a \rightarrow \sigma_{\text{flow}}s$. We need to show that a is in the image of σ_{flow} . Let $s' = \text{just } a \in S_{\text{flow}}$ (because S_{flow} is down-closed). $\sigma s' = a$ follows from a case analysis.

Assume now we have $\sigma_{\text{flow}}s = \sigma_{\text{flow}}s'$ for some $s, s' \in S_{\text{flow}}$. If $\sigma_{\text{flow}} \in \mathbb{X}$ then $s = s'$ because $\sigma_{\text{flow}}s = \sigma s$ in that case. Otherwise $\sigma_{\text{flow}}s = (1, (i, q_p, a))$. Then we know $\sigma s = (1, (i, a)) = \sigma s'$ thus $s = s'$.

- *Courtesy.* Consequence of the rigid inclusion
- *Strong-receptivity.* Let $x \in \mathcal{C}(S_{\text{flow}})$ such that σ_{flow} can extend x by a negative a . Then there exists $s \in S$ such that $x \cup \{s\} \in \mathcal{C}(S)$ and $\sigma s = a$. By definition of S_{flow} , $s \in S_{\text{flow}}$.

Strong-receptivity follows from that of S .

- *Visibility.* If q is a gcc of S_{flow} , then by rigidity of the inclusion it is a gcc of S and then $\sigma q \in \mathcal{C}(!A)$ from which the result follows by a reasoning similar as for the previous item.
- *Innocence.* Let $[s]$ be a prime configuration of S_{flow} , it is also a prime configuration of S and thus is normal in S and in S_{flow} (which is the same thing).
- *Well-bracketing.* Reasoning similar as for visibility and innocence.
- *Determinism.* Follows from the determinism of S .

□

Argument strategies To capture how σ answers when Opponent asks for the arguments to the functions called by σ , we describe the sub-strategies of σ responsible for providing Opponent with these arguments. Following a primary question $q \in Q_i$ there are $\omega \times p_i$ Opponent questions, one for each copy index and each argument to the call. For $k \in \omega$ and $1 \leq j \leq p_i$, we write $q_{q,j}^k$ for the corresponding move in S (which exists and is unique by receptivity). We set:

$$S_{q,j} = \{s \in S \mid \exists k \in \omega, s \geq q_{q,j}^k\}$$

This is an event structures with components inherited from S by hiding. In the rest of the paragraph, we consider a primary question $q \in Q_i$ fixed.

The symmetry on $S_{q,j}$ consists in those θ between configurations of $S_{q,j}$ such that $\theta \cup \text{id}_{[q]}$ is a symmetry on $S_{q,j}$. All the properties are easy to check but the extension. Let $\theta : x \cong y$ be an isomorphism in this family and let s be an extension of x in $S_{q,j}$. We know that $x \cup [q]$ can be extended by s , thus $[q] \cup y$

can be extended by s' , and θ can be extended by $\{s, s'\}$. We are left to check that $s' \in S_{q,j}$. Since $s \in S_{q,j}$ we have $s \geq q_{q,j}^k$. Because $\text{id}_{[q]} \cup \theta$ is an isomorphism in S , we must have

$$\text{just}((\text{id}_{[q]} \cup \theta)(q_{q,j}^k)) = (\text{id}_{[q]} \cup \theta)(\text{just}(q_{q,j}^k)) = q.$$

Then $\theta(q_{q,j}^k) = q_{q,j}^{k'}$ and since θ preserves the order we have $\theta s \geq q_{q,j}^{k'}$, $j \in S_{q,j}$ as wanted.

Lemma 3.21. $\sigma(S_{q,j})$ does not intersect $!\mathbb{X}$.

Proof. It cannot contain initial questions that are minimal in S . Suppose it contains a move s which is sent to an answer in \mathbb{X} , and consider a gcc of s in S that goes through some $q_{q,j}^k$ (and as a side-effect through q).

The question q cannot have answers in $[s]$ because $[s]$ needs to be normal by innocence. Thus when s is played q is still pending which contradicts the well-bracketing (s answers a question which is before a pending question). \square

To handle the reassignment, consider the following map

$$\varphi_{q,j} : \prod !A_i^\perp \rightarrow \prod !A_i^\perp \parallel !A_{i,j}$$

defined as follows:

- $\varphi_{q,j}(i, \alpha : [a] \rightarrow \omega) = (2, \alpha')$ if $a \in A_{i,j}$ and $\alpha(q_{A_i}) = \text{ind}(\sigma q)$ where q_{A_i} is the initial question of A_i (α' is the restriction of α to A_{i,G_j}).
- Otherwise: $\varphi_{q,j}(p, \alpha : [a] \rightarrow \omega) = (1, (p, \alpha))$

Lemma 3.22. $\varphi_{q,j}$ is a map of event structures.

Proof. Injectivity is clear. Assume we have $\beta = \varphi_{q,j}(p, \alpha)$ and $\beta_0 \leq \beta$. There are two cases:

- $\beta \in !A_{i,j}$: then $\alpha : [a]_{A_i} \rightarrow \omega$. Write $\beta_0 : [b]_{A_{i,j}} \rightarrow \omega$. By construction $[b]_{A_i} = [b]_{A_{i,j}} \cup \{q_{A_i}\}$ thus we let $\alpha_0 = \beta_0 \cup \{q_{A_i} \mapsto \alpha(q_{A_i})\}$. By construction we have $\varphi_{q,j}(\alpha_0) = \beta_0$.
- $\beta \in \prod !A_i^\perp$, then $\beta = (1, \alpha)$ and $\varphi_{q,j}(1, \beta_0) = (1, (p, \alpha))$.

\square

The map $\varphi_{q,j}$ is also surjective but not rigid though, it breaks the link $\sigma q \rightarrow \sigma q_{q,j}^k$. But it is on the image of σ :

Lemma 3.23. Let $s, s' \in S_{q,j}$ such that $\sigma s \leq \sigma s'$. Then $\varphi_{q,j}\sigma s \leq \varphi_{q,j}\sigma s'$.

Proof. By case analysis. The only non-trivial case is when $\sigma s = \sigma q$ but that cannot happen if $s \in S_{q,j}$ by local injectivity of σ . \square

Thus we define the j th argument strategy for q as $\sigma_{q,j} = \sigma; \varphi_{q,j} : S_{q,j} \rightarrow \prod A_i^\perp \parallel A_{i,j}$.

Lemma 3.24. $\sigma_{q,j}$ is a PCF_\parallel -strategy.

Proof. • *Map of event structures.* Consequence of Lemma 3.23.

- *Courtesy.* Consequence of Lemma 3.23
- *Strong-receptivity.* Let $x \in \mathcal{C}(S_{q,j})$ such that $\sigma_{q,j}x$ can be extended by a . Because $\varphi_{q,j}$ is surjective, we know that $[x]_S$ can be extended by s to match a through $\varphi_{q,j}$. The only thing we need to check is that s actually lives in $S_{q,j}$. If a is minimal, then it must be that $s = q_{q,j}^k$ for some k and thus lives in $S_{q,j}$. Otherwise, we use the fact that $S_{q,j}$ is upward-closed. Strong-receptivity comes from that of σ .
- *Visibility.* Let ϱ be a gcc of $S_{q,j}$. We can complete it into a gcc in S and we know that $\sigma[\varrho]_S$ is a configuration of $\prod !A_i^\perp \parallel \mathbb{X}$. Thus $\sigma\varrho$ is a down-closed subset of $\prod !A_i^\perp$ and by Lemma 3.23, $\varphi_{q,j}\sigma\varrho$ is down-closed.
- *Innocence.* Let $s \in S_{q,j}$. $[s]_S$ is normal by innocence of σ thus so is $[s]_{S_{q,j}}$.
- *Well-bracketing.* All conditions are trivial and do not talk about the projection on the game (beyond the labels) so this clearly comes from the properties on σ .
- *Determinism.* Consequence from that of S .

□

Decomposition of configurations of S Before showing we can recompose σ from its flow and argument strategies we show that there is a unique decomposition of configurations of S using those components:

Lemma 3.25. *The sets S_{flow} and $(S_{q,j})_{q,j}$ form a partition of S , thus any configuration of S can be uniquely split in a decomposition with components into those sets. Moreover such a decomposition $(x_f \in \mathcal{C}(S_{\text{flow}}), (x_{q,j} \in \mathcal{C}(S_{q,j}))_{q,j})$ yields a configuration of S (by doing the union of the components) if and only if the following condition holds:*

$$\forall q,j, x_{q,j} \neq \emptyset \Rightarrow q \in x_f$$

Proof. We first notice that all those sets are disjoint:

- $S_{\text{flow}} \cap S_{q,j} = \emptyset$: assume we have a minimal $s \in S_{\text{flow}} \cap S_{q,j}$. Then $s > q_{q,j}^k$ (for some k) and by Lemma 3.21, s has to be a negative answer to some primary question. By considering a causal chain of s inside $S_{q,j}$ and another inside S_{flow} from q to s , by minimality they have to be disjoint. Thus we can get two gccs of S forking at q containing s . By well-bracketing.(c) the two branches must be complete, in particular the $q_{q,j}^k$ must be answered below s which is absurd because it would be a non-maximal positive answer which cannot be because we are playing on a PCF type.

- $S_{q,j} = S_{q',j'} = \emptyset$ ($j \neq j'$): direct consequence of innocence as the prime configuration of any event of the intersection would not be normal.
- $S_{q,j} \cap S_{q',j'} = \emptyset$ ($q \neq q'$). Consequence of well-bracketing.(c). Any event s in $S_{q,j} \cap S_{q',j'}$ would induce two gccs forking at the initial question and containing q and q' as well as some $q_{q,j}^k$ and $q_{q',j'}^{k'}$ that would have to be answered below s – but this is not possible as seen above.

Moreover assume we have an event $s \in S$ such that $s \notin S_{\text{flow}}$. So there is a minimal $s_0 \in [s]$ such that s_0 is not sent to the first-order part of A . By minimality, it has to be a question of the form $q_{q,j}^k$ (with q a primary question), because those are the minimal events not sent to $!A^0$. From that it follows that $s \in S_{q,j}$.

Finally assume we have a decomposition $(x_f, (x_{q,j}))$ satisfies the condition of the lemma, we want to show the union $x_f \cup \bigcup x_{q,j}$ is down-closed. x_f is down-closed in S so there is no problem here. The only problem is if $q_{q,j}^k \in x_{q,j}$. Then its only predecessor (as it is a negative move) is the primary question q that does not belong to $x_{q,j}$. But by assumption since $x_{q,j} \neq \emptyset$ in that case, $q \in x_f$. \square

Thus, the set of such decompositions, ordered by pairwise inclusion forms an order isomorphic to $(\mathcal{C}(S), \subseteq)$. We use that in the following paragraph to recover σ .

Putting it together We have our decomposition and the following lemmas show it is actually correct. First, given $q \in Q_i$ we form σ_q as the following composition (in Cho):

$$\prod A_i \xrightarrow{\langle \pi_i, \langle \sigma_{q,j} \mid j \leq p_i \rangle \rangle} A_i \times \prod_j A_{i,j} \xrightarrow{ev} \mathbb{X}_i.$$

A few notations:

- If $x \in \mathcal{C}(\mathbb{C}_A)$, x is a pair $(x_1, x_2) \in \mathcal{C}(!A_i)$
- If $x \in \mathcal{C}(A_i)$ we write x^j for the projection of x to $A_{i,j}$ and similarly $x^{\mathbb{X}}$ for the projection to \mathbb{X}_i of x .

The configuration domain of S_q is actually easy to concretely represent. Define the following order \mathcal{D} as the set of tuples

$$(w, x_1, \dots, x_{p_i}, z) \in \mathcal{C}(\mathcal{C}(A_i)) \times \mathcal{C}(S_{q,1}) \times \mathcal{C}(S_{q,p_i}) \times \mathcal{C}(!A_i)$$

satisfying:

- $z^j = \sigma_{q,j}^2 x$ where $\sigma_{q,j}^2$ is the second component of $\sigma_{q,j}$
- $z \sqsubseteq w$

ordered by pairwise inclusion. Moreover define a map $d : \mathcal{D} \rightarrow \mathcal{C}(!A_i^\perp \parallel \mathbb{X}X_i)$ as $(w, x_1, \dots, x_n, z) \mapsto w \parallel z^\mathbb{X}$. In this paragraph we use the Lemma 2.20 and the characterization of configurations of the pullback (Proposition 2.13 of [CCW14]).

Lemma 3.26. *There is an order-isomorphism φ between $(\mathcal{C}(S_q), \subseteq)$ and \mathcal{D} that satisfies $\varphi; d = \sigma_q$.*

Proof. Given a configuration x of a $S_{q,j}$ we write x^* for the maximal sub-configuration of x containing no maximal negative events sent to $!A_{i,j}$.

Consider a configuration of S_q . It corresponds to a configuration of the interaction such that all maximal events are visible. As we have seen it is a tuple $(w \in \mathcal{C}(\mathbb{C}_{!A_i}), x_1 \in \mathcal{C}(S_{q,1}), \dots, x_{p_i} \in \mathcal{C}(S_{q,p_i}), z \in \mathcal{C}(\mathbb{C}_{A_i}))$ with the synchronization condition. The visible events are those projected to $!A_i^\perp \parallel !\mathbb{X}$, in other terms w_1 and $z_2 \cap \mathbb{X}$. Moreover, since $z_1 = w_2$ we have $z_2 \sqsubseteq z_1 = w_2 \sqsubseteq w_1$. Because maximal events are visible we must have $z_1 = w_2 = z_2 \cap w_1$. For the same reason, we must have $z_2^j \supseteq^- \sigma^2(x_j)$. By receptivity there are $x'_j \supseteq x_j$ such that $\sigma^2 x'_j = z_2^j$.

Define $\varphi(w, x_1, \dots, x_n, z) = (w_1, x'_1, \dots, x'_{p_i}, z_2)$. We have $\varphi; d = \sigma_q$. It is easy to see that φ preserves and reflects the order. It is also injective because given the x'_1, \dots, x'_{p_i} , the x_1, \dots, x_{p_i} are fixed ($x_j = x_j^*$). It is easy then easy to see that φ is injective. The inverse of φ is given by

$$\varphi^{-1}(w, x_1, \dots, x_{p_i}, z) = ((w, w \cap z), x_1^*, \dots, x_{p_i}^*, ((w \cap z), z))$$

□

Writing $Q = \cup_i Q_i$, consider:

$$\sigma' = \prod A_i \xrightarrow{\langle \sigma_q | q \in Q \rangle} \prod_{i, q \in Q_i} \mathbb{X}_i \xrightarrow{\sigma_{\text{flow}}} \mathbb{X}$$

Lemma 3.27. *Configurations of σ' are order-isomorphic to the decomposition of configurations of S seen in the last paragraph.*

$$((x_{q,j} \in \mathcal{C}(S_{q,j})_{q \in Q, i, j \leq p_i}, x_f \in \mathcal{C}(S_{\text{flow}}))$$

Proof. Configurations of σ' are given by tuples

$$(((w_q, x_{q,1}, \dots, x_{q,p_i}, z_q) \in \mathcal{C}(S_q))_{q \in Q}, x_f \in \mathcal{C}(S_{\text{flow}}))$$

such that $\prod_{q \in Q} z_q^\mathbb{X} = \sigma_{\text{flow}}^1 x_f$ with no maximal invisible events. Using that, by a similar argument as for the previous lemma, we extend the $x_{q,j}$ to $x'_{q,j}$ so that $w_q^j = x'_{q,j}$. Moreover, w_q can contain negative answers that are not synchronized back to x_f , again we can extend x_f to x'_f to include all those

answers. To show $(x'_f, (x'_{q,j}))$ is a valid decomposition, we assume $x'_{q,j}$ is not empty. Then as it is down-closed, it contains a question $q^k_{q,j}$. Thus $\sigma q^k_{q,j} \in w_q$ and thus $\sigma q \in w_q$ because it is down-closed. σq is negative in $!A_i$ and since $z_q \sqsubseteq w_q$ we have $\sigma q \in z_q^{\mathbb{X}} \subseteq \sigma^1 x_f$ and $q \in x_f$.

Conversely, if we are given x_f and all the $x_{q,j}$, whose union is the whole configuration $x \in \mathcal{C}(S)$, we can let $w_q = \sigma(x) \cap !A_i$ and z_q to be the maximal sub-configuration of w_q with no maximal positive answers. □

We have the following result:

Proposition 3.28. *σ and σ' are weakly equivalent strategies.*

Proof. We use Lemma 2.51 not to have to deal with the symmetry. By Lemma 3.27, it follows that their domain of configurations is isomorphic and it is easy to check the projection on the game commutes with that isomorphism. Thus $\sigma \simeq \sigma'$. □

Proposition 3.29 (Finite definability). *Let A be a type of PCF and $\sigma : S \rightarrow ![A]$ a PCF_{\parallel} -strategy whose reduced form is finite (S is always infinite by receptivity). There exists a term M of PCF such that $\llbracket M \rrbracket \simeq_{\text{obs}} \sigma$.*

Proof. If A is first-order, we can conclude by Proposition 3.18. Otherwise, we reason by induction on the size of the reduced form of σ .

First, we consider a term $M_{\text{flow}} : \prod_{i,q \in Q_i} \mathbb{X}_i \rightarrow \mathbb{X}$ such that $\llbracket M_{\text{flow}} \rrbracket \simeq \sigma_{\text{flow}}$ (again by Proposition 3.18).

Consider the argument strategies $\sigma_{q,j}$. Their reduced forms are included in that of S , and it does not contain q or the initial question of \mathbb{X} so it is strictly smaller. Thus by induction, we get a term $M_{q,j}$ such that $\llbracket M_{q,j} \rrbracket \simeq_{\text{obs}} \sigma_{q,j}$.

Because \simeq_{obs} is a congruence and preserves pairing, it follows that (by Proposition 3.28) :

$$\llbracket M_{\text{flow}} \rrbracket (\llbracket M_{q,j} \rrbracket | q \in Q_i, j \leq p_i) \simeq_{\text{obs}} \sigma_{\text{flow}} \odot \langle \sigma_{q,j} | q, j \rangle \simeq_{\text{obs}} \sigma$$

Thus the term $M(M_{q,j} | q \in Q_i, j \leq p_i)$ is a term defining σ up to \simeq_{obs} . □

3.3 Full abstraction

In this section we prove the full abstraction theorem (Theorem 3.30), which relates the observational equivalence of the syntax and the semantics.

Theorem 3.30 (Full abstraction). *If $\Gamma \vdash M, N : A$ then $M \simeq_{\text{obs}} N$ iff $\llbracket M \rrbracket \simeq_{\text{obs}} \llbracket N \rrbracket$.*

Proof. Assume $\llbracket M \rrbracket \simeq_{\text{obs}} \llbracket N \rrbracket$, and let $C : A \rightarrow \mathbb{X}$ be a context. By definition $\llbracket C \rrbracket : \llbracket A \rrbracket \xrightarrow{\text{PcfPar}} \llbracket \mathbb{X} \rrbracket$ is a test, thus $\llbracket C \rrbracket \odot \llbracket M \rrbracket \simeq \llbracket C \rrbracket \odot \llbracket N \rrbracket$. There are two cases:

- $\llbracket C \rrbracket \odot \llbracket M \rrbracket \simeq \llbracket v \rrbracket$ for some $v \in \mathbb{X}$. Then $C(M) \Downarrow v$ by adequacy (Proposition 3.12) and for the same reason $C(N) \Downarrow v$

- If $\llbracket C \rrbracket \odot \llbracket M \rrbracket$ is empty, then $\llbracket C \rrbracket \odot \llbracket N \rrbracket$ is empty too, thus for every v , $C(M)$ and $C(N)$ do not converge to v (by soundness).

Thus M and N both pass the same tests.

Conversely, assume $M \simeq_{\text{obs}} N$. Assume we do not have $\llbracket M \rrbracket \simeq_{\text{obs}} \llbracket N \rrbracket$.

Then by contraposition of Lemma 3.13, there exists $\alpha : A \xrightarrow{\text{PcfPar}} \mathbb{X}$ whose reduced form is finite such that $\alpha \odot \llbracket M \rrbracket \not\approx \alpha \odot \llbracket N \rrbracket$. By Proposition 3.29, there exists $M_\alpha : A \rightarrow \mathbb{X}$ such that $\llbracket M_\alpha \rrbracket \simeq_{\text{obs}} \alpha$. Thus we have $\llbracket M_\alpha M \rrbracket \not\approx \llbracket M_\alpha N \rrbracket$. It is a direct consequence of adequacy that $M_\alpha M \not\approx_{\text{obs}} M_\alpha N$ since both have a base type, hence a contradiction. \square

References

- [CCW14] Simon Castellan, Pierre Clairambault, and Glynn Winskel. Concurrent hybrid-ong games, 2014. On <http://arxiv.org/abs/1409.7542>.