Implicit automata in typed λ-calculi

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Simply typed functions on Church numerals

Church encodings of (unary) natural numbers:

- $\mathsf{Nat} = (o \to o) \to o \to o$
- $n \in \mathbb{N} \rightsquigarrow \overline{n} = \lambda f.\ \lambda x.\ f\,(\dots\ (f\,x)\dots) : \mathsf{Nat}$ with n times f
- all inhabitants of Nat are equal to some $\overline{n}$ up to $=_{\beta\eta}$

Theorem (Schwichtenberg 1975)

The functions $\mathbb{N} \to \mathbb{N}$ definable by simply-typed λ-terms of type $\mathsf{Nat} \to \mathsf{Nat}$ are the extended polynomials.

(generated by $0, 1, +, \times$, `id` and `ifzero`)

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Theorem (Schwichtenberg 1975)

The functions $\mathbb{N} \to \mathbb{N}$ definable by simply-typed λ-terms of type $\mathsf{Nat} \to \mathsf{Nat}$ are the extended polynomials.

(*generated by* $0, 1, +, \times,$ `id` *and* `ifzero`)

Let's add a bit of (meta-level) polymorphism: $t = \mathsf{Nat}[A] \to \mathsf{Nat}$
where $\mathsf{Nat}[A] = \mathsf{Nat}[A/o] = (A \to A) \to A \to A$

Open question

Choose some simple type A and some term $t : \mathsf{Nat}[A] \to \mathsf{Nat}$.
What functions $\mathbb{N} \to \mathbb{N}$ can be defined this way?

Simply typed functions on Church-encoded strings

To gain more insight, let's *generalize*! $\mathsf{Nat} = \mathsf{Str}_{\{1\}}$

Church encodings of *strings* over alphabet $\Sigma = \{a, b\}$:

- $\mathsf{Str}_{\{a,b\}} = (o \to o) \to (o \to o) \to o \to o$
- $abb \in \{a, b\}^* \rightsquigarrow \overline{abb} = \lambda f_a.\ \lambda f_b.\ \lambda x.\ f_a\ (f_b\ (f_b\ x)) : \mathsf{Str}_\Sigma$

More generally $\mathsf{Str}_\Sigma = (o \to o) \to \ldots |\Sigma| \text{ times} \ldots \to (o \to o) \to o \to o$

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Without input type substitutions, an answer is known [Zaionc 1987].

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An answer for predicates [Hillebrand & Kanellakis 1996]

A subset of Σ^* is decidable by some $t : \mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$ if and only if it is a *regular language*.

Note: unary regular languages $\cong$ ultimately periodic subsets of $\mathbb{N}$

λ-definable functions are regular

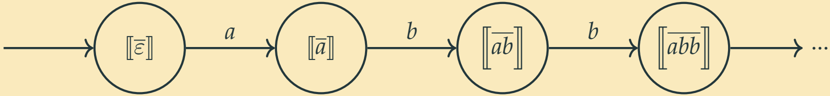
Theorem [Hillebrand & Kanellakis, LICS'96]

For any type A and any simply typed λ-term $t : \mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$, $\{w \in \Sigma^* \mid t\,\overline{w} =_\beta \mathtt{true}\}$ is *regular*.

Proof by semantic evaluation.

Let $[\![-]\!]$ stand for a denotational semantics in the *CCC of finite sets*. (determined by $[\![o]\!]$)

We build an automaton with *finite* set of states $Q = [\![\mathsf{Str}_\Sigma[A]]\!]$



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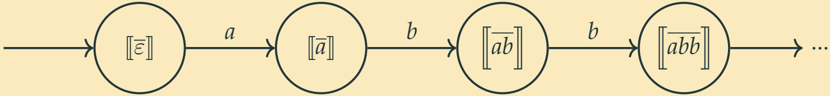

$$t\,\overline{w} =_\beta \mathtt{true} \iff [\![t]\!]([\![\overline{w}]\!]) = [\![\mathtt{true}]\!] \iff w \text{ accepted}$$

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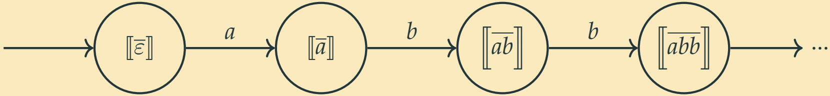

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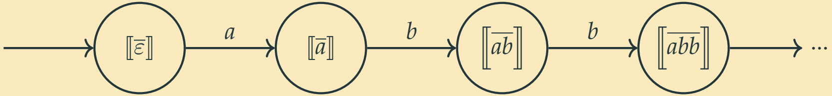

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Similar ideas in higher-order model checking (see e.g. Grellois & Melliès)

Regular functions

Assume a λ-calculus for linear intuitionistic logic with additives

- $\lambda^{\to} x.\, t : A \to B$ unrestricted function
- $\lambda^{\circ} x.\, t : A \multimap B$ linear function (exactly one x in t)
- coproducts $A \oplus B$ and products $A \,\&\, B$

Church encoding with linear types [Girard 1987]:

$$\overline{abb} = \lambda^{\to} f_a.\ \lambda^{\to} f_b.\ \lambda^{\circ} x.\ f_a\ (f_b\ (f_b\ x)) : \mathsf{Str}_{\{a,b\}} = (o \multimap o) \to (o \multimap o) \to o \multimap o$$

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Today's main theorem [Nguyễn & P.]

$f : \Gamma^* \to \Sigma^*$ is a *regular function*

$\iff$

f is defined by some $t : \mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ in the intuitionistic linear λ-calculus
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Regular functions are a classical topic, many equivalent definitions...
One of them: **copyless** *streaming string transducers* [Alur & Černý 2010]
$\rightsquigarrow$ sounds suspiciously like affine types!

Single-state streaming string transducers

Definition

- Finite set of Σ^*-valued *registers* e.g. $R = \{X, Y\}$
- Initial values $R \to \Sigma^*$ e.g. $X_{\text{init}} = Y_{\text{init}} = \varepsilon$
- *Register update function* e.g. $a \mapsto \begin{cases} X := Xa \\ Y := aY \end{cases}$ $b \mapsto \begin{cases} X := Xb \\ Y := bY \end{cases}$ $c \mapsto \begin{cases} X := aba \\ Y := YabaX \end{cases}$
- "output function" e.g. $\text{out} = XY$

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Execution over *abaa*: start with

$$X = \varepsilon \qquad Y = \varepsilon$$

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Execution over *abaa*:

$$X = ab \qquad Y = ba$$

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Execution over *abaa*:

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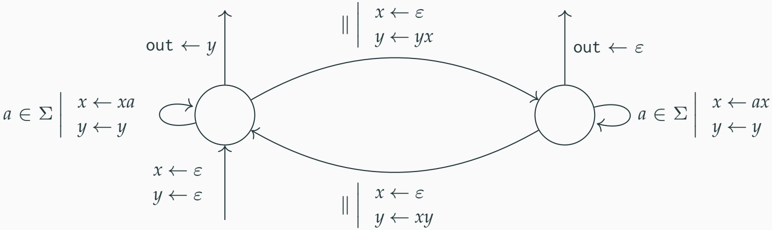
f restricted to $\{a, b\}^*$: corresponds to $w \mapsto w \cdot \mathtt{reverse}(w)$

SSTs can also have *states*: their memory is $Q \times (\Sigma^*)^R$ (with $|Q| < \infty$)



Stateful streaming string transducers

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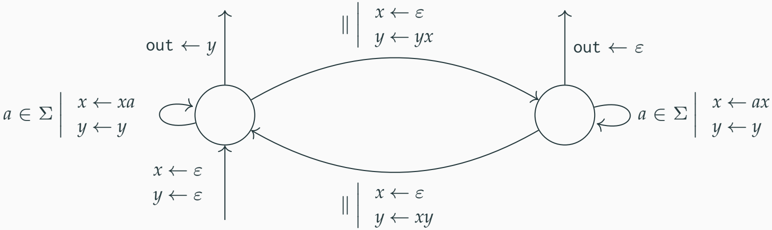

Copylessness restriction

Each register appears *at most once* on RHS of $\leftarrow$

(for each fixed input letter, at most once among all the associated $\leftarrow$)

Intuition: memory $M = Q \otimes \Sigma^* \otimes \ldots \otimes \Sigma^*$, transitions $M \multimap M$

$(Q \cong 1 \oplus \ldots \oplus 1, \mathsf{concat} : \Sigma^* \otimes \Sigma^* \multimap \Sigma^*)$

A framework for "single-pass" automata [Colcombet & Petrişan 2017]

- internal memory = object of a *category* $\mathcal{C}$
- transitions = morphisms (and [letter $\mapsto$ transition] = functor $\mathcal{T}_\Sigma \to \mathcal{C}$)

$$\mathcal{T}_\Sigma \quad = \quad \bullet \longrightarrow \overset{a \in \Sigma}{\overset{\circlearrowright}{\bullet}} \longrightarrow \bullet \qquad \longrightarrow \qquad \mathcal{C}$$

- DFA = automata over the category of finite sets
- Copyless SSTs $\approx$ start from a category $\mathcal{R}$ of copyless register updates
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Formally

A streaming setting $\mathfrak{C}$ with output X is a tuple $(\mathcal{C}, \top\!\!\top, \perp\!\!\perp, \mathit{out})$ with

- $\mathcal{C}$ a category
- $\top\!\!\top$ and $\perp\!\!\perp$ objects of $\mathcal{C}$
- $\mathit{out} : \mathrm{Hom}_{\mathcal{C}}(\top\!\!\top, \perp\!\!\perp) \to X$ a set-theoretic-map

Notion of $\mathfrak{C}$-automaton (abusively called $\mathcal{C}$-automata in the sequel)

SSTs as categorical automata

The register category with output alphabet Σ

- **Objects:** finite sets R, S — think register variables
- **Morphisms:** $\mathrm{Hom}_{\mathcal{R}}(R, S) = $ maps $S \to (R + \Sigma)^*$ corresponding to copyless register affectations $\sum_{s \in S} |f(s)|_r \leq 1$

- Monoidal with $\otimes = +$
- Free affine monoidal category over an object $\Sigma^* = \{\bullet\}$, morphisms $\varepsilon, a : \mathbf{I} \to \Sigma^*$ for $a \in \Sigma$ and $cat : \Sigma^* \otimes \Sigma^* \to \Sigma^*$
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Definition of the free finite coproduct completion $\mathcal{C}_\oplus$

- **Objects:** formal finite sums $\bigoplus_{u \in U} C_u$ of objects of $\mathcal{C}$ formally pairs $(U, (C_u)_{u \in U})$, U a finite set, $C_u \in \mathcal{C}_0$
- **Morphisms:** $\mathrm{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_u C_u, \bigoplus_v D_v\right) = \prod_u \sum_v \mathrm{Hom}_{\mathcal{C}}(C_u, D_v)$ $\cong \sum_f \prod_u \mathrm{Hom}_{\mathcal{C}}\left(C_u, D_{f(u)}\right)$

- Morphisms $\bigoplus_{q \in Q} R \quad \to \quad \bigoplus_{q \in Q} R$ correspond to transitions in a SST
- Canonical embedding $\mathcal{C} \to \mathcal{C}_\oplus$ allows to lift streaming settings

Compiling into higher-order transducers

Transductions definable in linear λ-calculus can be turned into automata over a category $\mathcal{L}$ of purely linear λ-terms (w/ const $f_c : o \multimap o$ for $c \in \Sigma$)

Claim

$\mathcal{L}$-automata compute the same string functions as λ-terms.

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Proof strategy for linear λ-definable $\implies$ regular function

Define a *functor* $\mathcal{L} \to \mathcal{R}_\oplus$ preserving enough structure

Useful fact: there is a canonical functor from $\mathcal{L}$ to any *symmetric monoidal closed category* with (co)products

Unfortunately $\mathcal{R}_\oplus$ is **not** monoidal closed...

Toward a monoidal closed category

So far, we encountered:

- $\mathcal{L}$: category of purely linear λ-terms (w/ const $f_c : o \multimap o$ for $c \in \Sigma$)
- $\mathcal{R}$: category of finite sets of registers and copyless assignments
- $\mathcal{R}_\oplus$: free finite coproduct completion of the latter (add states)

Now consider:

- the free finite *product* completion: $\mathcal{C} \mapsto \mathcal{C}_\& = ((\mathcal{C}^{\mathrm{op}})_\oplus)^{\mathrm{op}}$

  Objects: formal products $\&_x C_x$

- the composite completion $\mathcal{C} \mapsto \mathcal{C}_\& \mapsto (\mathcal{C}_\&)_\oplus$

  Objects: formal sums of products $\bigoplus_u \&_x C_{u,x}$

  similar to de Paiva's *Dialectica* categories **DC**, think $\exists u.\ \forall x.\ \varphi(u, x)$

Goals toward our main theorem

- Structure: $(\mathcal{R}_\&)_\oplus$ has finite products and is monoidal closed
- Conservativity: $(\mathcal{R}_\&)_\oplus$-automata and $\mathcal{R}_\oplus$-automata are equivalent

Structure (1): generic remarks $(\mathcal{C}_{\&})_{\oplus}$

Tensorial products can be lifted to the completions

- The new tensorial products satisfy the additional laws

$$A \otimes (B \,\&\, C) \equiv (A \otimes B) \,\&\, (A \otimes C) \qquad A \otimes (B \oplus C) \equiv (A \otimes B) \oplus (A \otimes C)$$

- In particular, $(\mathcal{C}_{\&})_{\oplus}$ has distributive cartesian products

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Lemma (folklore observation about dependent Dialectica categories?)

If $\mathcal{C}$ is symmetric monoidal and $(\mathcal{C}_{\&})_{\oplus}$ has the internal homs $A \multimap B$ for all $A, B \in \mathcal{C}$, then $(\mathcal{C}_{\&})_{\oplus}$ is symmetric monoidal closed.

$$\left(\bigoplus_{u \in U} \bigwith_{x \in X_u} A_x\right) \multimap \left(\bigoplus_{v \in V} \bigwith_{y \in Y_v} B_y\right) = \bigwith_{u \in U} \bigoplus_{v \in V} \bigwith_{y \in Y_v} \bigoplus_{x \in X_u} A_x \multimap B_y$$

Structure (2): combinatorics on strings

Lemma

$\mathcal{R}_\oplus$ has the internal homs $A \multimap B$ for all $A, B \in \mathcal{R}$.

The construction appears in the original SST paper [Alur & Černý 2010] without the categorical vocabulary.

$$\begin{cases} X := abXcY \\ Y := ba \end{cases} \quad \rightsquigarrow \quad \text{shape} \begin{cases} X := Z_1 X Z_2 Y \\ Y := Z_3 \end{cases} \quad + \quad \text{parameters } Z_1 = ab, \ldots$$

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Conclusion

$(\mathcal{R}_\&)_\oplus$ is symmetric monoidal closed (and almost affine).

Conservativity

Lemma

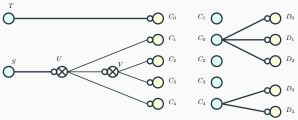
$(\mathcal{C}_{\&})_{\oplus}$ automata are equivalent to non-deterministic $\mathcal{C}_{\oplus}$ automata.

A uniformization ($\sim$ determinization) theorem is enough to conclude

Conservativity

$(\mathcal{R}_{\&})_{\oplus}$-automata are equivalent to standard SSTs.

- Uniformization already known [Alur & Deshmuk 2011]
- Argument implicitly based on monoidal closure!



Theorem

For any monoidal category $\mathcal{C}$, if $\mathcal{C}_{\oplus}$ has all the internal homsets $A \multimap B$ for $A, B \in \mathcal{C}$, then $(\mathcal{C}_{\&})_{\oplus}$-automata and $\mathcal{C}_{\oplus}$-automata are equivalent.

equivalently: ND $\mathcal{C}_{\oplus}$-automata can be uniformized

Main results

I have just discussed

Today's main theorem [Nguyễn & P.]

regular string function $\iff$ definable by some $t : \mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ in ILL with A purely linear

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Specific ingredients:

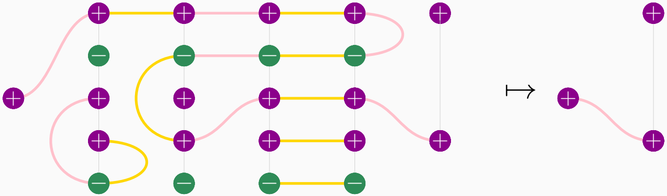
- Bottom-up categorical tree automata over SMCs
- A reasonably elegant multicategory of tree registers transition $\mathcal{R}$
  - Generated from the correponding PROP in a principled way (reminiscent from the notion of clone)
  - Argument for $\mathcal{R}$-monoidal closure argument generalizes to trees
- Regular functions already known to correspond to $\mathcal{R}_{\oplus\&}$-automata!

Dropping the additives

- Allows GoI-style interpretation in categories of diagrams ($\cong$ **Int**(FinPartInj))

$\rightsquigarrow$ Interpretation as two-way automata [Hines 2003]

Define regular languages



Consequence (not interesting)

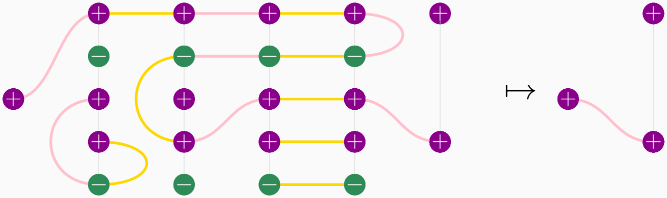
Every linear term $t : \mathsf{Str}_\Sigma[A] \multimap \mathsf{Bool}$ with A $\to$-free defines a regular language.

Dropping the additives and commutativity

- Allows GoI-style interpretation in categories of **planar** diagrams

$\rightsquigarrow$ Interpretation as two-way **planar** automata [Hines 2003,2006]

Define **star-free** languages



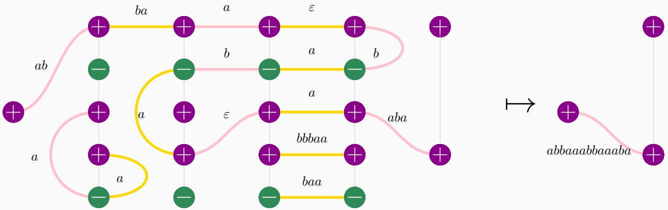
Consequence [Nguyễn, P. 2020]

Every **planar** linear term $t : \mathsf{Str}_\Sigma[A] \multimap \mathsf{Str}$ with A $\to$-free defines a star-free language.

- Allows GoI-style interpretation in categories of planar **labelled** diagrams

$\rightsquigarrow$ Interpretation as two-way planar **transducers** (2DFTs; w/o registers) [Hines 2003,2006]

Define **first-order** regular functions



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Every **planar** linear term $t : \mathsf{Str}_\Sigma[A] \multimap \mathsf{Str}$ with A $\to$-free defines a FO-transduction.

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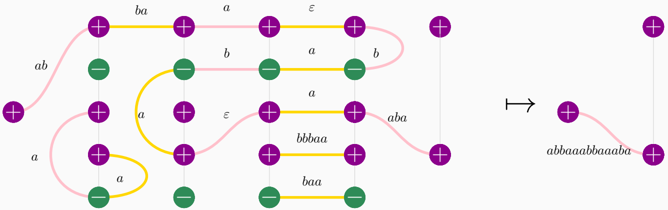

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Alas, planar linear terms are much weaker than FO-transductions (preserve Parikh images)

Dropping the additives and commutativity

- Allows GoI-style interpretation in categories of planar **labelled** diagrams

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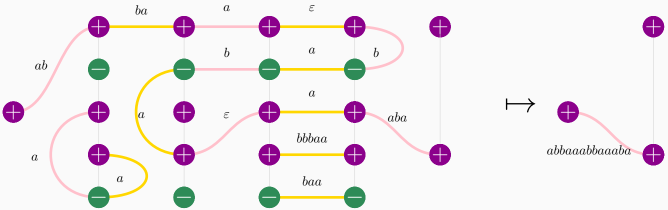

Conjecture

Every planar **affine** term $t : \mathsf{Str}_\Sigma[A] \multimap \mathsf{Str}$ with A $\rightarrow$-free defines a FO-transduction.

The converse holds (main ingredient for the proof: the Krohn-Rhodes theorem)

Conclusion

What happened here:

- Connections between Church encodings and automata
- Application of categorical semantics (Dialectica, geometry of interaction (GoI))
- A generic uniformization-like construction $(\mathcal{C}_{\&})_{\oplus} \to \mathcal{C}_{\oplus}$ for monoidal $\mathcal{C}$ with certain homsets

Some take-aways:

- Important ingredient in uniformization: monoidal closure

- Additive connectives are important for trees
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  - Further links with tree-walking automata?

Conclusion

Broader picture

$\mathsf{Str}_\Sigma[A] \multimap \mathsf{Bool}$ with A linear (adapted as needed):

λ-calculus	languages	status
simply typed	regular	✓ [Hillebrand & Kanellakis 1996]
linear or affine	regular	✓
non-commutative linear or affine	star-free	✓

$\mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ with A affine (adapted as needed):

λ-calculus	transducers	status
linear (without additives)	weird (?)	✓ (?)
affine	regular functions	✓
non-commutative affine	first-order regular fn.	✓?
linear/affine with additives	regular functions	✓
parsimonious	polyregular	??
simply typed	variant of CPDA???	???

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Thanks for listening! Questions?

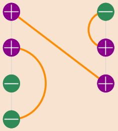
A category of planar diagrams

- Interpret purely linear non-commutative λ-terms in a monoidal closed category
- We consider a non-commutative refinement of Geometry of Interaction

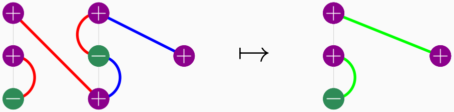
(well-known model of linear logic)

A compact closed category of planar diagrams

- **Objects:** words in $\{+,-\}^*$
- **Morphisms** $u \to v$ **:** graphs over $|u| + |v|$ with
  - degree ≤ 1 for every node
  - polarity restrictions
  - planarity restriction



To compute the composition of two morphisms, follow the paths (and forget the middle component)

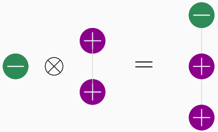


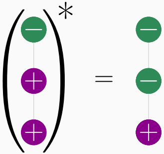
Compact-closure and interpretation of the λ-calculus

Structure to interpret the linear λ-calculus

- Monoidal product $A \otimes B$ given by concatenation
- Duals A^*: reverse and flip polarities
- Monoidal closure by setting $A \multimap B \; = \; A^* \otimes B$
- Interpretation of types $[\![A]\!]$ by induction with $[\![o]\!] = +$

(injective interpretation of booleans)

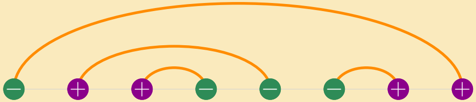




Examples

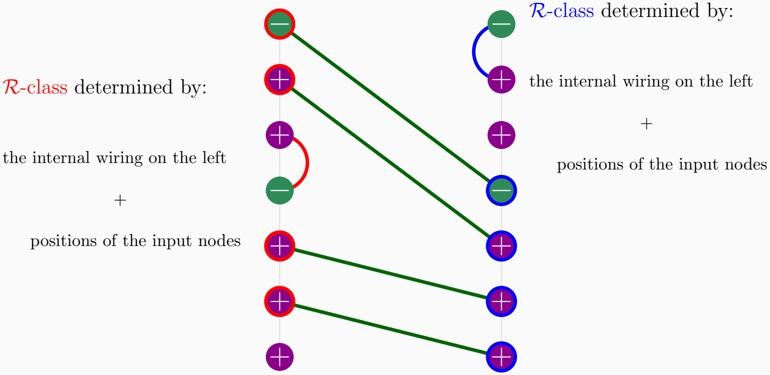
$$[\![((o \multimap o) \multimap o \multimap o) \multimap ((o \multimap o) \multimap o) \multimap o]\!] \; = \; - + + - - - + +$$

$$[\![\lambda f.\lambda g.\, f\,(\lambda x.\, x)\,(g\,(\lambda x.\, x))]\!] \; =$$



Aperiodicity

To conclude, we need to show that every $(\mathsf{Hom}(A, A), \circ)$ is finite and aperiodic for every A



Therefore: planar $\Longrightarrow$ $\mathcal{H}$-trivial

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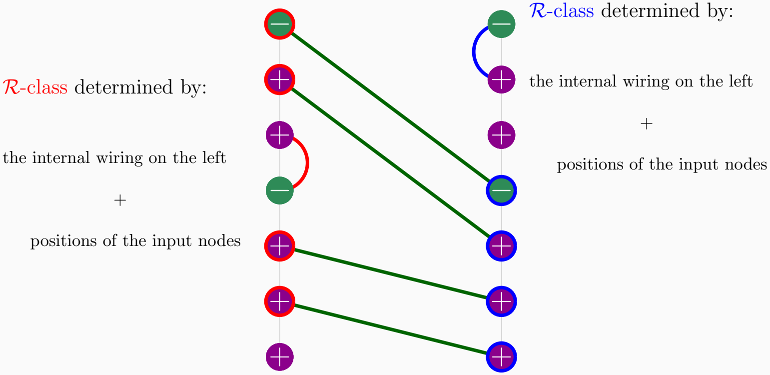
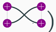
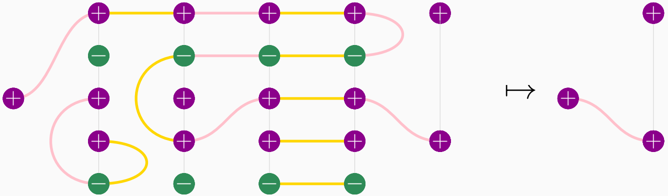


Therefore: planar $\Longrightarrow$ $\mathcal{H}$-trivial

- More elementary proofs w/o Green relations possible (e.g. order+Kleene's theorem)
- Planarity restriction is essential (consider )

Diagrams and two-way automata

Non-planar diagrams (with crossings): reminiscent of runs in 2DFAs!



- Transition functions $\delta : \Sigma \to \mathsf{Hom}(Q, Q)$ for some object Q $\qquad$ $Q \approx$ set of directed states
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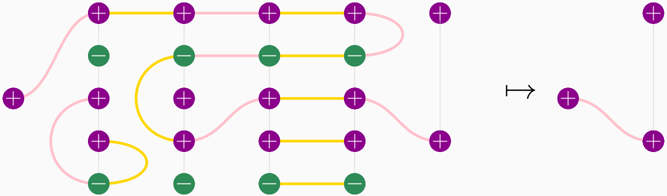

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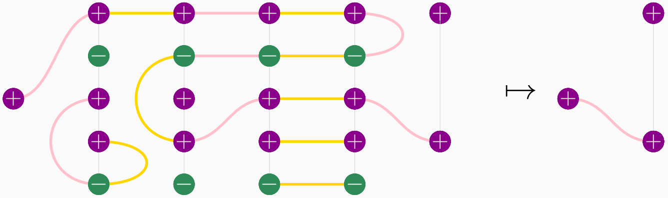

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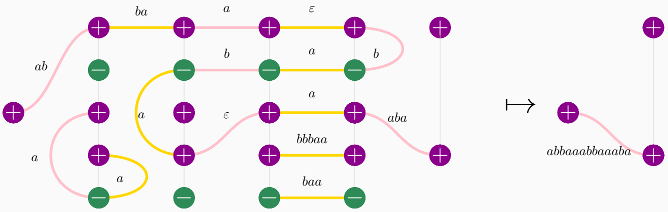
Theorem

Star-free languages are exactly those recognized by planar 2DFAs.

More generally: first-order transductions

Consider a richer category of diagrams where edges are labelled by output words

(labels of compositions given by concatenation)



Much like before, corresponding notion of (planar) 2DFTs.

Theorem

First-order transduction (FO regular functions) are those computed by reversible planar 2DFTs.

- 2DFTs with aperiodic transition monoid = FO regular functions [Carton&Dartois 2015]

  (hence reversible planar 2DFTs ⊆ FO-transductions)

- FO transduction ⊆ reversible planar 2DFTs: closure by composition and Krohn–Rhodes

  (see `http://nguyentito.eu/2021-01-links.pdf`)