Implicit automata in typed λ-calculi

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The research that we present here started out by exploring connections between the languages (resp. functions) recognized by automata (resp. transducers, i.e., automata with output) and those definable by programs in certain typed λ-calculi. While we are relating logic and automata, much like programming language theory and proof theory are linked via the Curry–Howard correspondence, our work does not fit in the "logics as specification languages" paradigm, exemplified by the equivalence of finite-state automata and Monadic Second-Order Logic (MSO). One could sum up the difference by analogy with the two main approaches to machine-free complexity: *implicit computational complexity (ICC)* and *descriptive complexity*. Both aim to characterize complexity classes without reference to a machine model, but the methods of ICC have a more computational flavor.

programming paradigm	declarative	functional
complexity classes	Descriptive Complexity	Implicit Computational Complexity
automata theory	subsystems of MSO	**our work**

To our knowledge, very few works had previously looked at this kind of "type-theoretic" or "proof-theoretic" ICC for automata. Let us mention a few recent papers concerning transducers [5, 2] and multi-head automata [19, 10]. Most importantly, our starting point is a remarkable result dating back to 1996:

▶ **Theorem 1** (Hillebrand & Kanellakis [8, Theorem 3.4]). *A language $L \subseteq \Sigma^*$ can be defined in the* simply typed λ-calculus *by some* closed λ-term *of type* $\mathtt{Str}_\Sigma[A] \to \mathtt{Bool}$ *for some type A (that may depend on L) if and only if it is a* regular language*.*

Let us explain this statement. We consider a grammar of simple types with a single base type: $A, B ::= o \mid A \to B$, and use the *Church encodings* of booleans $\mathtt{Bool} = o \to o \to o$ and strings $\mathtt{Str}_\Sigma = (o \to o) \to \ldots \to (o \to o) \to o \to o$ with $|\Sigma|$ arguments of type $(o \to o)$ where Σ is a finite alphabet. For types A and B, we write $B[A]$ for the substitution $B\{o := A\}$ of every occurrence of o in B by A.

Although little-known, Hillebrand and Kanellakis's theorem is not surprising in retrospect: strong connections between Church encodings and automata (see e.g. [18, 11]) have been exploited, in particular in *higher-order model checking*.

Linear and non-commutative types While there exist some results in implicit complexity based on the simply typed λ-calculus (e.g. [8]), many works in that area have taken inspiration from *linear logic* to design more sophisticated type systems, starting with two characterizations of polynomial time [7, 6]. We followed the same idea to characterize two transduction classes in Intuitionistic Linear Logic with additives (ILL). From now on, $\mathtt{Str}_\Sigma$ denotes the linearized Church encoding $\mathtt{Str}_\Sigma = (o \multimap o) \to \ldots \to (o \multimap o) \to o \to o$.

▶ **Theorem 2** ([14]). *A function $\Gamma^* \to \Sigma^*$ is* regular (see e.g. [12]) (resp. comparison-free polyregular [13]) *if and only if it can be defined by a closed term of type* $\mathtt{Str}_\Gamma[A] \multimap \mathtt{Str}_\Sigma$ (resp. $\mathtt{Str}_\Gamma[A] \to \mathtt{Str}_\Sigma$) *in ILL for some* purely linear *type A (that may depend on f).*

Here "purely linear" means that A does not contain any non-linear function arrow '$\to$'. We also provide a similar characterization of regular *tree* functions in [14]. What might be more surprising is our additional use of *non-commutativity* (a function must use its arguments in the same order that they are given in) to characterize a subclass of regular languages.

▶ **Theorem 3** ([15]). *A language $L \subseteq \Sigma^*$ is* star-free *if and only if it can be defined by a closed term of type* $\mathtt{Str}_\Sigma[A] \multimap \mathtt{Bool}$ *in an affine variant of* Intuitionistic Non-Commutative Linear Logic [17] *for some* purely linear *type A (that may depend on L).*

Denotational semantics meets categorical automata theory The key conceptual insight in our proof of Theorem 2 is to relate the semantics of purely linear ILL in monoidal closed categories with the representation of transducers in Colcombet and Petrişan's categorical framework for automata [3]. More precisely, we show that a variant of *copyless* (i.e. affine) *streaming string transducers* [12, §2] – a machine model for regular functions – can be formulated as $\mathcal{C}$-automata where $\mathcal{C}$ is a Dialectica-like completion of a category of string-valued registers, so $\mathcal{C}$ is monoidal closed for the same reason as Dialectica categories [4].

The notion of monoidal closure has a relevance for automata theory that goes beyond Theorem 2; intuitively, this is due to the important role that function spaces often play in automata constructions. To illustrate that, in [14], we gave abstract generalizations of the arguments showing that copyless SSTs may be determinized and that the composition of two regular functions may be implemented by a copyless SST, in terms of internal homsets. Interestingly, it is not clear to us if there is a nice condition analogous to monoidal closure over classes of *transition monoids* allowing to carry out those generalized arguments without introducing automata over monoidal closed categories.

Some automata-theoretic consequences We were thus led to define the aforementioned *comparison-free polyregular functions* by considering expressible functions in linear logic. This class of function is a natural restriction of polyregular functions [2], a class of string transductions whose outputs are of size at most polynomial in the output. We studied that class in [13] from the point of view of automata theory. While all arguments are rather unsurprising, the connection with λ-calculus helped us realize that the class was closed under composition, and provides an alternative proof of this fact leveraging the material of [14].

Furthermore, we also took inspiration from Theorem 3 and from the planar geometry of interaction (GoI) semantics [1] of non-commutative linear logic to design a new machine model for star-free languages and aperiodic regular functions (see [12, §3] for the latter): *planar two-way automata/transducers* [16]. It was previously known that two-way automata could be expressed as automata over a GoI category (a reformulation of the results of [9] in the framework of [3]), and that two-way transducers compute regular functions [12, §2].

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