Polynomial χ-Boundedness of Bounded Twin-Width Classes

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Joint work with Stéphan Thomassé

1st Twin-Width Workshop, Aussois, 2023

χ-boundedness & Twin-width

Definition (χ-bounded)

If $\mathcal{C}$ is a class of graphs, $\mathcal{C}$ is *χ-bounded* if $\forall G \in \mathcal{C}, \chi(G) \leq f(\omega(G))$ for some function f.
If f is a polynomial, $\mathcal{C}$ is *polynomially χ-bounded*.

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Theorem [BBDGTT'23]

For every t, there exists a triangle-free graph with twin-width t and chromatic number $t+1$.

Main result

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- Gives an efficient coloring algorithm.

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  With the product coloring: $\chi(G) \leq \chi(G_1) \cdot \chi(G_2)$.

Modular decomposition

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If $G = (V, E)$ is a graph, we say that $X \subseteq V$ is a *module* if $\forall v \in V \setminus X$, either v is complete to X or v is anti-complete to X.

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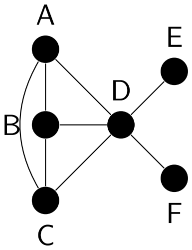
Theorem [Gallai '67]

If G and its complement are connected, the maximal proper modules of G form a partition of V, and the quotient graph is prime.

Tree for modular decomposition



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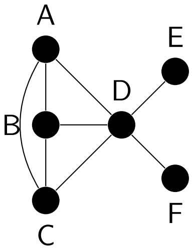

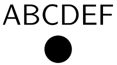

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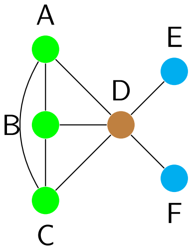

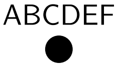

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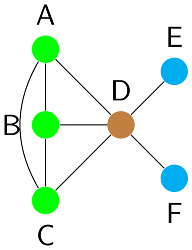

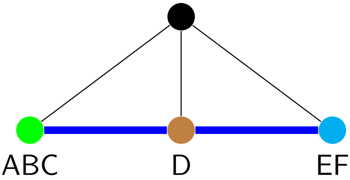

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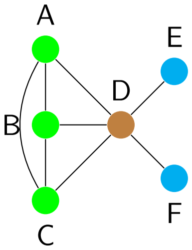

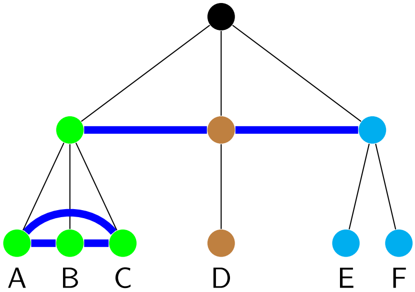

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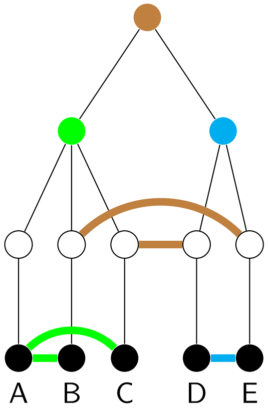
Theorem (Chudnovsky, Penev, Scott, Trotignon '13)

If $\mathcal{C}$ is polynomially χ-bounded, so is $\mathcal{C}_s$.

Delayed substitution



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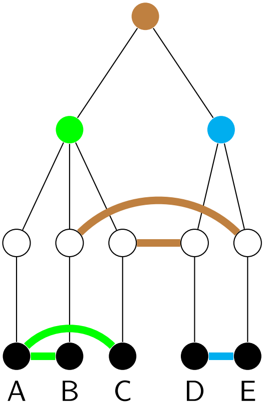

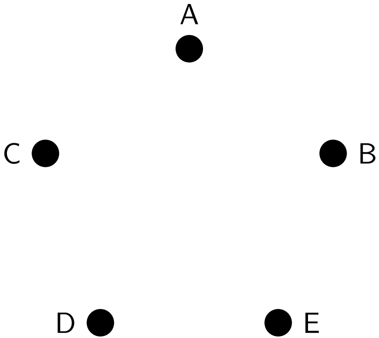

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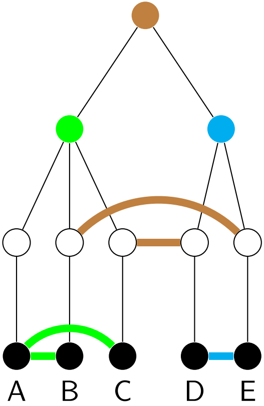

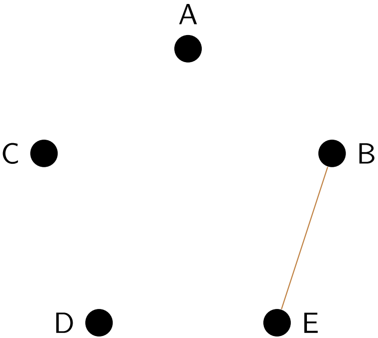

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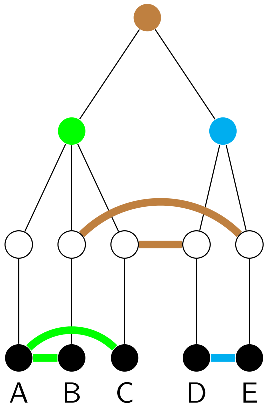

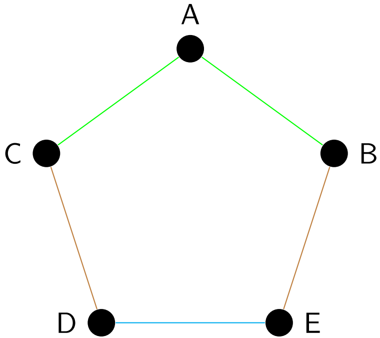

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Theorem

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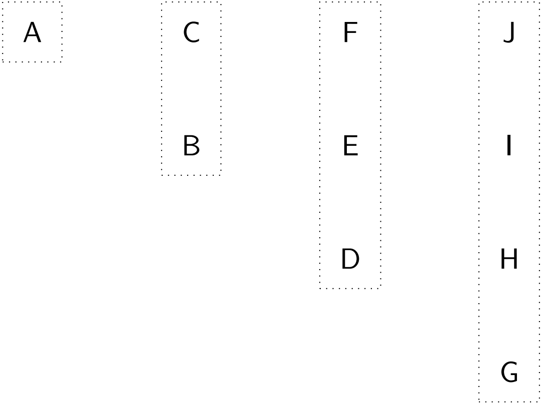

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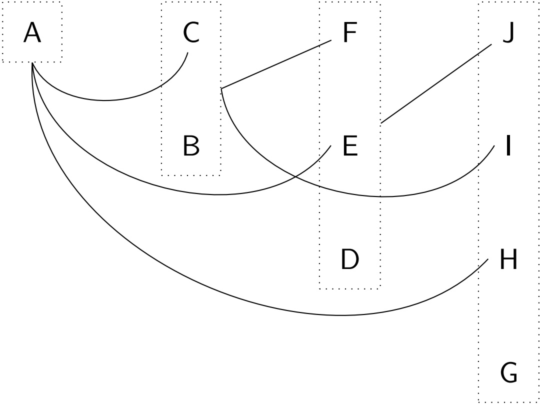

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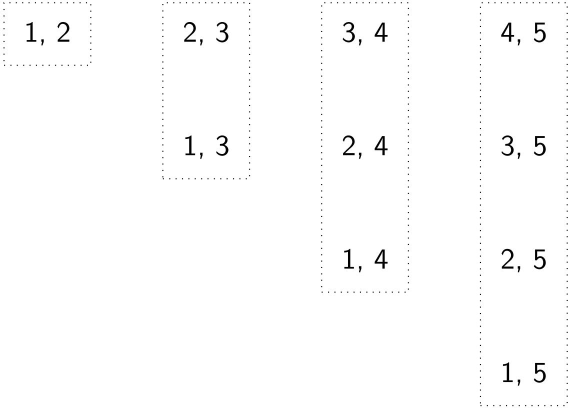

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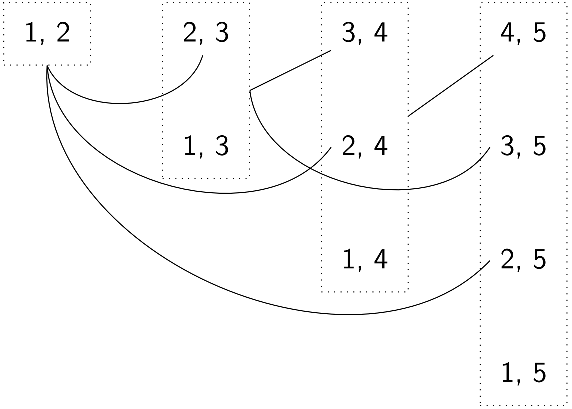

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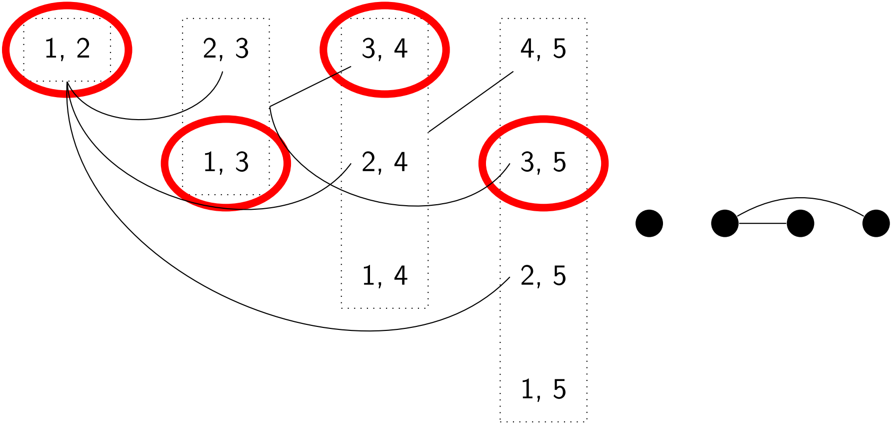

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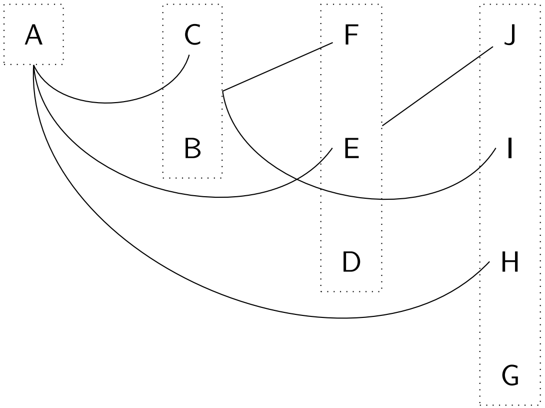

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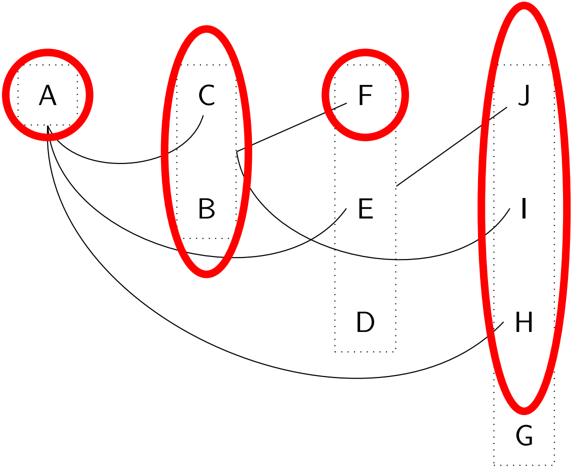

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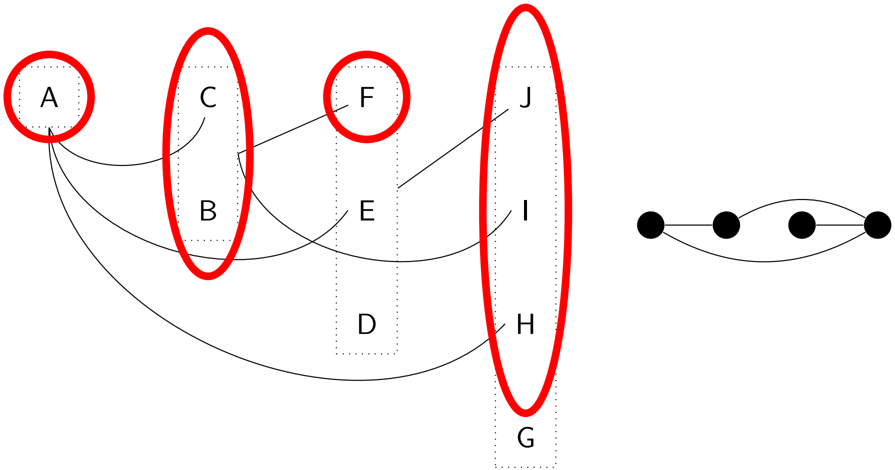

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If $\mathcal{C}$ is χ-bounded, then so is $RM(\mathcal{C})$.

Right extension and twin-width

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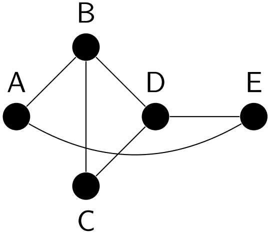
Theorem (from [PS'22])

If $\mathcal{C}$ is polynomially χ-bounded, the set of graphs with a d-nice $\mathcal{C}$-RMP is also polynomially χ-bounded.

How to obtain a delayed decomposition?

Start from an ordered graph G.



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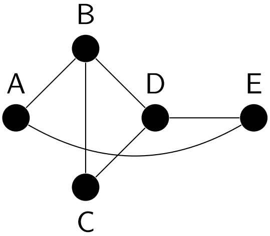

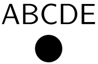

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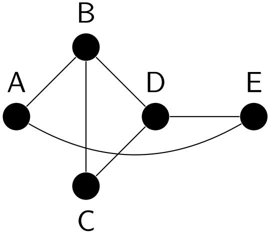

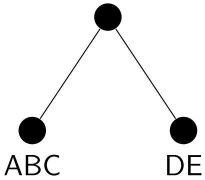

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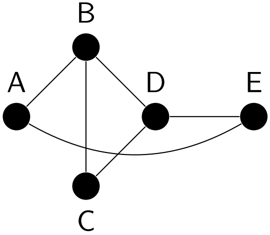

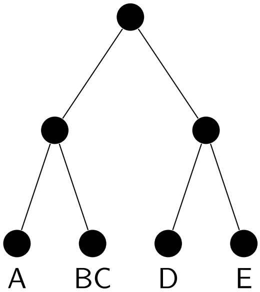

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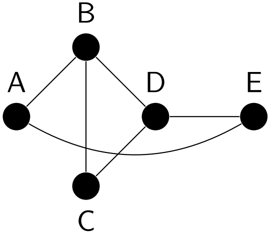

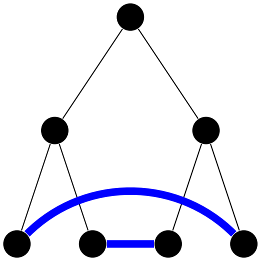

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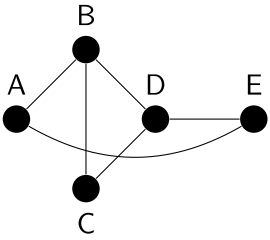

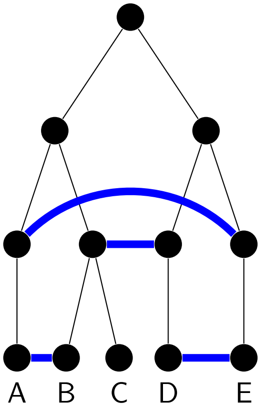

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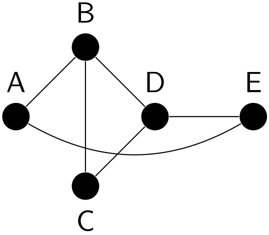

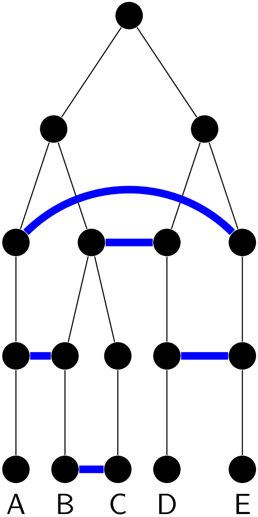

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- Find a suitable RMP.
- Prove that the *RMP* is $2t$-nice.
- Prove that all transversal minors are $t-1$-almost mixed-free.

Open questions

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- For triangle-free graphs of twin-width t, do we have $\chi \leq t + 1$?