Discrepancy and Some Applications in TCS

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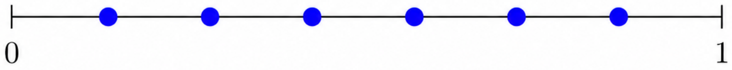

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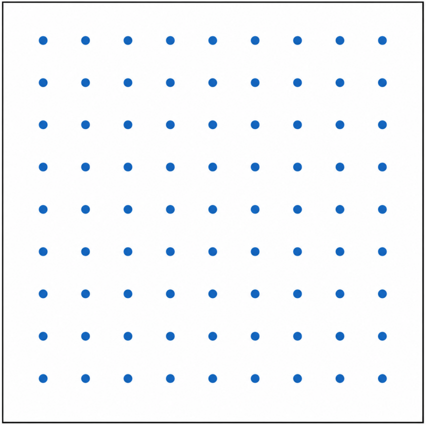

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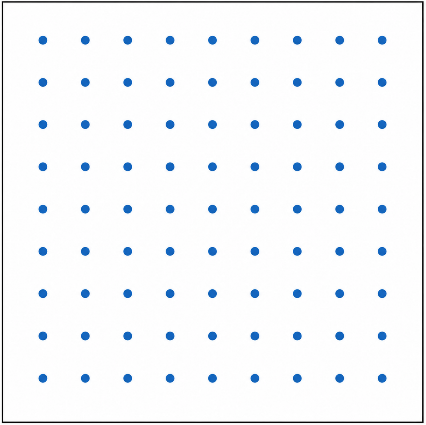

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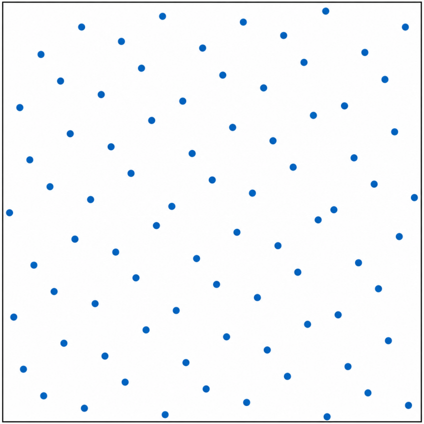

Van der Corput

Rectangle discrepancy in the plane

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Definition

$P \subseteq [0,1]^2$. For every rectangle $R \subseteq [0,1]^2$,

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$$\mathrm{disc}(P) = \max_R \mathrm{disc}(P,R).$$

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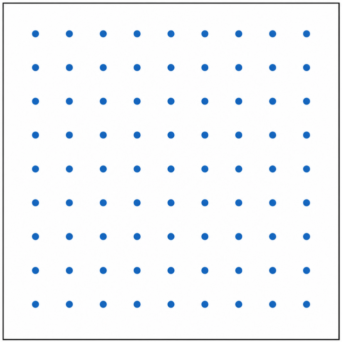

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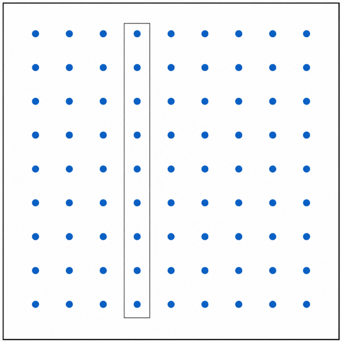

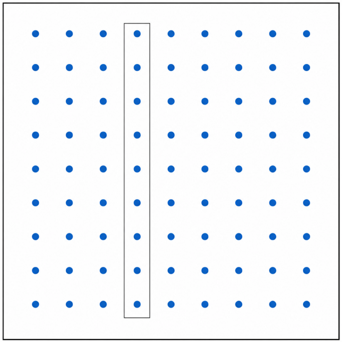

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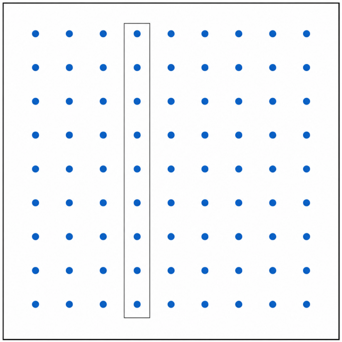

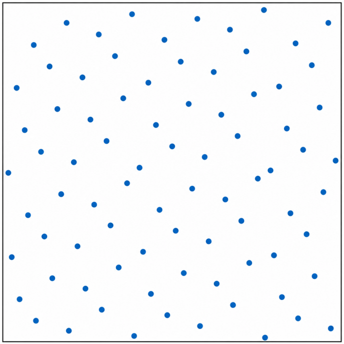

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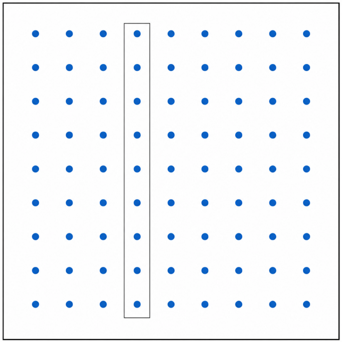

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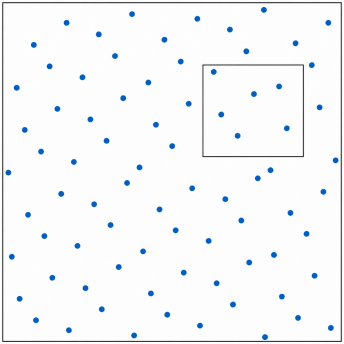

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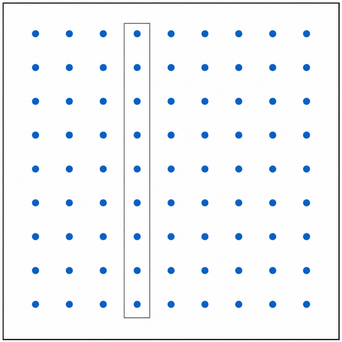

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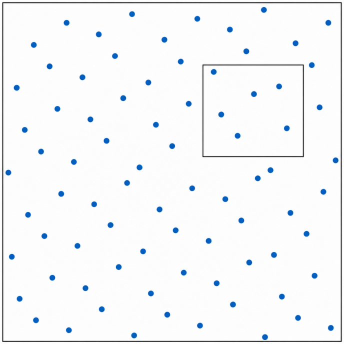

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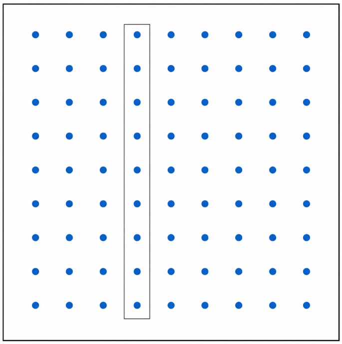

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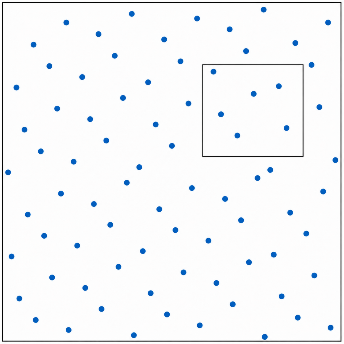

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Theorem [Schmidt '72]

Every set of n points in $[0,1]^2$ has discrepancy $\Omega(\log n)$.

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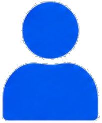
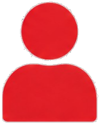
- $H_n(H_n)^\intercal = 2^n I$.
- After replacing the '1' entries by '0' entries and the '-1' entries by '1' entries, we get the matrix of the inner products in $\{0, 1\}^n$ modulo 2 (ordered lexicographically).

Communication complexity of the inner product



Alice

$a \in \{0,1\}^n$



Bob

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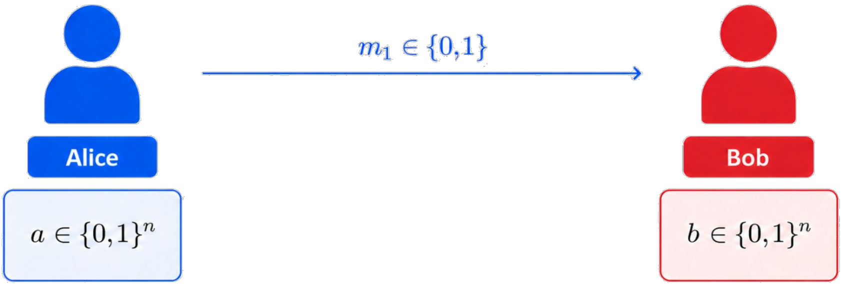

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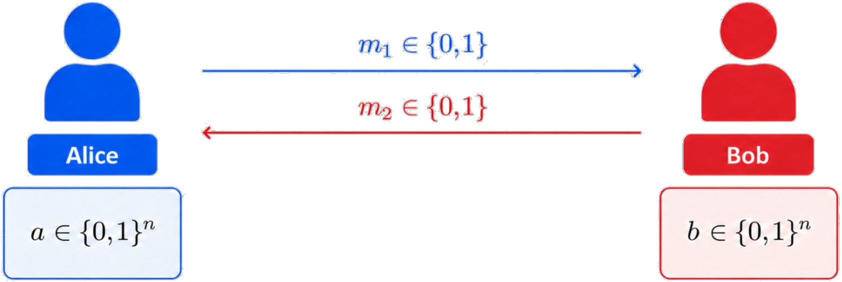

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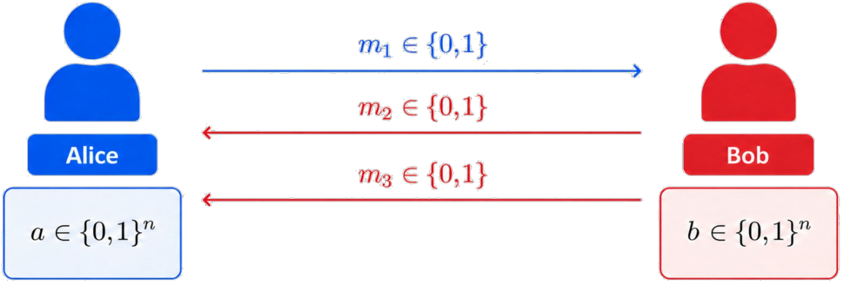

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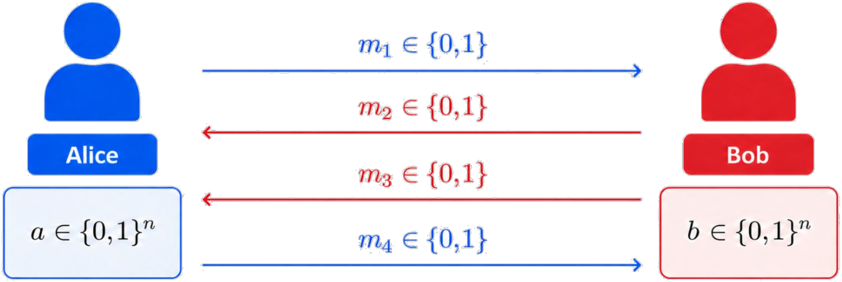

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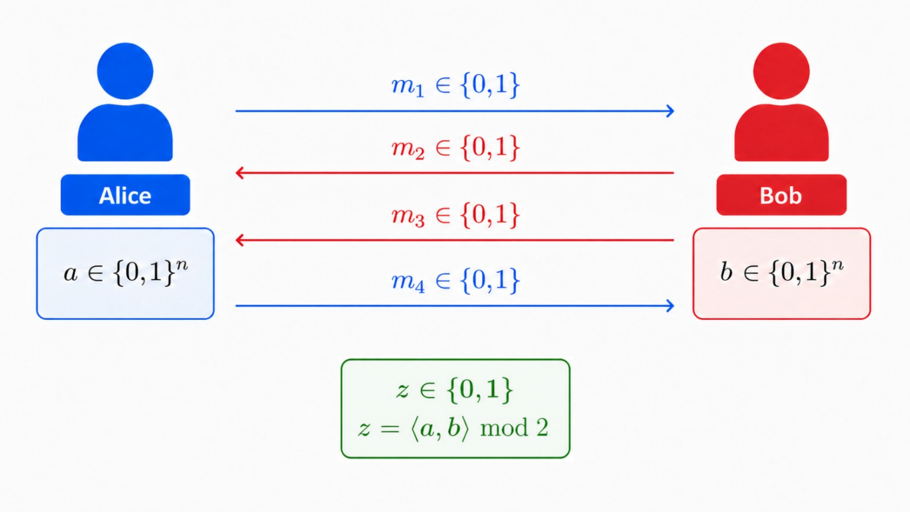

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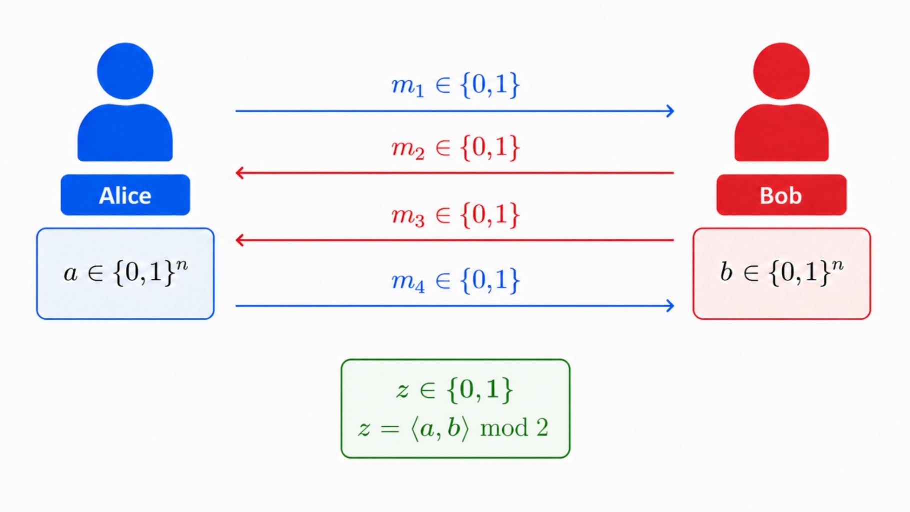

Cost: Number of bits sent in the worst case

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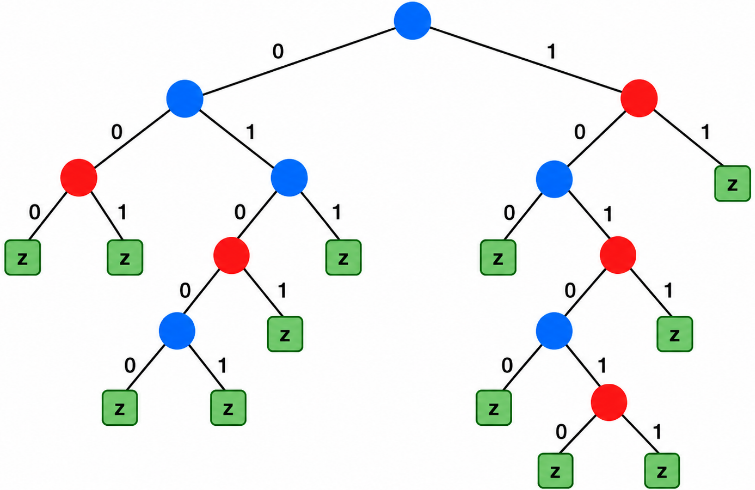

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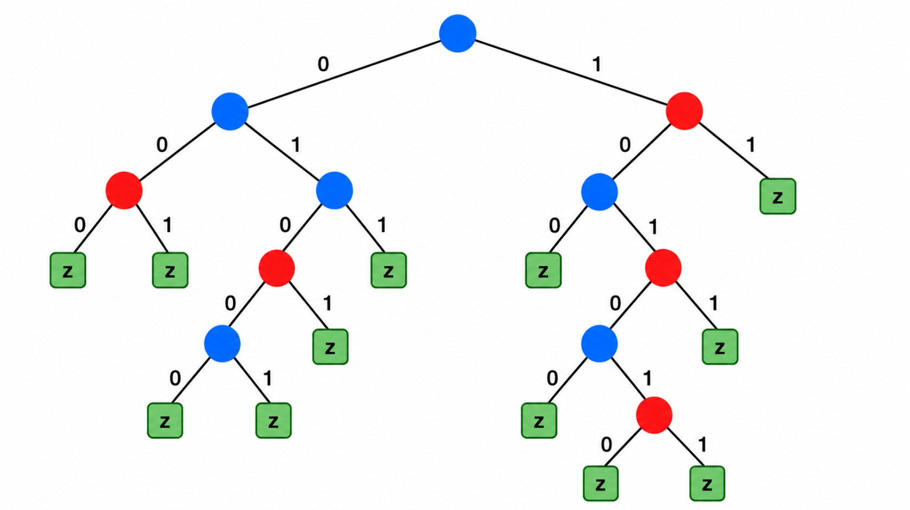

Key observation: The leaves give a partition of H_n into monochromatic rectangles.

Bound on the communication complexity

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Remark: A similar result holds if we just require the output bit to be correct with probability $1/2 + \varepsilon$.

Towards combinatorial discrepancy

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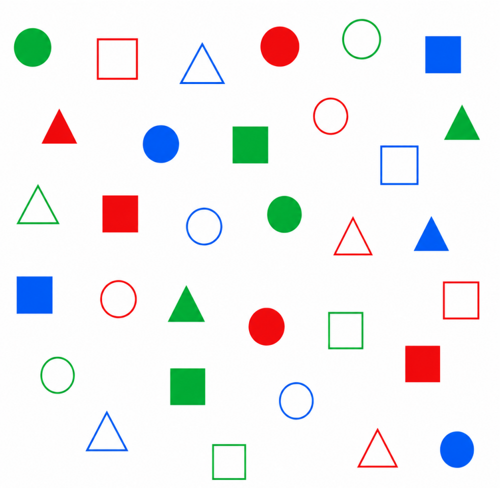

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Given $A \in \mathbb{R}^{m \times n}$, $\text{disc}(A) = \min_{x \in \{-1,1\}^n} \|Ax\|_\infty$.

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$\to \mathbb{P}[\mathrm{disc}(S, \chi) \leq O(\sqrt{n \log m})] \geq \frac{1}{2} > 0$.

Theorem [Spencer '85]

If $m \geq n$ then $\mathrm{disc}(\mathcal{S}) \leq O(\sqrt{n \log \frac{2m}{n}})$.
If $m = n$ then $\mathrm{disc}(\mathcal{S}) \leq 6\sqrt{n}$.

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There exists a set system $\mathcal{S}$ with $m = n$ sets and $\text{disc}(\mathcal{S}) = \Omega(\sqrt{n})$.

Proof: Let $\mathcal{S}$ be the set system with "incidence matrix" H_n, set $N = 2^n$.
Every element is in $N/2$ sets, every two elements share $N/4$ sets.
Let $\chi : V \to \{-1, 1\}$.

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$\to$ There exists $S \in \mathcal{S}$ with $\mathrm{disc}(S, \chi) = \Omega(\sqrt{n})$.

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Let $A \in \mathbb{R}^{m \times n}$ and $w \in (\mathbb{R}^+)^n$. There exists $x \in \mathbb{N}^n$ such that

$$\|A(w - x)\|_\infty \leq \text{lindisc}(A).$$

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Remark: There are also efficient constructive proofs.

Beck-Fiala

Theorem [Beck, Fiala '81]

Let $\mathcal{S}$ be a set system where every $v \in V$ is in $\leq \Delta$ sets. Then, $\text{herdisc}(\mathcal{S}) \leq 2\Delta - 1$.

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Let $\mathcal{S}$ be a set system where every $v \in V$ is in $\leq \Delta$ sets. Then, $\text{herdisc}(\mathcal{S}) \leq 2\Delta - 1$.

Remark: Beck and Fiala conjectured that $\text{herdisc}(\mathcal{S}) = O(\sqrt{\Delta})$. This is still a central open question in discrepancy theory.

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Find a solution $\overline{y}$ of the following LP:

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\sum_{j \in [m]} y_{i,j} &\geq 1, & \forall i \in [n] \\
\sum_{i \in [n]} y_{i,j} \cdot \frac{t_{i,j}}{T} &\leq 1, & \forall j \in [m] \\
y_{i,j} &\in [0,1], & \forall (i,j) \in [n] \times [m]
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Thus, there exists $\overline{x}$ such that:

$$\begin{aligned} \sum_{j \in [m]} x_{i,j} \geq 1, & \quad \forall i \in [n] \\ \sum_{i \in [n]} x_{i,j} \cdot \frac{t_{i,j}}{T} \leq 13, & \quad \forall j \in [m] \\ x_{i,j} \in \mathbb{N}, & \quad \forall (i,j) \in [n] \times [m] \end{aligned}$$

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Thank you!