A Dense Neighborhood Lemma, with Applications to Domination and Chromatic Number

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Joint work with Pierre Charbit and Stéphan Thomassé

April 24, 2025

Dense Neighborhood Lemma

Theorem

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- Similar statements in many other settings.

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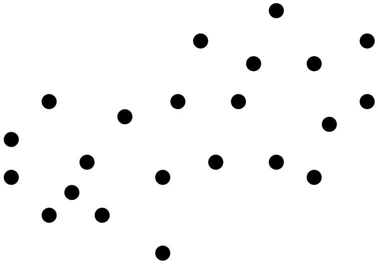

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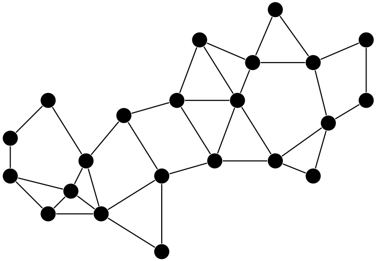

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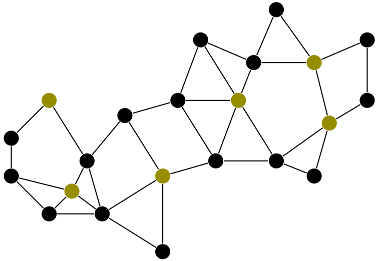

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Want: dominating set X of size $f(\delta)$.

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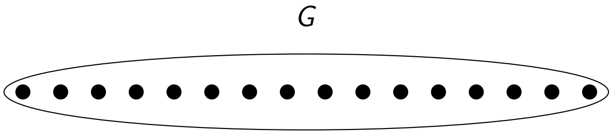

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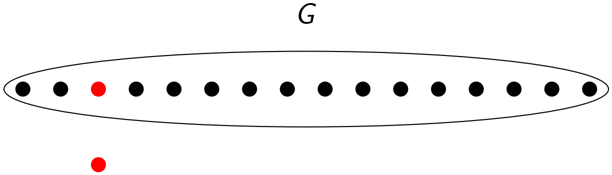

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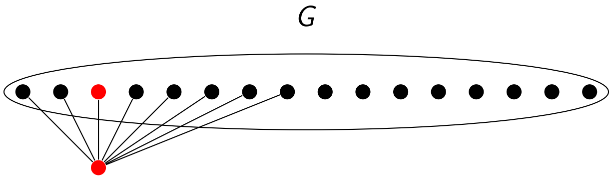

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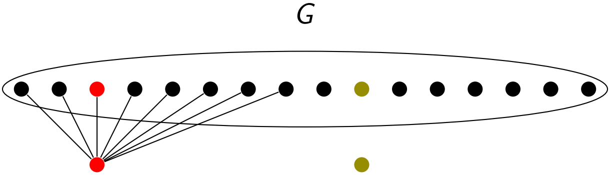

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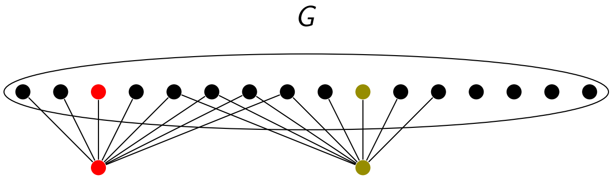

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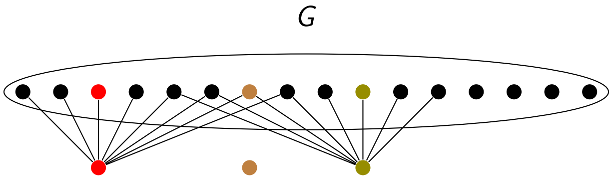

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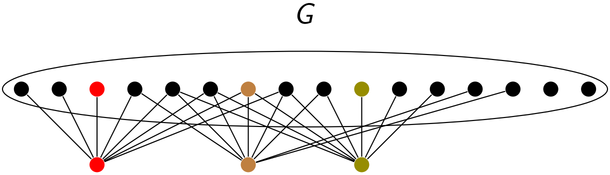

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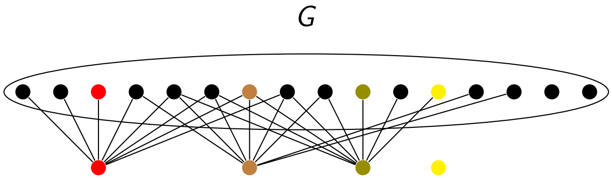

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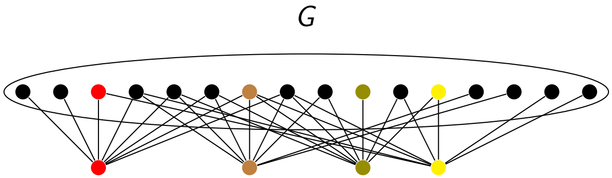

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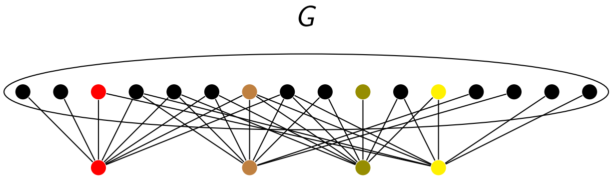

If $G \sim G(n/2, n/2, 1/2)$ then $\delta(G) \approx n/4$ but $\gamma(G) \approx \log(n)$.

VC-dimension

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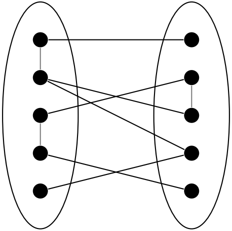
Definition (Vapnik, Cervonenkis '71)

A class $\mathcal{C}$ of graphs has bounded *VC-dimension* if it does not contain some bipartite graph as a semi-induced subgraph.



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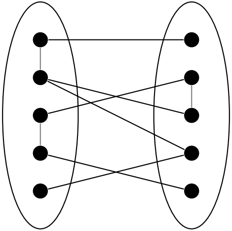

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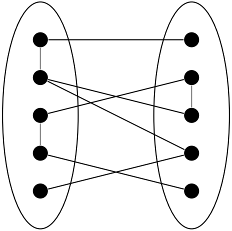

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Theorem [Haussler, Welzl '89]

If G has VC-dimension d and minimum degree δn then G has a dominating set of size $f(\delta, d)$.

VC-dimension of threshold graphs

Proposition [Folklore]

VC-dim$\big($threshold graphs in $\mathbb{R}^N\big) = N + 1$.

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Every n-vertex graph is a threshold graph in $\mathbb{R}^n$ $\Rightarrow$ needs to depend on N.

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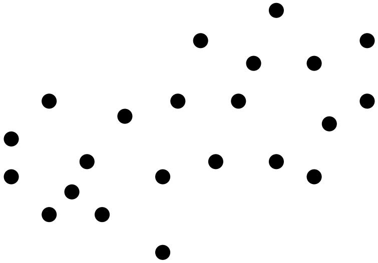

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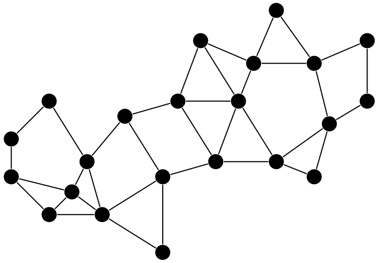

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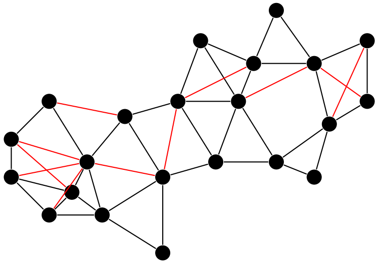

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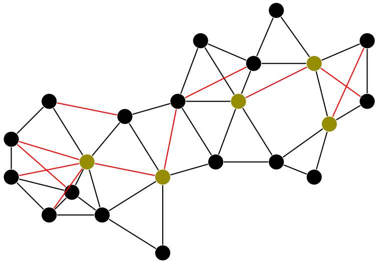

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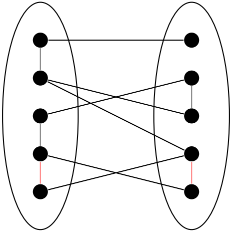

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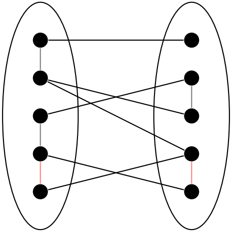

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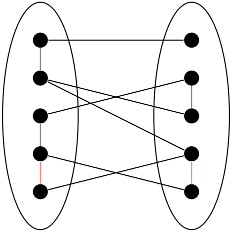

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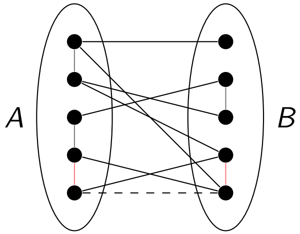

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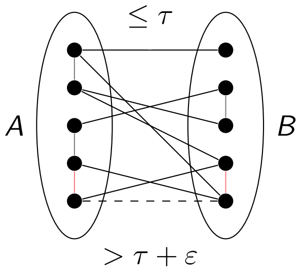

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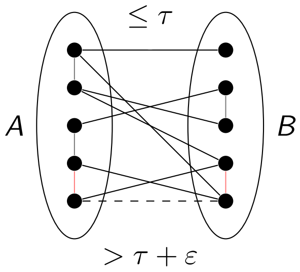

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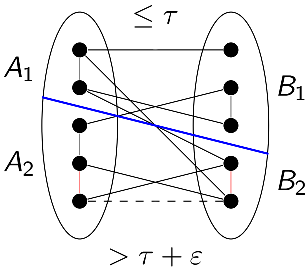

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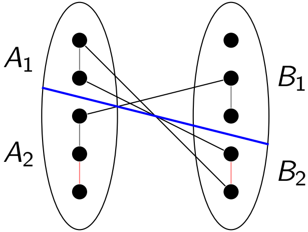

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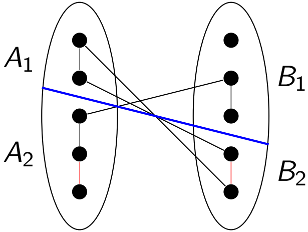

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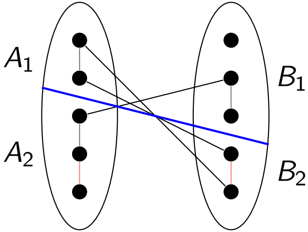

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$T[A, B]$ has a cut which kills *significantly* more edges than non-edges.

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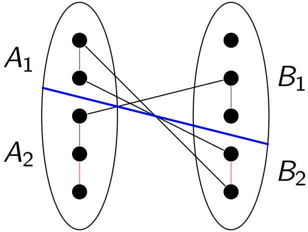

Take a random hyperplane $H \subseteq \mathbb{R}^{N+1}$.
Edges survive with probability $\leq \tau/\pi$,
non-edges with probability $> (\tau + \varepsilon)/\pi$.
$T[A, B]$ has a cut which kills *significantly* more edges than non-edges.
Some bipartite graphs have no such cuts.

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$V \subseteq \mathbb{R}^N$, finite set of points with $|B(v,1) \cap V| \geq \delta |V|$ for every $v \in V$. There is a set $X \subseteq V$ of size $f(\delta, \varepsilon)$ such that $V \subseteq \bigcup_{x \in X} B(x, 1+\varepsilon)$.

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Every ε-disjointness trigraph with minimum black degree δn has a black/red dominating set of size $f(\delta, \varepsilon)$.

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Theorem [Brandt, Thomassé '04]

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If u, v are dominated by $x \in X$ and are neighbors in G then $|N(x)\cup N(u)\cup N(v)| \geq 3\cdot(1/3+\varepsilon)\cdot n - 2\cdot\varepsilon n > n$, a contradiction.

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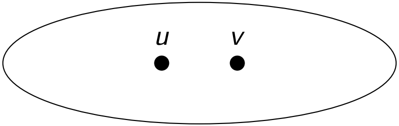

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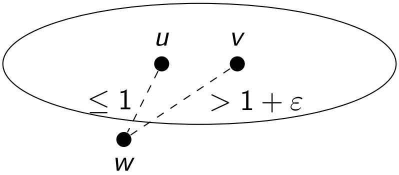

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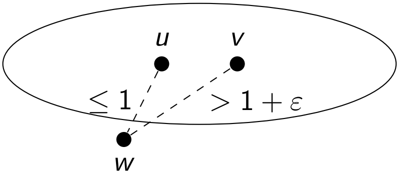

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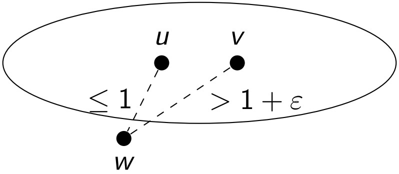

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Consider u, v in the same cluster. Every neighbor w of u satisfies $N(u) \cap N(w) = \emptyset$ so $(1/4 + \varepsilon/2)n$ of them satisfy $|N(v) \cap N(w)| \leq \varepsilon \cdot n/2$.

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Then, $|N(u) \cup N(v) \cup N(w) \cup N(w')| \geq 4 \cdot (1/4 + \varepsilon) \cdot n - \binom{4}{2} \cdot \varepsilon \cdot n/2 > n$, a contradiction.

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Proof sketch

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Thank you!