A Dense Neighborhood Lemma, with Applications to Domination and Chromatic Number

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Joint work with Pierre Charbit and Stéphan Thomassé

May 20, 2025

Dense Neighborhood Lemma

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- Similar statements in many other settings.

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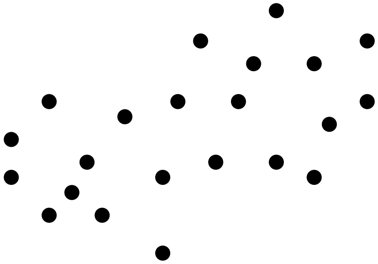

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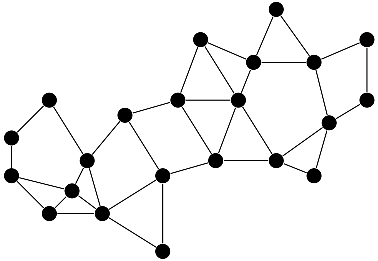

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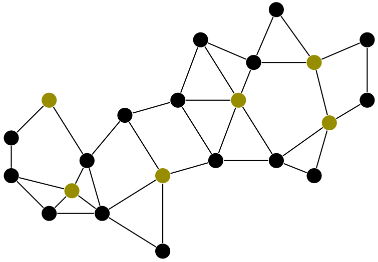

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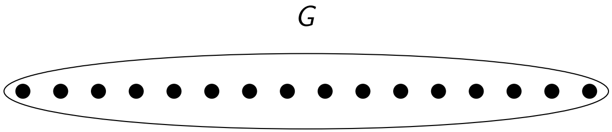

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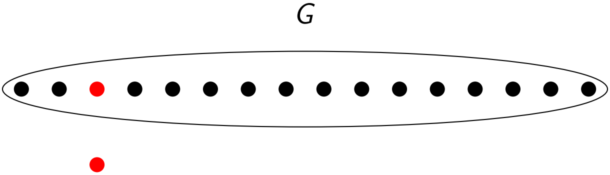

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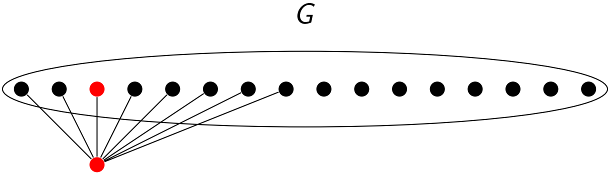

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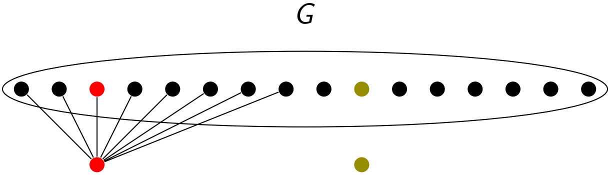

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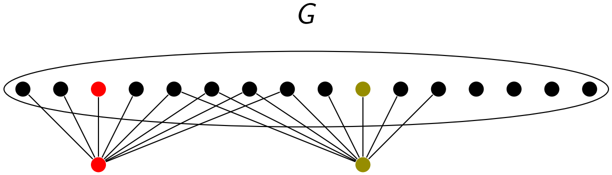

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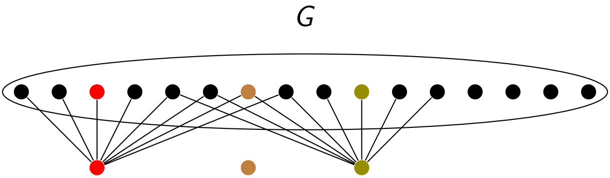

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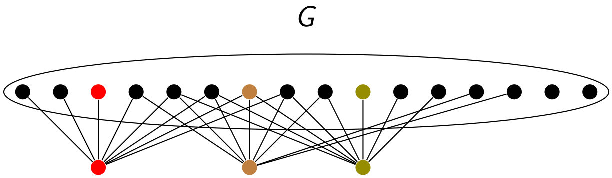

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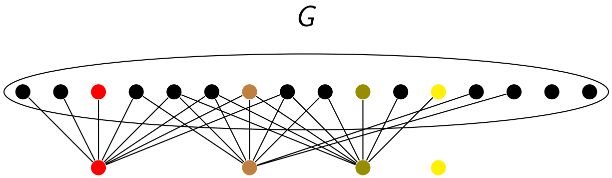

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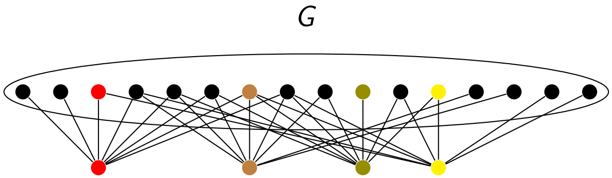

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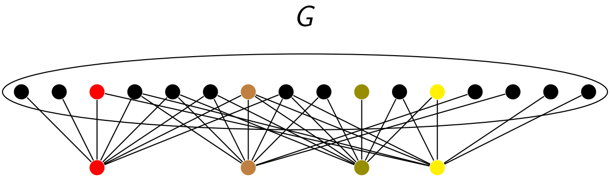

If $G \sim G(n/2, n/2, 1/2)$ then $\delta(G) \approx n/4$ but $\gamma(G) \approx \log(n)$.

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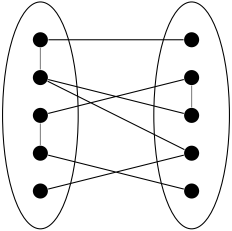
Definition (Vapnik, Cervonenkis '71)

A class $\mathcal{C}$ of graphs has bounded *VC-dimension* if it does not contain all bipartite graphs as semi-induced subgraphs.



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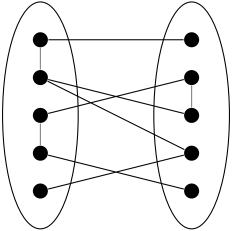

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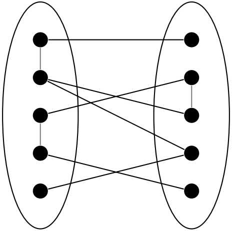

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Theorem [Haussler, Welzl '89]

If G has VC-dimension d and minimum degree δn then G has a dominating set of size $f(\delta, d)$.

VC-dimension of threshold graphs

Proposition [Folklore]

VC-dim$\big($threshold graphs in $\mathbb{R}^N\big) = N + 1$.

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Every n-vertex graph is a threshold graph in $\mathbb{R}^n$ $\Rightarrow$ needs to depend on N.

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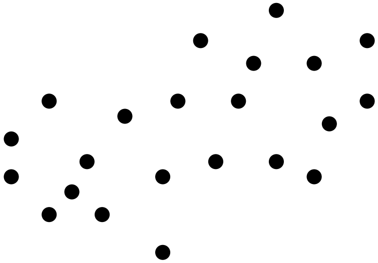

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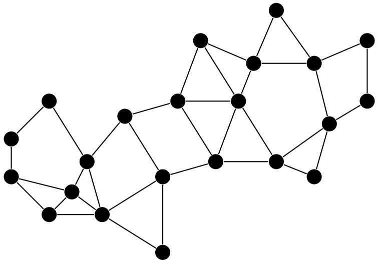

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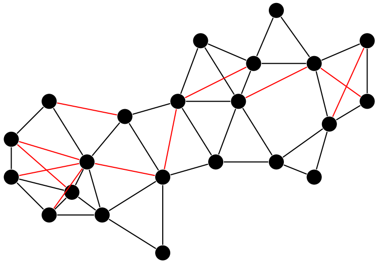

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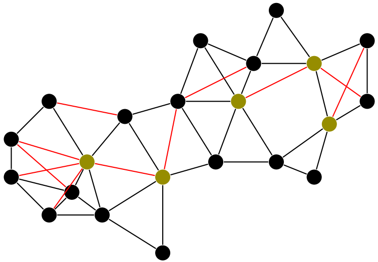

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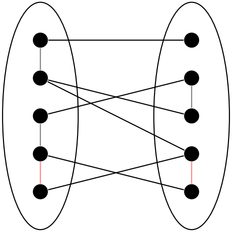

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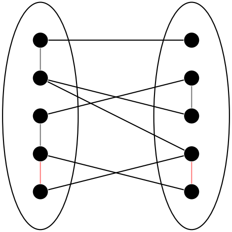

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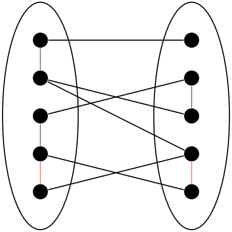

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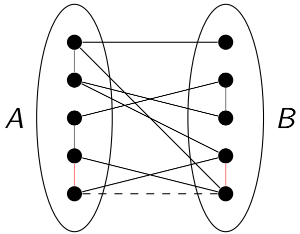

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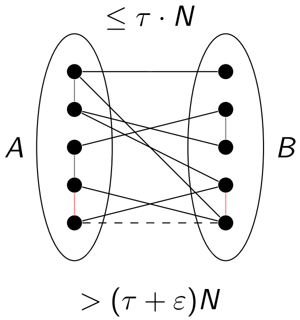

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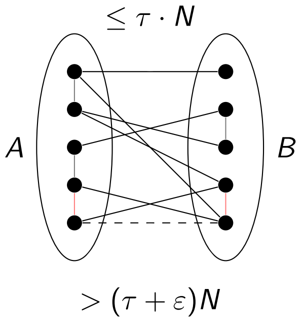

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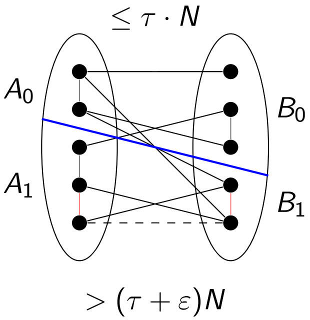

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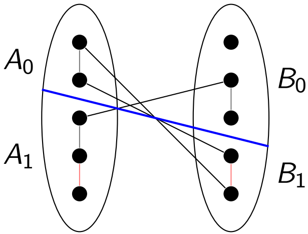

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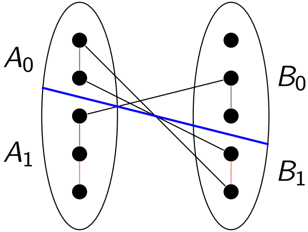

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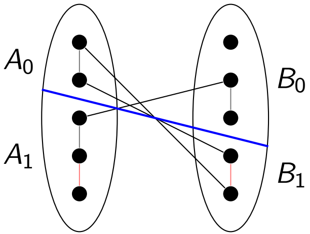

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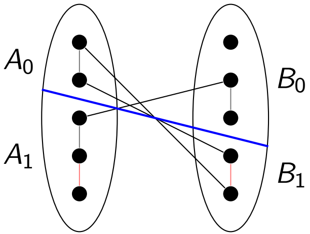

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$T[A, B]$ has a cut which kills *significantly* more edges than non-edges.
Some bipartite graphs have no such cuts.

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$V \subseteq \mathbb{R}^N$, finite set of points with $|B(v,1) \cap V| \geq \delta |V|$ for every $v \in V$.
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Theorem [Brandt, Thomassé '04]

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A quick proof

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If u, v are dominated by $x \in X$ and are neighbors in G then $|N(x) \cup N(u) \cup N(v)| \geq 3 \cdot (1/3 + \varepsilon) \cdot n - 2 \cdot \varepsilon n > n$, a contradiction.

More results

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Theorem

Every tournament with fractional dichromatic number at most k has domination at most $g(k)$.

Domination in majority digraphs

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Theorem

There is always an ε-fair set of recipients of size $f(\varepsilon)$.

Proof sketch

Form a tri-directed graph $T = (V, A, R)$ with $u \to v \in A$ if u is preferred to v by at least $k+1$ referees, and $u \to v \in R$ if u is preferred to v by at least $(1/2 - \varepsilon) \cdot (2k+1)$ referees.

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Using Farkas' Lemma, we can give weights to the vertices such that every vertex has black in-degree at least $n/2$ (keeping the total weight to n).
Then, there exists a black/red dominating set X of size $f(\varepsilon)$.

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Thank you!